Metric spaces related to Abelian groups

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Abstract

When working with a metric space, we are dealing with the additive group $\mathbb{(R,+)}$. Replacing $\mathbb{(R,+)}$ with an Abelian group $\mathbb{(G,\ast)}$, offers a new structure of a metric space. We call it a $G$-metric space and the induced topology is called the $G$-metric topology. In this paper, we are studying $G$-metric spaces based on L-groups (i.e., partially ordered groups which are lattices). Some results in $G$-metric spaces are obtained. The $G$-metric topology is defined which is further studied for its topological properties. We prove that if $G$ is a densely ordered group or an infinite cyclic group, then every $G$-metric space is Hausdorff. It is shown that if $G$ is a Dedekind-complete densely ordered group, $(X,d)$ a $G$-metric space, $A \subseteq X$ and $d$ is bounded, then $f : X \rightarrow G$ with $f(x) = d(x,A) := \inf\{d(x,a) : a \in A\}$ is continuous and further $x \in \text{cl}_X A$ if and only if $f(x) = e$ (the identity element in $G$). Moreover, we show that if $G$ is a densely ordered group and further a closed subset of $\mathbb{R}$, $K(X)$ is the family of nonempty compact subsets of $X$, $e < g \in G$ and $d$ is bounded, then $d'(A,B) < g$ if and only if $A \subseteq N_d(B,g)$ and $B \subseteq N_d(A,g)$, where $N_d(A,g) = \{x \in X : d(x,A) < g\}$, $d_B(A) = \sup\{d(a,B) : a \in A\}$ and $d'(A,B) = \sup\{d_A(B),d_B(A)\}$.

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1. Introduction

In this article, a group \((G, \ast)\) (briefly, \(G\)) is an Abelian group and for readability, we use \(g_1g_2\) instead of \(g_1 \ast g_2\). Let \(X\) be a set and \(\leq\) relation on \(X\), we recall that the pair \((X, \leq)\) is a partially ordered set (in brief, a poset) if the following conditions hold: \(x \leq x\), if \(x \leq y\) and \(y \leq x\), then \(x = y\); if \(x \leq y\) and \(y \leq z\), then \(x \leq z\). In a poset, the symbol \(a \lor b\) denotes \(\sup\{a, b\}\), i.e., the smallest element \(c\), if one exists, such that \(c \geq a\) and \(c \geq b\). Likewise, \(a \land b\) stands for \(\inf\{a, b\}\). When both \(a \lor b\) and \(a \land b\) exist, for all \(a, b \in A\), then \(A\) is called a lattice. A subset \(S\) is a sublattice of \(A\) provided that, for all \(x, y \in S\), the elements \(x \lor y\) and \(x \land y\) of \(A\) belong to \(S\). (Thus, it is not enough that \(x\) and \(y\) have a supremum and an infimum in \(S\).) For instance, \(C(X)\), the ring of real-valued continuous functions on the topological space \(X\) is a lattice. If \(f, g \in C(X)\), then \(f \lor g = \frac{f + g + |f - g|}{2} \in C(X)\) (note, \(f \land g = -(f \lor -g) = \frac{f + g - |f - g|}{2} \in C(X)\)). In fact, \(C(X)\) is a sublattice of \(\mathbb{R}^X\), the ring of real-valued functions on the set \(X\) (note, the partial ordering on \(\mathbb{R}^X\) is: \(f \leq g\) if and only if \(f(x) \leq g(x)\) for all \(x \in X\)). A poset in which every nonempty subset has both a supremum and an infimum is said to be a lattice-complete. For example, \(P(X)\), the set of all subsets of \(X\) with inclusion is lattice-complete. Union (resp. intersection) of sets is the supremum (resp. the infimum) of them. A totally (or linearly) ordered set is a poset in which every pair of elements is comparable, i.e., \(x \leq y\) or \(y \leq x\) for all \(x\) and \(y\) in \(X\). We use “ordered sets” instead of “totally ordered sets”. An ordered set is often referred to as a chain. A lattice need not be an ordered set, necessarily, but the converse is always true. We notice that \(C(X)\) and \(C_\varepsilon(X)\), its subalgebra consisting of elements with countable image, are lattices, while they are not ordered sets. Moreover, they are not lattice-complete necessarily. An ordered set is said to be Dedekind-complete if every nonempty subset with an upper bound has a supremum, and conversely, every nonempty subset with a lower bound has an infimum. (For example, \(\mathbb{R}\), the set of real numbers is Dedekind-complete, but not lattice-complete.) An ordered field \(F\) is said to be archimedean if \(\mathbb{Z}\), the set of integers is cofinal, i.e., for every \(x \in F\), there exists \(n \in \mathbb{Z}\) such that \(n \geq x\). For instance, \(\mathbb{Q}(\sqrt{n}) := \{a + b\sqrt{n} : a, b \in \mathbb{Q}, \sqrt{n} \notin \mathbb{Q}\}\) is an archimedean field.

Theorem 1.1 ([3, Theorem 0.21]). An ordered field is archimedean if and only if it is isomorphic to a subfield of the ordered field \(\mathbb{R}\).

A brief outline of this paper is as follows. In Section 2, we introduce the \(G\)-metric spaces related to \(L\)-groups (i.e., partially ordered groups which are lattices) and study them further. In Section 3, the basic topological properties based on the notion of \(g\)-disk are studied. We prove that if \(G\) is a densely ordered group or an infinite cyclic group, then every \(G\)-metric space is Hausdorff. It is shown that if \(G\) is a Dedekind-complete densely ordered group, \((X, d)\) a \(G\)-metric space, \(d\) is bounded and \(A \subseteq X\), then \(f : X \to G\) given by \(f(x) = d(x, A) := \inf \{d(x, a) : a \in A\}\) is continuous and further \(x \in \text{cl}_X A\) if and only if \(f(x) = e\) (the identity element of \(G\)). Moreover, let
\[ F(X) \] be the family of nonempty closed sets in \( X \), \( e \leq g \in G \), \( A, B \in F(X) \) and \( d_A(B) = \sup \{ d(b, A) : b \in B \} \). Then for the \( G \)-metric space \( (F(X), d') \) (note, \( d'(A, B) = \sup \{ d_A(B), d_B(A) \} \)), we have \( d'(A, B) \leq g \) if and only if \( A \subseteq N_d(B, \bar{g}) \) and \( B \subseteq N_d(\bar{A}, \bar{g}) \), where \( N_d(A, \bar{g}) = \{ x \in X : d(x, A) \leq g \} \). Particularly, if \( G \) is a densely ordered group and further a closed subset of \( \mathbb{R} \), \( X \) is a \( G \)-metric space and \( K(X) \) is the family of nonempty compact sets in \( X \), then \( d'(A, B) < g \) if and only if \( A \subseteq N_d(B, A) \) and \( B \subseteq N_d(A, B) \), where \( N_d(A, B) = \{ x \in X : d(x, B, A) < g \} \).

