From interpolative contractive mappings to generalized Ćirić-quasi contraction mappings

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Abstract

In this article we consider a restricted version of Ćirić-quasi contraction mapping for showing that this mapping generalizes several known interpolative type contractive mappings. Also here we introduce the concept of interpolative strictly contractive type mapping $T$ and prove a fixed point theorem for such mapping over a $T$-orbitally compact metric space. Some examples are given in support of our established results. Finally we give an observation regarding $(\lambda, \alpha, \beta)$-interpolative Kannan contractions introduced by Gaba et al.

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1. Introduction and preliminaries

In the year 1922, S. Banach had established a remarkable fixed point theorem, known as 'Banach Contraction Principle' which is given as follows:

**Theorem 1.1** ([2]). If a mapping $T$ from a complete metric space $(X, d)$ to itself satisfies the following condition

\[ d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X, \]

for some $\alpha \in [0, 1)$ then $T$ possesses a unique fixed point in $X$.  

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Several generalizations of this theorem have been made by researchers, working in the area of fixed point theory, by means of different new type contractive mappings.

Recently E. Karapinar [7] proposed a new Kannan-type contractive mapping via the notion of interpolation and proved a fixed point theorem over metric space. In his paper, Karapinar assumed that the interpolative Kannan-type contractive mapping $T$ over a metric space $X$ satisfies the contractive condition for all $x, y \in X$ with $x \neq Tx$. But in this situation it is to be noted that if this mapping $T$ has a fixed in $X$ then it will be a constant mapping and therefore $T$ has a unique fixed point trivially. To remove such triviality the authors in [8] assumed that interpolative type mappings satisfy the contractive condition for all $x, y \in X \setminus \text{Fix}(T)$, where $\text{Fix}(T)$ is the set of all fixed points of $T$. Though in this case an interpolative contractive type mapping may possess more than one fixed point.

**Definition 1.2** ([7]). In a metric space $(X, d)$, a mapping $T : X \to X$ is said to be interpolative Kannan-type contractive mapping if it satisfies
\[
d(Tx, Ty) \leq \lambda[d(x, Tx)]^\alpha[d(y, Ty)]^{1-\alpha} \quad \text{for all } x, y \in X \setminus \text{Fix}(T),
\]
for some $\lambda \in [0, 1)$ and for some $\alpha \in (0, 1)$.

**Theorem 1.3**. [7] Let $(X, d)$ be a complete metric space and $T : X \to X$ be an interpolative Kannan-type contractive mapping. Then $T$ has at least one fixed point in $X$.

As an extension of interpolative Kannan-type contractive mappings, Karapinar et al. introduced interpolative Reich-Rus-Ćirić type contractions (See [8]). The definition is given below.

**Definition 1.4** ([8]). In a metric space $(X, d)$, a mapping $T : X \to X$ is called interpolative Reich-Rus-Ćirić type contraction mapping if it satisfies
\[
d(Tx, Ty) \leq \lambda[d(x, y)]^{\beta}[d(x, Tx)]^\alpha[d(y, Ty)]^{1-\alpha-\beta} \quad \text{for all } x, y \in X \setminus \text{Fix}(T),
\]
for some $\lambda \in [0, 1)$ and for some $\alpha, \beta \in (0, 1)$.

**Theorem 1.5** ([8]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be an interpolative Reich-Rus-Ćirić type contraction mapping. Then $T$ has a fixed point in $X$.

Further extension of interpolative Kannan-type contractive mappings has been given by Karapinar et al. [9], which is known as interpolative Hardy-Rogers type contraction. The definition is given as follows.

**Definition 1.6** ([9]). In a metric space $(X, d)$, a mapping $T : X \to X$ is said to be interpolative Hardy-Rogers type contraction mapping if it satisfies
\[
d(Tx, Ty) \leq \lambda[d(x, y)]^{\beta}[d(x, Tx)]^\alpha[d(y, Ty)]^{1-\alpha-\beta-\gamma}
\left[\frac{1}{2}(d(x, Ty) + d(y, Tx))\right]^{1-\alpha-\beta-\gamma}
\]
for some $\lambda \in [0, 1)$ and for some $\alpha, \beta, \gamma \in (0, 1)$.
for all $x, y \in X \setminus \text{Fix}(T)$, for some $\lambda \in [0, 1)$ and for $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$.

