$e_c$-Filters and $e_c$-ideals in the functionally countable subalgebra of $C^*(X)$

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ABSTRACT

The purpose of this article is to study and investigate $e_c$-filters on $X$ and $e_c$-ideals in $C^*_c(X)$ in which they are in fact the counterparts of $z_c$-filters on $X$ and $z_c$-ideals in $C_c(X)$ respectively. We show that the maximal ideals of $C^*_c(X)$ are in one-to-one correspondence with the $e_c$-ultrafilters on $X$. In addition, the sets of $e_c$-ultrafilters and $z_c$-ultrafilters are in one-to-one correspondence. It is also shown that the sets of maximal ideals of $C_c(X)$ and $C^*_c(X)$ have the same cardinality. As another application of the new concepts, we characterized maximal ideals of $C^*_c(X)$. Finally, we show that whether the space $X$ is compact, a proper ideal $I$ of $C_c(X)$ is an $e_c$-ideal if and only if it is a closed ideal in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.

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1. INTRODUCTION

All topological spaces are completely regular Hausdorff spaces and we shall assume that the reader is familiar with the terminology and basic results of [6]. Given a topological space $X$, we let $C(X)$ denote the ring of all real-valued continuous functions defined on $X$. $C_c(X)$ is the subalgebra of $C(X)$ consisting of functions with countable image and $C^*_c(X)$ is its subalgebra consisting of bounded functions. In fact, $C^*_c(X) = C_c(X) \cap C^*(X)$, where elements of $C^*(X)$
are bounded functions of $C(X)$. Recall that for $f \in C(X)$, $Z(f)$ denotes its zero-set:

$$Z(f) = \{x \in X : f(x) = 0\}.$$  

The set-theoretic complement of a zero-set is known as a cozero-set and we denote this set by $\text{coz}(f)$. Let us put $Z_e(X) = \{Z(f) : f \in C_c(X)\}$ and $Z_e^*(X) = \{Z(g) : g \in C_c^*(X)\}$. These two latter sets are in fact equal, since $Z(f) = Z(\frac{f}{1+|f|})$, where $f \in C_c(X)$. A nonempty subfamily $F$ of $Z_e(X)$ is called a $z_e$-filter if it is a filter on $X$. If $I$ is an ideal in $C_c(X)$ and $F$ is a $z_e$-filter on $X$ then, we denote $Z_e[I] = \{Z(f) : f \in I\}$, $\cap Z_e[I] = \cap \{Z(f) : f \in I\}$ and $Z_e^{-1}[F] = \{f : Z(f) \in F\}$. We see that $Z_e[I]$ is a $z_e$-filter and $Z_e^{-1}[Z_e[I]] \supseteq I$. If the equality holds, then $I$ is called a $z_e$-ideal. Moreover, $Z_e^{-1}[F]$ is a $z_e$-ideal and we always have $Z_e[Z_e^{-1}[F]] = F$. So maximal ideals in $C_c(X)$ are $z_e$-ideals. In [5], a Hausdorff space $X$ is called countably completely regular (briefly, $c$-completely regular) if whenever $F$ is a closed subset of $X$ and $x \notin F$, there exists $f \in C_c(X)$ such that $f(x) = 0$ and $f(F) = 1$. In addition, two closed sets $A$ and $B$ of $X$ are also called countably separated (in brief, $c$-separated) if there exists $f \in C_c(X)$ with $f(A) = 0$ and $f(B) = 1$. $c$-completely regular and zero-dimensional spaces are the same, see Theorem 1.1.

If we let $M_p^c = \{f \in C_c(X) : f(p) = 0\} \ (p \in X)$, then the ring isomorphism $\frac{C_c(X)}{M_p^c} \cong \mathbb{R}$ gives that $M_p^c$ is a maximal ideal, in fact, $M_p^c$ is a fixed maximal ideal. Moreover, $\cap Z_e[M_p^c] = \{p\}$.

Our concentration is on the zero-dimensional spaces since in [5] the authors proved that for any space $X$ there is a zero-dimensional Hausdorff space $Y$ such that $C_c(X)$ and $C_c(Y)$ are isomorphic as rings, see Theorem 1.2.

In section 2, we study and investigate the $e_e$-filters on $X$ and $e_e$-ideals in $C_c^*(X)$ which they are in fact the counterpart of [6, 2L]. We show that the maximal ideals of $C_c^*(X)$ are in one-to-one correspondence with the $e_e$-ultrafilters on $X$. Moreover, the sets of $e_e$-ultrafilters and $z_e$-ultrafilters are in one-to-one correspondence. By using the latter facts, it is shown that the sets of maximal ideals of $C_c(X)$ and $C_c^*(X)$ have the same cardinality. Finally, maximal ideals of $C_c^*(X)$ are characterized based on these concepts. In Section 3, our concentration is on the uniform norm topology on $C_c^*(X)$ which is the restriction of the uniform norm topology on $C^*(X)$. It is shown that whenever the space $X$ is compact, a proper ideal $I$ of $C_c(X)$ is an $e_e$-ideal if and only if it is a closed ideal in $C_c(X)$ and if and only if it is an intersection of maximal ideals of $C_c(X)$.