2. G-METRIC SPACES

**Definition 2.1.** A group \( G \) with a partial ordering relation \( \leq \) is called a partially ordered group (in brief, a poset group) if the binary operation of \( G \) preserves the order, i.e.,

\[
 g_1 \leq g_2 \text{ implies } g_1g_3 \leq g_2g_3 \text{ for all } g_1, g_2, g_3 \in G. \tag{R_1}
\]

Moreover, if a poset group \( G \) is a lattice then \( G \) is called an L-group.

From the above definition, the following facts are evident: \( g_1 \geq g_2 \) if and only if \( g_1^{-1} \leq g_2^{-1} \); \( g \leq e \) if and only if \( g^{-1} \leq e^{-1} \); if \( g_1 \geq g_2 \) and \( g_2 \geq g_3 \), then \( g_1g_2 \geq g_3g_4 \). For example, every archimedean field with the addition is an L-group. But \( \mathbb{Z}_n \) with the addition of modulo \( n \) is not a poset group yet, since this addition does not preserve the order. For an L-group \( G \) and \( g \in G \), we let \( |g| = \sup \{ g, g^{-1} \} = g \lor g^{-1} = |g^{-1}| \).

**Example 2.2.** Consider the group \( G := \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z} \) (\( k \)-times) with the usual addition and the identity element \( e = (0, 0, \ldots, 0) \). Let \( g_1 = (m_1, m_2, \ldots, m_k) \), \( g_2 = (n_1, n_2, \ldots, n_k) \in G \). Define

\[
g_1 \leq g_2 \text{ if and only if } m_i \leq n_i \text{ for all } i = 1, 2, \ldots, k.
\]

We see that \( \leq \) is a partial ordering relation on \( G \). Also, the condition \( (R_1) \) in the above definition is satisfied, i.e., \( G \) is a poset group. Let \( z_i = \max \{ m_i, n_i \} \) and \( z_i' = \min \{ m_i, n_i \} \), where \( i = 1, 2, \ldots, k \). Let \( g_3 = (z_1, z_2, \ldots, z_k) \) and \( g_4 = (z'_1, z'_2, \ldots, z'_k) \). Then we obtain \( g_1 \lor g_2 = g_3 \) and \( g_1 \land g_2 = g_4 \). Hence, \( G \) is an L-group.

By an ordered group, we mean a poset group which is a totally ordered set by its partial ordering relation. It is clear that an ordered group is an L-group. Finally, by Dedekind-complete group, we mean an ordered group which is a Dedekind-complete set with its partial ordering relation, i.e., every nonempty subset with an upper bound has a supremum, or equivalently, every nonempty subset with a lower bound has an infimum. For example, every archimedean field with the addition is a Dedekind-complete group.

**Corollary 2.3.** If \( G \) is an L-group and \( g_1, g_2 \in G \), then \( |g_1g_2| \leq |g_1||g_2| \).

**Proof.** We note that \( g_1g_1^{-1} \leq |g_1| \) and \( g_2g_2^{-1} \leq |g_2| \). Definition 2.1 now gives \( g_1g_2 \leq |g_1||g_2| \) and also \( g_1^{-1}g_2^{-1} \leq |g_1||g_2| \). So we have that \( |g_1g_2| = \sup \{ g_1g_2, (g_1g_2)^{-1} \} \leq |g_1g_2| \), and the result holds. \( \square \)
Definition 2.4. Let $G$ be a poset group and $X$ a nonempty set. We say the function $d : X \times X \to G$ is a $G$-metric on $X$, whenever the following conditions hold, for every $x, y, z \in X$.

(i) $d(x, y) \geq e$ ($e$ is the identity element in $G$),
(ii) $d(x, y) = e$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, y) \leq d(x, z)d(z, y)$ (triangle inequality).

The pair $(X, d)$ (briefly, $X$) is called a $G$-metric space. Evidently, every metric space is a $G$-metric space, when $(G, \ast) = (\mathbb{R}, +)$. If all axioms but the second part of Definition 2.4 are satisfied, we call $d$ a $G$-pseudometric and then $X$ a $G$-pseudometric space. Defining $d(x, y) = e$ for all $x$ and $y$ in $X$, gives a $G$-pseudometric on $X$, called the trivial $G$-pseudometric, in this case, $d$ is a $G$-metric if and only if $X$ is the singleton set $\{x\}$. Although all the material of this section is developed for $G$-metric spaces, the basic results remain true for $G$-pseudometric spaces as well. If $(X, d)$ is a $G$-metric on $X$ and $A$ is a subset of $X$, then $A$ inherits a $G$-metric structure from $X$ in an obvious way, making $A$ a $G$-metric space.

In the following example, we will present some examples of $G$-metric spaces.

Before it, let $X = \mathbb{R}^n$, $G_1 = (\mathbb{R}, +), G_2 = ((0, +\infty), \cdot), G_3 = (\mathbb{R} - \{0\}, \cdot)$ and $G_4 = (\mathbb{Z}_2, \oplus)$, where $+, \cdot,$ are usual addition and multiplication, the symbol $\oplus$ is the addition of modulo 2 and $\ast$ is defined as follows: $x \ast y = x + y + xy$. In $G_3$, the identity element is 0 and the inverse of $x$ is $x^{-1} = -\frac{x}{1-x}$. Checking of the associative property of $\ast$ is easy.