**Theorem 1.7** ([9]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be an interpolative Hardy-Rogers type contraction mapping. Then $T$ has at least one fixed point in $X$.

Recently C. B. Ampadu [1] has defined interpolative Berinde weak operator in his paper. The definition is given as follows:

**Definition 1.8** ([1]). Let $(X, d)$ be a metric space. We say $T : X \to X$ is

(i) an interpolative Berinde weak operator if it satisfies

$$(1.5) \quad d(Tx, Ty) \leq \lambda [d(x, y)]^{\alpha} [d(x, Tx)]^{1-\alpha}$$

for all $x, y \in X \setminus \text{Fix}(T)$, for some $\lambda \in (0, 1)$ and for some $\alpha \in (0, 1)$.

(ii) an alternate interpolative Berinde Weak operator if it satisfies

$$(1.6) \quad d(Tx, Ty) \leq \lambda \sqrt{d(x, y)d(x, Ty)}$$

for all $x, y \in X \setminus \text{Fix}(T)$, where $\lambda \in (0, 1)$.

Any interpolative Berinde weak operator is an alternate interpolative Berinde Weak operator.

**Theorem 1.9** ([1]). In a complete metric space $(X, d)$ an interpolative Berinde weak operator $T$ always possesses a fixed point.

As a generalization of 'Banach Contraction Principle', Ćirić [3] had introduced a new contractive mapping known as Ćirić-quasi contraction mapping and proved a fixed point theorem for such mappings.

**Theorem 1.10** ([3]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a self mapping. If $T$ satisfies the contractive condition

$$(1.7) \quad d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

for all $x, y \in X$, then $T$ has a unique fixed point in $X$.

In the next section we find some new forms of interpolative contractive mappings and show that these interpolative contractive mappings are nothing but Ćirić-quasi contraction mappings.

## 2. Main results

Let $(X, d)$ be a metric space, $\Delta_{IK}$ be the set of all interpolative Kannan type contractions on $X$ and $\Delta_{SK} = \{T : X \to X : d(Tx, Ty) \leq \lambda \sqrt{d(x, Tx)d(y, Ty)} \text{ for all } x, y \in X \setminus \text{Fix}(T), \text{ where } \lambda \in (0, 1)\}$. 

**Theorem 2.1.** In a metric space $(X, d)$, $\Delta_{IK} = \Delta_{SK}$.

**Proof.** Clearly $\Delta_{SK} \subset \Delta_{IK}$. Now let $T \in \Delta_{IK}$ be chosen as arbitrary. Then there exists $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$(2.1) \quad d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^{1-\alpha}$$

for all $x, y \in X \setminus \text{Fix}(T)$. 

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Now for any \( x, y \in X \setminus \text{Fix}(T) \) we have
\[
(2.1) \quad d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^{1-\alpha}
\]
and also due to symmetry
\[
(2.2) \quad d(Tx, Ty) = d(Ty, Tx) \leq \lambda d(y, Ty)^\alpha d(x, Tx)^{1-\alpha}.
\]
Multiplying the inequalities (2.1) and (2.2) it follows that
\[
\text{d}(Tx, Ty) \leq \sqrt{\lambda d(x, Tx)d(y, Ty)},
\]
which proves that \( T \in \Delta_{SK} \) and hence \( \Delta_{IK} = \Delta_{SK} \). \( \square \)

In a metric space \((X, d)\), let \( \Delta_{IR} \) be the set of all interpolative Reich-Rus-Ćirić type contractions on \( X \) and \( \Delta_{SR} = \{ T : X \to X : d(Tx, Ty) \leq \lambda d(x, y)^\alpha (d(x, Tx)d(y, Ty))^{1-\alpha} \} \) for all \( x, y \in X \setminus \text{Fix}(T) \), where \( \lambda \in [0, 1], \alpha \in (0, 1) \).

**Theorem 2.2.** In a metric space \((X, d)\), \( \Delta_{IR} = \Delta_{SR} \).

**Proof.** It is clearly seen that \( \Delta_{SR} \subset \Delta_{IR} \). Now let \( T \in \Delta_{IR} \) be chosen arbitrarily. Then there exists \( \lambda \in [0, 1] \) and \( \alpha, \beta \in (0, 1) \) such that
\[
(2.3) \quad d(Tx, Ty) \leq \lambda d(x, y)^\alpha d(x, Tx)^\beta d(y, Ty)^{1-\alpha-\beta}
\]
and also due to symmetry we get
\[
(2.4) \quad d(Tx, Ty) = d(Ty, Tx) \leq \lambda d(y, x)^\alpha d(y, Ty)^\beta d(x, Tx)^{1-\alpha-\beta}.
\]
Multiplying the inequalities (2.3) and (2.4) it follows that
\[
d(Tx, Ty) \leq \lambda d(x, y)^\alpha (d(x, Tx)d(y, Ty))^{\frac{1-\alpha}{\beta}},
\]
which proves that \( T \in \Delta_{SR} \) and hence \( \Delta_{IR} = \Delta_{SR} \). \( \square \)

**Remark 2.3.** From the Theorem 2.2 we observe that, \( \beta \) has no importance to define interpolative Reich-Rus-Ćirić type contraction mappings.

Let us take \( \Delta_{IH} \) as the set of all interpolative Hardy-Rogers type contractions and \( \Delta_{SH} = \{ T : X \to X : d(Tx, Ty) \leq \lambda d(x, y)^\alpha (d(x, Tx)d(y, Ty))^\xi \left( \frac{1}{2} d(x, Ty) + d(y, Tx) \right)^{1-\alpha-2\xi} \} \) for all \( x, y \in X \setminus \text{Fix}(T) \), where \( \lambda \in [0, 1], \alpha, \xi \in (0, 1) \) such that \( \alpha + 2\xi < 1 \).

**Theorem 2.4.** In a metric space \((X, d)\), \( \Delta_{IH} = \Delta_{SH} \).

**Proof.** \( \Delta_{SH} \subset \Delta_{IH} \) trivially. Now let \( T \in \Delta_{IH} \) be taken as arbitrary. Then there exists \( \lambda \in [0, 1] \) and \( \alpha, \beta, \gamma \in (0, 1) \) with \( \alpha + \beta + \gamma < 1 \) such that
\[
d(Tx, Ty) \leq \lambda d(x, y)^\alpha d(x, Tx)^\beta d(y, Ty)^\gamma \left( \frac{1}{2} [d(x, Ty) + d(y, Tx)] \right)^{1-\alpha-\beta-\gamma},
\]
From interpolative contractive mappings...

for all \( x, y \in X \setminus Fix(T) \). Now for any \( x, y \in X \setminus Fix(T) \) we have

\[
(2.5) \quad d(Tx, Ty) \leq \lambda d(x,y)^\alpha d(x,Tx)^\beta d(y,Ty)^\gamma \left( \frac{1}{2} [d(x,Ty) + d(y,Tx)] \right)^{1-\alpha-\beta-\gamma}
\]

and also due to the symmetry of \( d \) we get

\[
(2.6) \quad d(Tx, Ty) = d(Ty, Tx) \leq \lambda d(y,x)^\alpha d(y,Ty)^\beta d(x,Tx)^\gamma \left( \frac{1}{2} [d(y,Tx) + d(x,Ty)] \right)^{1-\alpha-\beta-\gamma}.
\]

Multiplying the inequalities (2.5) and (2.6) it follows that

\[
\begin{align*}
\quad d(Tx, Ty) & \leq \lambda d(x,y)^\alpha \{d(x,Tx)d(y,Ty)\}^{\frac{\beta+\gamma}{\gamma+\alpha}} \left( \frac{1}{2} [d(y,Tx) + d(x,Ty)] \right)^{1-\alpha-\beta-\gamma},
\end{align*}
\]

which proves that \( T \in \Delta_{SH} \) and hence \( \Delta_{IH} = \Delta_{SH} \).

\[ \square \]

Remark 2.5. From Theorem 2.1, 2.2 and 2.4 it is clear that in each of the Definitions, \( T \) can be expressed by fewer constants used as powers in the R.H.S.

Now we consider a version of Ćirić-quasi contraction mapping and show that interpolative contractive mappings are special cases of such type of mappings.