Moreover, let $\varphi_i : G_i \times G_i \to G_i$, where $i = 1, 2, 3, 4$, such that $\varphi_1(x, y) = x - y,$ $\varphi_2(x, y) = \frac{x}{y}, \varphi_3(x, y) = \frac{-xy}{1+x}$ and $\varphi_4(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. Then, since each $\varphi_i$ is continuous, each $G_i$ is a topological group as subspaces of $\mathbb{R}$ with the usual topology.

Example 2.5. Let $X = \mathbb{R}^n$, $G_i$, where $i = 1, 2, 3, 4$, be as defined in the previous discussion. For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in X$, $\|x - y\|$ is the usual norm, i.e., $\|x - y\| = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}$. We claim that each $d_i$, the functions below, is a $G_i$-metric and therefore $(X, d_i)$ is a $G_i$-metric space.

We only check that $d_2$ and $d_3$ satisfy (iv) of Definition 2.4. Other conditions are routine.

1. Let $d_1 : X \times X \to G_1$ such that $d_1(x, y) = \|x - y\|$.
2. Let $d_2 : X \times X \to G_2$ such that $d_2(x, y) = e^{\|x-y\|}$.
3. Let $d_3 : X \times X \to G_3$ such that $d_3(x, y) = e^{\|x-y\|} - 1$.
4. Let $d_4 : X \times X \to G_4$ such that $d_4(x, y) = 0$ if $x = y$; and 1 if $x \neq y$. $d_4$ is called a discrete $G$-metric.

We notice that the identity elements in $G_2$ and $G_3$ are 1 and 0 respectively. Moreover, $d_2(x, y) \geq 1$ and $d_3(x, y) \geq 0$. Now,

$$d_2(x, y) = e^{\|x-y\|} \leq e^{\|x-z\|+\|z-y\|} = e^{\|x-z\|}e^{\|z-y\|} = d_2(x, z)d_2(z, y).$$
Also, we have
\[
\begin{align*}
d_3(x, y) &= e^{\|x-y\|-1} - e^{\|x-z\|e^{\|z-y\|-1}} \\
&= (e^{\|x-z\|-1} + e^{\|z-y\|-1} + e^{\|z-x\|-1})(e^{\|z-y\|-1} - 1) \\
&= d_3(x, z) + d_3(z, y) + d_3(x, z)d_3(z, y) \\
&= d_3(x, z) + d_3(z, y) + d_3(x, y).
\end{align*}
\]
So \(d_2\) and \(d_3\) satisfy the triangle inequality of Definition 2.4.

A \(G\)-metric \(d\) on a set \(X\) is called bounded if \(d(x, y) \leq g_0\), for all \(x, y \in X\) and some \(g_0 \in G\). Thus, the next result is now immediate.

**Corollary 2.6.** Let \(G\) be an \(L\)-group, \(d\) a \(G\)-metric on \(X\), \(e < g_1\) a fixed element in \(G\) and \(d_1(x, y) = \inf\{d(x, y), g_1\}\). Then \(d_1\) is a bounded \(G\)-metric.

**Lemma 2.7.** Let \(G\) be an \(L\)-group; \(A\) and \(B\) are finite subsets of \(G\) such that \(A \geq e\) (i.e., \(a \geq e\), for all \(a \in A\)) and \(B \geq e\). Then

(i) if \(A \leq B\) (i.e., for each \(a \in A\) there is \(b \in B\) such that \(a \leq b\)) and \(e \leq g \in G\), then \(\sup(gA) = g\sup A \leq g\sup B = \sup(gB)\), where \(gA = \{ga : a \in A\}\).

(ii) if \(A \geq B\) (i.e., for each \(a \in A\) there is \(b \in B\) such that \(b \leq a\)) and \(e \leq g \in G\), then \(\inf(gB) = g\inf B \leq g\inf A = \inf(gA)\).

(iii) \(\sup(AB) = \sup A\sup B\), and also \(\inf(AB) = \inf A\inf B\), where \(AB = \{ab : a \in A, b \in B\}\).

**Proof.** First, we note that by definition of an \(L\)-group, each of the finite sets \(A, B\) and \(AB\) has a supremum and an infimum in \(G\). The proofs of (i) and (ii) are routine. (iii). Let \(\sup A = \alpha, \sup B = \beta\) and \(\sup(AB) = \gamma\). Since \(G\) is an \(L\)-group, it is a poset group. So by Definition 2.1, we have \(ab \leq \alpha\beta\), for all \(a \in A\) and \(b \in B\). Evidently, \(\gamma \leq \alpha\beta\). Now, we are ready to show that \(\gamma = \alpha\beta\). For the reverse inclusion, let \(a \in A\) be fixed. Then \(ab \leq \gamma\) implies \(b \leq a^{-1}\gamma\). Therefore, \(B\) is bounded by \(a^{-1}\gamma\). So \(\beta = \sup B = a^{-1}\gamma\), in other words, \(a \leq \beta^{-1}\gamma\). Since \(a \in A\) is arbitrary, we deduce that \(A\) is bounded by \(\beta^{-1}\gamma\). Thus, \(\alpha \leq \beta^{-1}\gamma\). This yields \(\alpha\beta \leq \gamma\), and we are through. The proof of another assertion (infimum) is done similarly. \(\square\)

**Proposition 2.8.** Let \(G\) be an ordered group. Then defining
\[
d : G \times G \to G\text{ given by } d(g_1, g_2) = |g_1g_2^{-1}|,
\]
turns \(G\) into a \(G\)-metric space.