Definition 2.6. Let \((X,d)\) be a metric space. A non-identity mapping \( T : X \to X \) is said to be restricted Ćirić-quasi contraction mapping if there exists \( \lambda \in [0,1) \) such that

\[
(2.7) \quad d(Tx, Ty) \leq \lambda M(x,y) \quad \text{for all} \quad x, y \in X \setminus Fix(T),
\]

where \( M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2} [d(y,Tx) + d(x,Ty)]\} \).

Theorem 2.7. In a complete metric space \((X,d)\), a restricted Ćirić-quasi contraction mapping possesses at least one fixed point in \( X \).

Proof. The proof is straightforward so we omit the proof. \[ \square \]

Clearly any Ćirić-quasi contraction mapping is also a restricted Ćirić-quasi contraction mapping but the converse is not true in general. The following examples proves our assertion.

Example 2.8. (i) Let \( X = [0,1] \) be the metric space endowed with the usual metric and \( T : X \to X \) be defined by

\[
T(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\frac{x}{1-x} & \text{if } 0 < x < 1 \\
1 & \text{if } x = 1.
\end{cases}
\]

Then it can be easily checked that \( T \) is a restricted Ćirić-quasi contraction mapping but it is not a Ćirić-quasi contraction mapping, because \( T \) has three fixed points 0, \( \frac{1}{1} \) and 1.
(ii) Let $X = [1, 2]$ together with the usual metric and $T : X \to X$ be defined by

$$T(x) = \begin{cases} 
\frac{x + 1}{2} & \text{if } 1 \leq x < 2 \\
\frac{x}{2} & \text{if } x = 2.
\end{cases}$$

Then it can be easily checked that $T$ is a restricted Ćirić-quasi contraction mapping but it is not a Ćirić-quasi contraction mapping, since $T$ has two fixed points 1 and 2.

(iii) Let $X = [-1, 1]$ be the metric space endowed with the usual metric and $T : X \to X$ be defined by

$$T(x) = \begin{cases} 
\frac{1}{2} & \text{if } x = -1 \\
x & \text{if } -1 < x < 1 \\
-\frac{1}{2} & \text{if } x = 1.
\end{cases}$$

Then it can be easily checked that $T$ is a restricted Ćirić-quasi contraction mapping but it is not a Ćirić-quasi contraction mapping, because $T$ has infinitely many fixed points in $X$.

Let $\Delta_{IH}$ and $\Delta_{IC}$ be the collections of all alternate interpolative Berinde weak mappings and restricted Ćirić-quasi contraction mappings respectively. Now we prove the following theorem.

**Theorem 2.9.** In a metric space $(X, d)$ if $T \in \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW}$ then $T \in \Delta_{IC}$.

**Proof.** Let $T \in \Delta_{IK}$. Then there exists $\lambda \in [0, 1)$ such that $d(Tx, Ty) \leq \lambda \sqrt{d(x, Tx)d(y, Ty)}$ for all $x, y \in X \setminus Fix(T)$. Thus for any $x, y \in X \setminus Fix(T)$ we have

$$d(Tx, Ty) \leq \lambda \sqrt{d(x, Tx)d(y, Ty)}$$

$$\leq \lambda \sqrt{M(x, y)^2} = \lambda M(x, y).$$

If $T \in \Delta_{IR}$ then there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^{\frac{1-\alpha}{2}}$ for all $x, y \in X \setminus Fix(T)$. Thus for any $x, y \in X \setminus Fix(T)$ we have

$$d(Tx, Ty) \leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^{\frac{1-\alpha}{2}}$$

$$\leq \lambda M(x, y)^\alpha \{M(x, y)^2\}^{\frac{1-\alpha}{2}} = \lambda M(x, y).$$

Choose $T \in \Delta_{IH}$. Then there exist $\lambda \in [0, 1)$ and $\alpha, \xi \in (0, 1)$ with $\alpha + 2\xi < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^\xi \left(\frac{1}{2} [d(x, Ty) + d(y, Tx)]\right)^{1-\alpha-2\xi}$ for all $x, y \in X \setminus Fix(T)$. Thus for any $x, y \in X \setminus Fix(T)$ we get

$$d(Tx, Ty) \leq \lambda d(x, y)^\alpha \{d(x, Tx)d(y, Ty)\}^\xi \left(\frac{1}{2} [d(x, Ty) + d(y, Tx)]\right)^{1-\alpha-2\xi}$$

$$\leq \lambda M(x, y)^\alpha \{M(x, y)^2\}^\xi (M(x, y))^{1-\alpha-2\xi} = \lambda M(x, y).$$
Consider \( T \in \Delta_{IW} \). Then there exists \( \lambda \in [0, 1) \) such that \( d(Tx, Ty) \leq \lambda \sqrt{d(x, y)d(x, Tx)} \) for all \( x, y \in X \setminus Fix(T) \). Thus for any \( x, y \in X \setminus Fix(T) \) we have

\[
d(Tx, Ty) \leq \lambda \sqrt{d(x, y)d(x, Tx)} \leq \lambda \sqrt{M(x, y)^2} = \lambda M(x, y). \tag{2.11}
\]

Hence from (2.8), (2.9), (2.10) and (2.11) we have in any case \( T \in \Delta_{IC} \). \( \square \)

Theorem 2.2 [7], Corollary 1 [8], Theorem 4 [9] and Theorem 1.2 [1] follow from our next corollary.

**Corollary 2.10.** In a complete metric space \((X, d)\) if \( T \in \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW} \) then \( T \) has a fixed point in \( X \).

**Proof.** From Theorem 2.9 we see that if \( T \in \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW} \) then \( T \in \Delta_{IC} \). Also Theorem 2.7 says that a mapping \( T \in \Delta_{IC} \) always possesses fixed point in \( X \). Hence the corollary. \( \square \)

Any mapping \( T \in \Delta_{IC} \) may not be a member of \( \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW} \).

The next example supports our contention.

**Example 2.11.** Let us consider \( X = [0, 1] \) equipped with the usual metric. Also let \( T : X \to X \) be defined by

\[
T(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\frac{x}{2} & \text{if } 0 < x < 1 \\
1 & \text{if } x = 1.
\end{cases}
\]

Then clearly \( T \) is a restricted Ćirić-quasi contraction mapping for \( \frac{1}{2} \leq \lambda < 1 \) but not an usual Ćirić-quasi contraction mapping. Also by taking \( x = \epsilon \) and \( y = 1 - \delta \) with \( 0 < \epsilon, \delta < 1 \) and letting \( \epsilon, \delta \to 0 \) we see that \( T \notin \Delta_{IK} \cup \Delta_{IR} \cup \Delta_{IH} \cup \Delta_{IW} \).

In metric fixed point theory, our main objective is to check whether a mapping \( T \) over a complete metric space \( X \) into itself possesses a fixed point in \( X \). In order to satisfy the interpolative Kannan type contractive condition (1.2) for a mapping \( T : X \to X \), we have to know the set \( Fix(T) \) and whenever we know the whole set \( Fix(T) \) why we bother about, whether the mapping \( T \) satisfies the contractive condition (1.2) ?

In one word, to check the existence of fixed points for an interpolative contractive mapping \( T \) in \( X \), we have to know the set \( Fix(T) \) in advance, which is quite absurd.

Moreover, Theorem 2.7 shows that, if any one of the contractive condition (like Banach, Kannan, Chatterjea) holds ”for all \( x, y \in X \setminus Fix(T) \)” instead of ”for all \( x, y \in X \)” then we can easily remove the part uniqueness from the like Theorems (Banach, Kannan, Chatterjea), but in each case we have to know first the set \( Fix(T) \).
From this point of view we can conclude that the Theorem 1.3 has no real significance.

To avoid such a situation we can redefine the contractive condition (1.2) in the way that is given below.

Remark 2.12. In a metric space \((X, d)\) if we define an interpolative mapping \(T : X \to X\) satisfying

\[
d(Tx, Ty) \leq \lambda \sqrt{\max\{d(x, Tx), d(x, y)\} \cdot \max\{d(y, Ty), d(y, x)\}}
\]

for all \(x, y \in X\) and for some \(\lambda \in [0, 1)\), then it is seen that \(T\) can be a non-constant function even if \(T\) has a fixed point in \(X\).