**Proof.** We claim that \(d\) is a \(G\)-metric on \(G\). First, we note that since \(G\) is an ordered group, it is an \(L\)-group and further \(g \in G\) gives \(g \geq e\) or \(g^{-1} \geq e\). So \(|g| = |g^{-1}| = \sup\{g, g^{-1}\} = g\) or \(|g| = |g^{-1}| = \inf\{g, g^{-1}\} = g\). Hence, \(|g| \geq e\) and therefore conditions (i)-(iii) of Definition 2.4 hold. Moreover, if \(g_1, g_2, g_3 \in G\), then Corollary 2.3 implies
\[
d(g_1, g_2) = |g_1g_2^{-1}| = |(g_1g_3^{-1})(g_3g_2^{-1})| \leq |g_1g_3^{-1}||g_3g_2^{-1}|
\]
\[
= d(g_1, g_3)d(g_3, g_2)
\]
This gives \( d \) satisfies the triangle inequality, i.e., it is a \( G \)-metric on \( G \) and hence \( G \) is a \( G \)-metric space.

\[ \square \]

**Corollary 2.9.** Let \( G \) be an ordered group, \( X \) a nonempty set and \( f : X \to G \) a function. Then \( d : X \times X \to G \) given by \( d(x, y) = |f(x)f^{-1}(y)| \) is a \( G \)-pseudometric on \( X \). Moreover, \( d \) is a \( G \)-metric on \( X \) if and only if \( f \) is one-one.

The next proposition generalizes Proposition 2.8.

**Proposition 2.10.** Let \( G \) be an ordered group. Then each of the following binary operations on \( G^n \) (the \( n \) product of \( G \)), turns it into a \( G \)-metric space, where \( g = (g_1, g_2, \ldots, g_n) \) and \( g' = (g'_1, g'_2, \ldots, g'_n) \) are arbitrary elements of \( G^n \).

(i) \( d_1 : G^n \times G^n \to G \) defined by \( d_1(g, g') = |g_1^{-1}g'_1||g_2^{-1}g'_2|\ldots|g_n^{-1}g'_n| \).

(ii) \( d_2 : G^n \times G^n \to G \) defined by

\[ d_2(g, g') = \sup\{|g_1^{-1}g'_1|, |g_2^{-1}g'_2|, \ldots, |g_n^{-1}g'_n|\} \]

**Proof.** We only check that the triangle inequality for \( d_1 \) and \( d_2 \). Other conditions are routine. (i). Let \( g'' = (g_1'', g_2'', \ldots, g_n'') \in G^n \). Then

\[ d_1(g, g'') = |g_1^{-1}g_1||g_2^{-1}g_2|\ldots|g_n^{-1}g_n| \]

\[ = |(g_1^{-1}g_1')(g_2^{-1}g_2')(g_3^{-1}g_2')|\ldots|g_n^{-1}g_n'(g_n^{-1}g_n'')| \]

\[ \leq |g_1^{-1}g_1||g_1^{-1}g_1'g_2^{-1}g_2'g_3^{-1}g_3'|\ldots|g_n^{-1}g_n''g_n'''| \]

\[ = |(g_1^{-1}g_1g_2^{-1}g_2g_3^{-1}g_3)|\ldots|g_n^{-1}g_n''g_n'''| \]

\[ = d_1(g, g')d_1(g', g'') \]

Notice that the above inequality is obtained by Corollary 2.3. (ii). Let \( g'' = (g_1'', g_2'', \ldots, g_n'') \in G^n \) and let

\[ A = \{|g_1^{-1}g_1|, |g_2^{-1}g_2'|, \ldots, |g_n^{-1}g_n''|\} \]

\[ B = \{|g_1^{-1}g_1||g_1^{-1}g_1'g_2^{-1}g_2'g_3^{-1}g_3'|\ldots|g_n^{-1}g_n''g_n'''g_n'''|\} \]

\[ B_1 = \{|g_1^{-1}g_1', |g_2^{-1}g_2'|, \ldots, |g_n^{-1}g_n''|\} \]

\[ B_2 = \{|g_1^{-1}g_1', |g_2^{-1}g_2'', |g_3^{-1}g_3''|, \ldots, |g_n^{-1}g_n'|\} \]

We notice that \( G \) is an \( L \)-group. Now, according to Lemma 2.7, we have \( A \leq B \leq B_1B_2 \). Therefore,

\[ d_2(g, g'') = \sup A \leq \sup B \leq \sup(B_1B_2) = \sup B_1 \sup B_2 = d_2(g, g')d_2(g', g'') \]

which completes the proof. \[ \square \]
3. Basic topological concepts in G-metric spaces and some related results

We begin with the following definition.

**Definition 3.1.** Let $G$ be a poset group, $(X, d)$ a G-metric space and $x$ a point of $X$. Given $e < g \in G$, we let

$$N_d(x, g) = \{y \in X : d(x, y) < g\},$$

and call it the $g$-disk centered at $x$. Also, we put $N_d(x, [g]) = \{y \in X, d(x, y) \leq g\}$.

A subset $U$ of $X$ is said to be open in $X$ if either $U = \emptyset$ or for every $x \in U$ there is a $g \in G$ such that $N_d(x, g) \subseteq U$. Here, $x$ is called an interior point of $U$. The set of all interior points of $U$ is called the interior of $U$, denoted by $U^\circ$ (or $\text{int}_X U$). Also, a set $F$ is called closed if and only if its set-theoretic complement is an open set in $X$. Evidently, a set $F$ is closed if and if every $g$-disk centered at $x$ meets $F$, then $x \in F$.

**Corollary 3.2.** Every $g$-disk $N_d(x, g)$ is an open set in $X$ (and hence $X \setminus N_d(x, g) = \{y \in X : d(x, y) \geq g\}$ is a closed set in $X$).

**Proof.** Let $y \in N_d(x, g)$. Then $g_1 = d(x, y) < g$. We claim that $N_d(y, gg_1^{-1}) \subseteq N_d(x, g)$ (note, $gg_1^{-1} > e$). To see this, assume that $z \in N_d(y, gg_1^{-1})$. Hence, $d(z, x) \leq d(z, y)d(y, x) < gg_1^{-1}g_1 = g$. This yields $z \in N_d(x, g)$, i.e., $N_d(y, gg_1^{-1}) \subseteq N_d(x, g)$, and we are done. \qed

**Definition 3.3.** Let $X$ be a G-metric space and $A \subseteq X$. The closure of $A$ in $X$ is denoted by $\text{cl}_X A$ (or briefly $\text{cl} A$) and defined by the set

$$\text{cl} A = \cap \{F \subseteq X : F \text{ is closed in } X \text{ and } A \subseteq F\}.$$

By the above definition, $A$ is closed if and only if $A = \text{cl} A$.

**Corollary 3.4.** Let $G$ be a poset group, $(X, d)$ a G-metric space and $A = \{x \in X : N_d(x, g) \cap A \neq \emptyset \text{ for all } e < g \in G\}$, where $A \subseteq X$. Then

1. $\overline{A} = \text{cl} A$.
2. If $x \in X$ and $g \in G$, then $N_d(x, g) \subseteq N_d(x, [g])$.

**Proof.** (1). Let $x \notin \text{cl} A$. Then $x \notin F$, for some closed set $F$ containing $A$. Now, since $X \setminus F$ is open, there exists $e < g \in G$, such that $x \in N_d(x, g) \subseteq X \setminus F$. So $x \notin A$. Conversely, suppose that $x \notin A$. So for some $e < g \in G$, $N_d(x, g) \cap A = \emptyset$. Therefore, the closed set $X \setminus N_d(x, g)$ contains $A$ but not $x$. This gives $x \notin \text{cl} A$, and we are done. (2) Suppose that $y \notin N_d(x, [g])$. So $d(x, y) > g$ and hence $g_1 = d(x, y)g^{-1} > e$. We claim that $N_d(y, g_1) \cap N_d(x, g) = \emptyset$. Otherwise, for some $z \in N_d(y, g_1) \cap N_d(x, g)$, we have $d(x, y) \leq d(x, z)d(z, y) < gg_1 = d(x, y)$, a contradiction. Hence, $y \notin N_d(x, g)$ and we are done. \qed

**Proposition 3.5.** Let $G$ be an ordered group and $X$ a G-metric space. Then the open sets in $X$ have the following properties:

1. $X$ and $\emptyset$ are both open.
(ii) Every union of open sets is open.
(iii) Every finite intersection of open sets is open.

Proof. (i) and (ii) are clear. (iii). Let \( x \in \bigcap_{i=1}^{n} U_i \), where \( U_i \) is an open set in \( X \). Take \( g_i \in G \) such that \( x \in N_d(x, g_i) \subseteq U_i \). Since \( G \) is an ordered group, there exists \( g \in G \) such that \( g = \inf \{ g_i \}_{i=1}^{n} \) (note, the elements \( g_i \) form a chain and hence \( g \) is one of them). Thus, \( x \in N_d(x, g) \subseteq \bigcap_{i=1}^{n} N_d(x, g_i) \subseteq \bigcap_{i=1}^{n} U_i \), which completes the proof. \( \square \)

By the above proposition, every \( G \)-metric \( d \) on a set \( X \) defines a topology \( \tau_d \) on \( X \); members of \( \tau_d \), or, open subsets of \( X \) are unions of \( g \)-disks. Clearly, the family of all \( g \)-disks is a base for \( (X, \tau_d) \). We call \( \tau_d \) the topology induced by the \( G \)-metric \( d \) (or \( G \)-metric topology).

Remark 3.6. Even if \( G \) is a Dedekind-complete group, a countable intersection of open sets in a \( G \)-metric space need not be an open set necessarily. To see this, consider \( \mathbb{R} \) as a \( G \)-metric space, where \( G \) is \((\mathbb{R}, +)\) or \((\mathbb{R}, \leq)\). Also, recall the fact that every point \( a \) of \( \mathbb{R} \) is a \( G \)-set, i.e., \( \{ a \} = \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, a+\frac{1}{n}) \).

In [2, 1.3], an ordered set \( X \) is called a densely ordered set, if no cut of \( X \) is a jump, or equivalently, for every pair \( x, y \) of elements of \( X \) satisfying \( x < y \), there exists a \( z \in X \), such that \( x < z < y \).

Definition 3.7. An ordered group \( G \) is called a densely ordered group if it is a densely ordered set with its total ordering relation.

It is clear that densely ordered groups are infinite. For example, every archimedean field like as \( \mathbb{R} \) and \( \mathbb{Q}(\sqrt{p}) := \{ a + b\sqrt{p} : a, b \in \mathbb{Q}, \sqrt{p} \notin \mathbb{Q} \} \) with the addition is a densely ordered group. \( \mathbb{Z} \), the group of integers is an ordered group which is not a densely ordered group while \( \mathbb{Z}_n \) with the addition of modulo \( n \), is not a poset group yet.

From now on, the group \( G \) is assumed to be a densely ordered group.

Proposition 3.8. Let \( G \) be a densely ordered group, \( (X, d) \) a \( G \)-metric space, \( x \in X \) and \( g \in G \) be fixed, and \( A = \{ y \in X : d(x, y) > g \} \). Then \( A \) is an open set in \( X \) (and hence \( N_d(x, y) := \{ y \in G : d(x, y) \leq g \} \) is closed).

Proof. Let \( y \in A \) be fixed. Then \( d(x, y) > g \). We must show that \( y \) is an interior point of \( A \). Let \( g_1 = d(x, y)g^{-1} \). Then \( g_1 > e \) and \( d(x, y) = gg_1 \). Since \( G \) is a densely ordered group, we take \( e < g_2 < g_1 \) and claim that \( N_d(y, g_2) \) is contained in \( A \) entirely. To see this, let \( z \in N_d(y, g_2) \). Then we have \( d(y, z) < g_2 \) and so \( g_2^{-1} < d^{-1}(y, z) \). Now, the inequality \( d(x, y) \leq d(x, z)d(z, y) \) yields

\[
g < gg_1g_2^{-1} < d(x, y)d^{-1}(y, z) \leq d(x, z).
\]

(Notice that \( g < gg_1g_2^{-1} \) if and only if \( g_2 < g_1 \) ) Therefore, \( g < d(x, z) \), i.e., \( y \) is an interior point of \( A \). So \( A \) is an open set in \( X \), and we are done. \( \square \)
Proposition 3.9. Let $G$ be an ordered group and $(X,d)$ a $G$-metric space. Then the following statements hold.