Clearly the contractive condition (2.12) can also be taken as

\[
d(Tx, Ty) \leq \lambda [\max\{d(x, Tx), d(x, y)\}]^\alpha [\max\{d(y, Ty), d(y, x)\}]^{1-\alpha}
\]

for all \(x, y \in X\), for \(\alpha \in (0, 1)\) and for some \(\lambda \in [0, 1)\).

Remark 2.13. It is to be noted that if \(Ba(X)\), \(Mi(X)\) and \(Ci(X)\) are the set of all Banach contractions, interpolative contractive mappings satisfying condition (2.12) and \(\tilde{\text{C}}\text{iri}c\) quasi contractions on \(X\) respectively then \(Ba(X) \subset Mi(X) \subset Ci(X)\). Therefore it is clear that in a complete metric space \((X, d)\) an interpolative contractive mapping \(T\) satisfying condition (2.12) has a unique fixed point.

3. INTERPOLATIVE STRICTLY CONTRACTIVE MAPPINGS OVER A COMPACT METRIC SPACE

In this section we prove some fixed point theorems for interpolative strictly contractive type mappings in the framework of a metric space which is weaker than compact metric space. First we recall the definitions of \(T\)-orbitally compact metric space with respect to a self mapping \(T\) and orbital continuity of a self mapping over a metric space.

Definition 3.1 ([4]). A metric space \((X, d)\) is said to be \(T\)-orbitally compact with respect to a mapping \(T : X \to X\) if for all \(x \in X\), every sequence in the orbit of \(T\) at \(x \in X\) given by \(\mathcal{O}(x, T) = \{x, Tx, T^2x, \ldots\}\) has a convergent subsequence in \(X\).

Definition 3.2 ([5]). Let \((X, d)\) be a metric space. A mapping \(T : (X, d) \to (X, d)\) is said to be orbitally continuous if \(u \in X\) and such that \(u = \lim_{i \to \infty} T^{n_i}x\) for some \(x \in X\), then \(Tu = \lim_{i \to \infty} TT^{n_i}x\).

Theorem 3.3. Let \((X, d)\) be a metric space and \(T : X \to X\) be a mapping which satisfies

\[
d(Tx, Ty) < \Theta(x, y) \text{ for all } x, y \notin Fix(T),
\]

where \(\Theta(x, y) = \max\{\sqrt{d(x, Tx)d(y, Ty)}, d(x, y)^\mu (d(x, Tx)d(y, Ty))^{1/2}\}
\]

\[
d(x, y)\nu \{d(x, Tx)d(y, Ty)\}^{1-\nu} \cdot \left(\frac{1}{\theta} \frac{d(x, Ty) + d(y, Tx)}{1 + d(x, y)}\right)^{1-\xi}
\]

with \(\mu, \nu, \tau, \xi \in (0, 1)\) and \(\nu + 2\tau < 1\). If \(X\) is compact (or, \(T\)-orbitally compact)
then $T$ has atleast one fixed point in $X$, provided that $T$ is orbitally continuous in $X$.

Proof. Let $x_0 \in X$ be chosen as arbitrary. Let us construct an iterative sequence \( \{x_n\} \), where $x_n = T^nx_0$ for all $n \geq 1$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$ then $x_n$ will be a fixed point of $T$. So without loss of generality we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Now from the contractive condition (3.1) we have

\[ (3.2) \quad d(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1}) < \Theta(x_{n-1}, x_n) \quad \text{for all } n \geq 1. \]

Now we have to consider four cases.

**Case-I:** If $\Theta(x_{n-1}, x_n) = \sqrt{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})}$ then we get

\[ (3.3) \quad d(x_n, x_{n+1}) < \sqrt{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})} \]

\[ \Rightarrow d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \]

**Case-II:** If $\Theta(x_{n-1}, x_n) = d(x_{n-1}, x_n)^\mu \{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})\}^{\frac{1+\nu}{2}}$ then we have

\[ (3.4) \quad d(x_n, x_{n+1}) < d(x_{n-1}, x_n)^\mu \{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})\}^{\frac{1+\nu}{2}} \]

\[ \Rightarrow d(x_n, x_{n+1})^{\frac{1+\nu}{2}} < d(x_{n-1}, x_n)^{\frac{1+\nu}{2}} d(x_n, x_{n+1}) \]