(i) If $G$ is a densely ordered group, then $X$ is Hausdorff.

(ii) If $G$ is an infinite cyclic group, then $X$ is discrete (and so it is first countable).

Proof. (i). Let $x,y \in X$ and $d(x,y) = g > e$. By assumption, since $G$ is a densely ordered set, we can take $g_1, g_2 \in G$ such that $e < g_1 < g_2 < g$. Now, we claim that two disks $N_d(x,g_1)$ and $N_d(y,gg_2^{-1})$ are disjoint (note, $gg_2^{-1} > e$). Otherwise, for some $x' \in N_d(x,g_1) \cap N_d(y,gg_2^{-1})$, we have $d(x,x') < g_1$ and $d(x',y) < gg_2^{-1}$. Hence, $g = d(x,y) \leq d(x,x')d(x',y) < g_1g_2^{-1}$. Therefore, $e < g_1g_2^{-1}$, or equivalently, $g_2 < g_1$, a contradiction. So we are done.

(ii). Let $e < g \in G$ be the generator of $G$. Then $G = \{g^n : n \in \mathbb{Z}\}$, in fact, we have $G \cong \mathbb{Z}$, the additive group of integers with the generator $1$ (or $-1$). We note that the elements of $G$ form a chain. So we obtain

$$\ldots < g^{-3} < g^{-2} < g^{-1} < e < g < g^2 < g^3 < \ldots$$

Therefore, for each $x \in X$, $N_d(x,g) = \{y \in X : d(x,y) < g\} = \{y \in X : d(x,y) = e\} = \{x\}$. This yields $X$ is discrete (note, in this case $G$ is not a densely ordered group).

Remark 3.10. By Proposition 3.9(ii), if $G$ is an infinite cyclic group then $X$ is first countable. But the converse of that result may be false, since every metric space is first countable, whereas the additive group $(\mathbb{R}, +)$ is not even countably generated.

In general, the converse of the above proposition does not need to be true. In the next example, we give examples of Hausdorff $G$-metric spaces such that the group $G$ is neither a densely ordered group nor an infinite cyclic group.

Example 3.11. (i) Let $d_1 : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $d_1(m,n) = |m - n|$. Then, since $N_{d_1}(m,1) = \{m\}$, we obtain $\mathbb{Z}$ is a discrete $\mathbb{Z}$-metric space. So it is Hausdorff, whereas $\mathbb{Z}$ is not a densely ordered group. But if $\mathbb{Z}$ is considered as a $\mathbb{Q}$-metric space with the same definition, $d_1(m,n) = |m - n|$, it is a discrete $\mathbb{Q}$-metric space while $\mathbb{Q}$ is a densely ordered group.

(ii) Let $G := \mathbb{Z} \times \mathbb{Z}$ with the identity element $e = (0,0)$. Define $d_2 : \mathbb{Z} \times \mathbb{Z} \rightarrow G$ with $d_2(m,n) = (|m-n|,|m-n|)$. By example 2.2, $G$ is an $L$-group. It is easy to see that $d_2$ is a $G$-metric on $\mathbb{Z}$. Now, let $g = (1,1)$. Then

$$N_{d_2}(m,g) = \{n \in \mathbb{Z} : d_2(m,n) < g\} = \{m\}.$$ 

This yields $\mathbb{Z}$ is a discrete $G$-metric space, whereas $G = \mathbb{Z} \times \mathbb{Z}$ is not a cyclic group (note, it is a finitely generated group which is generated by the set $\{(0,1),(1,0)\}$).

Definition 3.12. If $(X,d_X)$ (resp. $(Y,d_Y)$) is a $G_1$- (resp. $G_2$-) metric space, a function $f : X \rightarrow Y$ is called continuous at $x_0 \in X$ if and only if for each $e_2 < g_2 \in G_2$ there is some $e_1 < g_1 \in G_1$ such that $d_Y(f(x_0),f(y)) < g_2$.
whenever \( d_X(x_0, y) < g_1 \). \( f \) is called \textit{continuous} on \( X \), if it is continuous at every \( x \in X \).

A simple translation of the above definition is:

\textbf{Corollary 3.13.} A function \( f : X \to Y \) is continuous at \( x_0 \in X \) if and only if for each \( g_2 \)-disk \( \mathcal{N}_{d_Y}(f(x_0), g_2) \) centered at \( f(x_0) \), there is some \( g_1 \)-disk \( \mathcal{N}_{d_X}(x_0, g_1) \) centered at \( x_0 \), such that \( f(\mathcal{N}_{d_X}(x_0, g_1)) \subseteq \mathcal{N}_{d_Y}(f(x_0), g_2) \).

\textbf{Theorem 3.14.} If \((X, d_X)\) and \((Y, d_Y)\) are \( G_1 \)- and \( G_2 \)-metric spaces respectively, a function \( f : X \to Y \) is continuous at \( x_0 \in X \) if and only if for each open set \( V \) of \( Y \) containing \( f(x_0) \), there exists an open set \( U \) of \( X \) containing \( x_0 \) such that \( f(U) \subseteq V \).

\textit{Proof.} (\( \Rightarrow \)) Suppose that \( f \) is continuous at \( x_0 \) and \( V \) is an open set in \( Y \) containing \( f(x_0) \). By definition of open sets, there is \( g_2 \in G_2 \) such that \( f(x_0) \in \mathcal{N}_{d_Y}(f(x_0), g_2) \subseteq V \). By Corollary 3.13, there exists a \( g_1 \)-disk \( \mathcal{N}_{d_X}(x_0, g_1) \) centered at \( x_0 \) such that \( f(\mathcal{N}_{d_X}(x_0, g_1)) \subseteq \mathcal{N}_{d_Y}(f(x_0), g_2) \subseteq V \), where \( g_1 \in G_1 \). It now suffices to choose \( U = \mathcal{N}_{d_X}(x_0, g_1) \).

(\( \Leftarrow \)) Consider \( e_2 < g_2 \in G_2 \) and \( \mathcal{N}_{d_Y}(f(x_0), g_2) \) as an open set in \( Y \) containing \( f(x_0) \). By hypothesis, there exists an open set \( U \) in \( X \) containing \( x_0 \) such that \( f(U) \subseteq \mathcal{N}_{d_Y}(f(x_0), g_2) \). Also, we can take \( e_1 \geq g_1 \in G_1 \) such that \( \mathcal{N}_{d_X}(x_0, g_1) \subseteq U \). So \( f(\mathcal{N}_{d_X}(x_0, g_1)) \subseteq f(U) \subseteq \mathcal{N}_{d_Y}(f(x_0), g_2) \), and we are done. \( \square \)

The following lemma is the counterpart of Lemma 2.7 for a Dedekind-complete group \( G \). The only difference is that there \( A \) and \( B \) were finite subsets of \( G \) but here these sets must be bounded.

\textbf{Lemma 3.15.} Let \( G \) be a Dedekind-complete group and; \( A \) and \( B \) are bounded subsets of \( G \) such that \( A \supseteq B \) (\( \text{i.e.,} \ a \geq e, \text{for all} \ a \in A \) and \( B \supseteq e \)). Then

(i) if \( A \subseteq B \) (\( \text{i.e.,} \ \text{for each} \ a \in A \ \text{there is} \ b \in B \ \text{such that} \ a \leq b \)) and \( e \leq g \in G \), then \( \sup(gA) = g \sup A \leq g \sup B = \sup(gB) \), where \( gA = \{ ga : a \in A \} \).

(ii) if \( A \supseteq B \) (\( \text{i.e.,} \ \text{for each} \ a \in A \ \text{there is} \ b \in B \ \text{such that} \ b \leq a \)) and \( e \leq g \in G \), then \( \inf(gB) = g \inf B \leq g \inf A = \inf(gA) \).

(iii) \( \sup(AB) = \sup A \sup B \), and also \( \inf(AB) = \inf A \inf B \), where \( AB = \{ ab : a \in A, b \in B \} \).

In the remainder of this article, \( G \) is assumed to be a Dedekind-complete densely ordered group (\( \text{i.e.,} \ \text{a densely ordered group in which every bounded nonempty subset has a supremum and an infimum in} \ G \)), \((X, d)\) a \( G \)-metric space, and \( d \) is bounded. The distance of a point \( x \) to a set \( A \subseteq X \) is defined by \( d(x, A) = \inf \{ d(x, a) : a \in A \} \), if \( A \neq \emptyset \), and \( d(x, \emptyset) = e \).

\textbf{Theorem 3.16.}

(i) The mapping \( f : X \to G \) defined by \( f(x) = d(x, A) \) is continuous.

(ii) \( x \in \text{cl}_X A \) if and only if \( f(x) = d(x, A) = e \), in fact, \( \text{cl}_X A = f^{-1}(e) \).
Proof. (i). First, by Proposition 2.8, we have \((G, d')\) is a \(G\)-metric space, where \(d'(g_1, g_2) = |g_1g_2^{-1}|\). Let \(x_0 \in X, g_0 \in G\) and \(N_{d'}(f(x_0), g_0)\) be an open set containing \(f(x_0)\). Then
\[
d(x_0, a) \leq d(x_0, x)d(x, a), \quad \text{and} \quad d(x, a) \leq d(x, x_0)d(x_0, a). \tag{R_2}
\]
Now, if we let \(G_1 = \{d(x_0, a) : a \in A\}\) and \(G_2 = \{d(x_0, x)d(x, a) : a \in A\}\), then \(G_1\) and \(G_2\) are two subsets of \(G\) with the same cardinality and \(G_2 \geq G_1\). By Lemma 3.15 (ii), we have \(\inf_{a \in A} G_1 \leq \inf_{a \in A} G_2\). In other words, taking infimum on both sides of each of the inequalities in \((R_2)\) with respect to \(a \in A\), we obtain
\[
\inf_{a \in A} d(x_0, a) \leq \inf_{a \in A} d(x_0, x) \inf_{a \in A} d(x, a), \quad \text{and} \quad \inf_{a \in A} d(x, a) \leq \inf_{a \in A} d(x_0) \inf_{a \in A} d(x_0, a).
\]
Thus, \(f(x_0) \leq d(x_0, x_0)f(x)\) and \(f(x) \leq d(x_0, x_0)f(x_0)\). Hence, \(f(x_0)f^{-1}(x) \leq d(x_0, x_0)\) and also \(f(x)f^{-1}(x_0) \leq d(x_0, x_0)\), i.e., \(d(x_0, x)\) is a common upper bound for \(f(x_0)f^{-1}(x)\) and \(f^{-1}(x_0)f(x)\). Therefore, \(d'(f(x_0), f(x)) = |f(x)f^{-1}(x_0)| = \sup\{f(x_0)f^{-1}(x), f^{-1}(x_0)f(x)\} \leq d(x_0, x_0)\).

Now, for the \(g_0\)-disk \(N_d(x_0, g_0)\) we have \(f(N_d(x_0, g_0)) \subseteq N_{d'}(f(x_0), g_0)\), and we are through.

(ii). Necessity: First, we note that by Corollary 3.4, \(\text{cl}A = \bar{A} = \{x \in X : N_d(x, g) \cap A \neq \emptyset, \text{for all } g \geq e\}\). If \(d(x, A) = g > e\) then \(d(x, a) \geq g > e\), for all \(a \in A\). By assumption, since \(G\) is a densely ordered group, we can take \(g_1 \in G\) such that \(g > g_1 > e\). Now, we observe that \(N_d(x, g_1) \cap A = \emptyset\). Hence, \(x \notin \bar{A}\).