**Case-III:** If $\Theta(x_{n-1}, x_n) = d(x_{n-1}, x_n)^\nu \{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})\}^\tau \left(\frac{1}{2} [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\right)^{1-\nu-2\tau}$ then we obtain that

\[ (3.5) \quad d(x_n, x_{n+1}) < d(x_{n-1}, x_n)^\nu \{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})\}^\tau \left(\frac{1}{2} [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\right)^{1-\nu-2\tau} \]

\[ \leq d(x_{n-1}, x_n)^\nu \{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})\}^\tau \left(\frac{1}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right)^{1-\nu-2\tau}. \]

If $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$ then from (3.5) it follows that

\[ (3.6) \quad d(x_n, x_{n+1}) < d(x_{n-1}, x_n)^\nu \{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+1})\}^\tau \left(\frac{1}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right)^{1-\nu-2\tau} \]

\[ \leq d(x_n, x_{n+1}), \text{ a contradiction.} \]

Which implies that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$.
Case-IV: If $\Theta(x_{n-1}, x_n) = d(x_{n-1}, x_n) \xi \left\{ \frac{d(x_{n-1}, x_n) + 1}{1 + d(x_{n-1}, x_n)} \right\}^{1-\xi}$ then we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \xi \left\{ \frac{d(x_{n-1}, x_n) + 1}{1 + d(x_{n-1}, x_n)} \right\}^{1-\xi}$$

$$= d(x_{n-1}, x_n) \xi d(x_n, x_{n+1})^{1-\xi}$$

$$\Rightarrow d(x_n, x_{n+1})^{\xi} < d(x_{n-1}, x_n)^{\xi}$$

Thus from equations (3.3), (3.4), (3.5) and (3.7) we see that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. So $\{d(x_{n-1}, x_n)\}$ is a monotonically decreasing sequence which is bounded below. Therefore there exists some $l \geq 0$ such that $d(x_{n-1}, x_n) \to l$ as $n \to \infty$.

Now since $X$ is compact (or, $T$-orbitally compact), $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, which converges to some $u \in X$. Due to the orbital continuity of $T$ it follows that $\{x_{n_k+1}\}$ converges to $Tu$ and $\{x_{n_k+2}\}$ converges to $T^2u$ respectively. Therefore the continuity of the metric $d$ implies that $\lim_{k \to \infty} d(x_{n_k}, x_{n_k+1}) = d(u, Tu)$ and $\lim_{k \to \infty} d(x_{n_k+1}, x_{n_k+2}) = d(Tu, T^2u)$. So $d(u, Tu) = l = d(Tu, T^2u)$. If $l > 0$ then $u, Tu \notin \text{Fix}(T)$ and therefore

(3.8) $d(Tu, T^2u) < \Theta(u, Tu)$ implies that $d(Tu, T^2u) < d(u, Tu)$, a contradiction.

Hence $l = 0$ and $Tu = u$ that is $u$ is a fixed point of $T$. \qed

From the above theorem we get the following immediate corollaries.

**Corollary 3.4.** Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping which satisfies

(3.9) $d(Tx, Ty) < d(x, Tx)^{\gamma} d(y, Ty)^{1-\gamma}$ for all $x, y \notin \text{Fix}(T)$, $\gamma \in (0, 1)$.

If $X$ is compact (or, $T$-orbitally compact) then $T$ has a fixed point in $X$, provided that $T$ is orbitally continuous in $X$.

**Example 3.5.** Let $X = [0, \infty)$ with the usual metric, $M = \{n + (n + \frac{1}{n})^2 : n \geq 2\}$ and $T : X \to X$ be defined by

$$T(x) = \begin{cases} n & \text{if } x = n + (n + \frac{1}{n})^2, n \geq 2 \\ x & \text{if } x \in X \setminus M. \end{cases}$$

Then $T$ satisfies the contractive condition (3.1) in particular the contractive condition (3.9). Also $X$ is $T$-orbitally compact and $T$ is orbitally continuous on $X$. Here we see that $T$ has infinitely many fixed points in $X$.

**Corollary 3.6.** Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping which satisfies

(3.10) $d(Tx, Ty) < d(x, y)^{\gamma} d(x, Tx)^{\delta} d(y, Ty)^{1-\gamma-\delta}$ for all $x, y \notin \text{Fix}(T)$, $\gamma, \delta \in (0, 1)$.
If \( X \) is compact (or, \( T \)-orbitally compact) then \( T \) has atleast one fixed point in \( X \), provided that \( T \) is orbitally continuous in \( X \).

**Corollary 3.7.** Let \((X, d)\) be a metric space and \( T: X \to X \) be a mapping which satisfies

\[
(3.11)
\]

\[
d(Tx, Ty) < d(x, y)^{\gamma} d(x, Tx)^{\delta} d(y, Ty)^{\zeta} \left( \frac{1}{2} [d(x, Ty) + d(y, Tx)] \right)^{1-\gamma-\delta-\zeta}
\]

for all \( x, y \notin \text{Fix}(T) \), where \( \gamma, \delta, \zeta \in (0, 1) \) with \( \gamma + \delta + \zeta < 1 \). If \( X \) is compact (or, \( T \)-orbitally compact) then \( T \) has a fixed point in \( X \), provided that \( T \) is orbitally continuous in \( X \).

**Corollary 3.8.** Let \((X, d)\) be a metric space and \( T: X \to X \) be a mapping which satisfies

\[
(3.12)
\]

\[
d(Tx, Ty) < d(x, y)^{\xi} \left[ \frac{d(x, Tx) + 1}{1 + d(x, y)} \right]^{1-\xi}
\]

for all \( x, y \notin \text{Fix}(T) \), \( \xi \in (0, 1) \).

If \( X \) is compact (or, \( T \)-orbitally compact) then \( T \) has atleast one fixed point in \( X \), provided that \( T \) is orbitally continuous in \( X \).

### 4. A Remark on Interpolative Kannan Contractivity Conditions

In [6] the authors have defined \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction and prove a fixed point theorem for such mappings. The definition of the mapping is given as follows:

**Definition 4.1 ([6]).** Let \((X, d)\) a metric space and \( T: X \to X \) be a self map. \( T \) is called a \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction, if there exist \( \lambda \in [0, 1) \) and \( \alpha, \beta \in (0, 1) \) with \( \alpha + \beta < 1 \) such that

\[
(4.1) \quad d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta} \text{ for all } x, y \in X \setminus \text{Fix}(T).
\]

**Theorem 4.2 ([6]).** Let \((X, d)\) a complete metric space and \( T: X \to X \) be a \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction with \( \lambda \in [0, 1) \) and \( \alpha, \beta \in (0, 1) \), \( \alpha + \beta < 1 \). Then \( T \) has a fixed point in \( X \).

Theorem 4.2 is not true in general. The next example proves our assertion.

**Example 4.3.** Let \( X = \{ \frac{1}{3}, \frac{1}{2} \} \) with usual metric and \( T: X \to X \) be given by

\[
T(x) = \begin{cases} 
\frac{1}{2} & \text{if } x = \frac{1}{3} \\
\frac{1}{3} & \text{if } x = \frac{1}{2}
\end{cases}
\]

Then \( T \) is a \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction with \( \lambda = \frac{3}{5} \) and \( \alpha = \beta = \frac{1}{3} \). Here \( X \) is complete but \( T \) has no fixed point in \( X \).
Comment/s: (The reason/s why the proof of Theorem 2 in [6] fails)
In the proof of Theorem 2 (See the line number 5 of Theorem 2 in Page 2 of [6]) the authors used the fact that
\begin{equation}
\tag{4.2}
d(x_n, x_{n+1})^{1-\beta} \leq \lambda d(x_{n-1}, x_n)^{\alpha} \leq \lambda d(x_{n-1}, x_n)^{1-\beta}
\end{equation}
whenever $\alpha < 1 - \beta$,
which is actually not true in case $0 < d(x_{n-1}, x_n) < 1$.

Therefore the contractive condition
\begin{equation}
\tag{4.1}
d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha}
\end{equation}
can not be replaced by the contractive condition (4.1).

E. Karapinar have pointed out a similar idea in Example 2 of [10], where he forewarned about the mappings $T : \{x_0, y_0\} \to \{x_0, y_0\}$ defined by $Tx_0 = y_0$ and $Ty_0 = x_0$. These particular type of mappings defined on two point sets satisfy the contractive condition (3) (See [10]) but are fixed-points free.

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