Sufficiency: Let \(x \notin \bar{A}\). Then \(N_d(x, g) \cap A = \emptyset\), for some \(e < g \in G\). Hence, \(d(x, a) \geq g\), for all \(a \in A\). Therefore, \(d(x, A) \geq g > e\). So \(d(x, A) \neq e\), and we are done.

\[\Box\]

**Theorem 3.17.** Let \(G\) be a Dedekind-complete densely ordered group, \((X, d)\) a \(G\)-metric space, \(d\) is bounded, \(g \in G\), and let \(\mathcal{F}(X)\) be the family of all nonempty closed subsets of \(X\). For \(A, B \in \mathcal{F}(X)\) define
\[
d_B(A) = \sup\{d(a, B) : a \in A\}, \quad \text{and} \quad d'(A, B) = \sup\{d(A, B), d_B(A)\}.
\]
Then the following statements hold.

(1) \(d'\) is a \(G\)-metric on \(\mathcal{F}(X)\). We call it the Hausdorff \(G\)-metric on \(\mathcal{F}(X)\).

(2) \(d'(A, B) \leq g\) if and only if \(A \subseteq N_d(B, g)\) and \(B \subseteq N_d(A, g)\), where \(N_d(A, g) = \{x \in X : d(x, A) \leq g\}\).

**Proof.** (1). (i) and (iii) of Definition 2.4 are evident. Let \(d'(A, B) = e\). Then \(d_B(A) = e = d_A(B)\). So \(d(a, B) = e\) for all \(a \in A\). By Theorem 3.16 (ii), \(a \in \text{cl}B = B\), i.e., \(A \subseteq B\). Similarly, \(B \subseteq A\). This proves (ii) of Definition 2.4. For the proof of triangle inequality, let \(A, B, C \in \mathcal{F}(X)\) and \(a \in A, b \in B, c \in C\). We notice that \(d(a, B) \leq d(a, b)\) and \(d(b, C) \leq d(c, B)\). Thus,
\[
d(a, B) \leq d(a, b) \leq d(a, c)d(c, b).
\]
Taking infimum on both sides of the above inequality with respect to \(c \in C\) plus Lemma 3.15 yield
d(a, B) ≤ \inf_{c \in C} \{d(a, c)d(c, b)\} = \inf_{c \in C} d(a, c) \inf_{c \in C} d(c, b).

Therefore, \( d(a, B) \leq d(a, C)d(b, C) \). Since \( d(b, C) \leq d_C(B) \), we have \( d(a, B) \leq d(a, C)d_C(B) \). Taking supremum on both sides of the latter inequality with respect to \( a \in A \), we obtain

\[
    d_B(A) \leq d_C(A)d_C(B). \quad (R_3)
\]

On the other hand, taking infimum over \( c \in C \) on both sides of the inequalities \( d(b, A) \leq d(a, b) \leq d(a, c)d(c, b) \) we obtain \( d(b, A) \leq d(a, C)d_C(B) \) (Lemma 3.15). Furthermore, \( d(a, C) \leq d_C(A) \) gives \( d(b, A) \leq d_C(A)d(b, C) \). Now, take supremum on both sides of the latter inequality respect to \( b \in B \). Thus,

\[
    d_A(B) \leq d_C(A)d_C(B). \quad (R_4)
\]

Combining \( (R_3) \) and \( (R_4) \) we get

\[
    d'(A, B) \leq d_C(A)d_C(B) \leq d'(A, C)d'(C, B).
\]

Hence, \( d' \) satisfies (iv) of Definition 2.4, and we are done.

(2). \( \Rightarrow \): Let \( d'(A, B) \leq g \). Then \( d_B(A) \leq g \) and \( d_A(B) \leq g \). Hence, \( d(a, B) \leq g \), for all \( a \in A \). So \( A \subseteq N_d(B, g) \). Similarly, \( B \subseteq N_d(A, g) \).

(\( \Leftarrow \)): Since \( A \subseteq N_d(B, g) \), it gives \( d(a, B) \leq g \), for all \( a \in A \), and therefore \( d_B(A) = \sup_{a \in A} d(a, B) \leq g \). The assertion \( d_A(B) \leq g \) is deduced similarly. So \( d'(A, B) \leq g \), and we are through. \( \square \)

**Corollary 3.18.** Let \( G \) be a densely ordered group and further a closed subset of \( \mathbb{R} \), \( K(X) \) the family of nonempty compact subsets of \( X \) and \( A, B \in K(X) \) such that \( X, d, g, d_A \), and \( d' \) be as defined in Theorem 3.17. Then \( d'(A, B) \leq g \) if and only if \( A \subseteq N_d(B, g) \) and \( B \subseteq N_d(A, g) \), where \( N_d(A, g) = \{ x \in X : d(x, A) < g \} \).

**Proof.** We first recall the fact that a nonempty subset of \( \mathbb{R} \) has the least-upper-bound property (equivalently, the greatest-lower-bound property) if and only if it is closed in \( \mathbb{R} \). So \( G \) has the least-upper-bound property and hence it is a Dedekind-complete densely ordered group. Moreover, by Proposition 3.9, \( X \) is Hausdorff and therefore every compact set in \( X \) is closed. Thus, the conditions of Theorem 3.17 are satisfied. The necessary condition is obvious. To prove the sufficiency, let us define

\[
    f_1, f_2 : X \rightarrow G \text{ with } f_1(x) = d(x, A) \text{ and } f_2(x) = d(x, B).
\]

Now, since \( A \) and \( B \) are compact subsets of \( X \) and further; \( f_1 \) and \( f_2 \) are continuous functions on \( X \) (Theorem 3.16), \( f_1(B) \) and \( f_2(A) \) are compact sets in \( G \) (note, since \( G \) is closed, \( f_1 \) and \( f_2 \) are well defined). Therefore, \( \sup f_1(B) \in f_1(B) \) and also \( \sup f_2(A) \in f_2(A) \). So we have

\[
    d_A(B) = \sup f_1(B) = f_1(b_1) = d(b_1, A), \text{ for some } b_1 \in B,
\]

and also

\[
    d_B(A) = \sup f_2(A) = f_2(a_2) = d(a_2, B), \text{ for some } a_2 \in A.
\]
By assumption, we now get $d_A(B) < g$ and $d_B(A) < g$. Hence, $d'(A, B) < g$, and we are through. 

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