We recite the following results from [5].

**Theorem 1.1** ([5, Proposition 4.4]). Let $X$ be a topological space. Then, $X$ is a zero-dimensional space (i.e., a $T_1$-space with a base consisting of clopen sets) if and only if $X$ is $c$-completely regular space.

**Theorem 1.2** ([5, Theorem 4.6]). Let $X$ be any topological space (not necessarily completely regular). Then, there is a zero-dimensional space $Y$ which is a continuous image of $X$ with $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$. 

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Remark 1.3 ([5, Remark 7.5]). There is a topological space $X$, such that there is no space $Y$ with $C_e(X) \cong C(Y)$.

The following results are the known facts about $C_e(X)$ and we are seeking to get similar results for $C^*_e(X)$.

Proposition 1.4. Let $I$ be a proper ideal in $C_e(X)$ and $\mathcal{F}$ a $z_e$-filter on $X$. Then:

(i) $Z(I)$ is a $z_e$-filter and $Z^{-1}[\mathcal{F}]$ is a $z_e$-ideal of $C_e(X)$.
(ii) If $I$ is maximal then $Z(I)$ is a $z_e$-ultrafilter, and the converse holds if $I$ is a $z_e$-ideal.
(iii) $\mathcal{F}$ is a $z_e$-ultrafilter if and only if $Z^{-1}[\mathcal{F}]$ is a maximal ideal.
(iv) If $\mathcal{F}$ is a $z_e$-ultrafilter and $Z \in Z_e(X)$ meets each element of $\mathcal{F}$, then $Z \in \mathcal{F}$.

Corollary 1.5. There is a one-to-one correspondence $\psi$ between the sets of $z_e$-ideals of $C_e(X)$ and $z_e$-filters on $X$, defined by $\psi(I) = Z(I)$. In particular, the restriction of $\psi$ to the set of maximal ideals is a one-to-one correspondence between the sets of maximal ideals of $C_e(X)$ and $z_e$-ultrafilters on $X$.

2. $e_c$-FILTERS ON $X$ AND $e_c$-IDEALS IN $C^*_e(X)$

For $f \in C^*_e(X)$ and $\epsilon > 0$, we define

$$E^*_e(f) = f^{-1}([\epsilon, \epsilon]) = \{x \in X : |f(x)| \leq \epsilon\}.$$  

Each such set is a zero set, since it is equal to $Z(|f| - \epsilon, 0)$. Conversely, every zero set is also of this form, since for $g \in C^*_e(X)$ we have $Z(g) = E^*_e(|g| + \epsilon)$. For a nonempty subset $I$ of $C^*_e(X)$ we denote $E^*_e(I) = \{E^*_e(f) : f \in I\}$, and $E_e(I) = \bigcup E^*_e[I]$. Moreover, if $\mathcal{F}$ is a nonempty subfamily of $Z_e(X)$, then we define $E^*_e[I]$ = $\{f \in C^*_e(X) : E^*_e(f) \in \mathcal{F}\}$ and $E^{-1}_e[\mathcal{F}] = \bigcap E^{-1}_e[I]$. So we have $E_e(I) = \{E_e(f) : f \in I \text{ and } \epsilon > 0\}$, and $E^{-1}_e[\mathcal{F}] = \{f \in C^*_e(X) : E_e(f) \in \mathcal{F}\}$, for all $\epsilon$. Moreover, $E^{-1}_e(E_e(I)) = \{g \in C^*_e(X) : E_e(g) \in E_e(I)\}$ for all $\delta > 0$ and $E_e(E^{-1}_e[\mathcal{F}]) = \{E_e(f) : E^{-1}_e[\mathcal{F}] \in \mathcal{F}\}$, for all $\delta > 0$.

The next result is now immediate.

Corollary 2.1. The following statements hold.

(i) $I \subseteq E^{-1}_e(E_e(I))$ and $E_e(E^{-1}_e[\mathcal{F}]) \subseteq \mathcal{F}$.
(ii) The mappings $E_e$ and $E^{-1}_e$ preserve the inclusion.
(iii) If $f \in I$ then for each positive integer $n$, $E^*_e(f) = E^*_e(f^n)$.
(iv) If $I$ is an ideal, then $E_e(I)$ is a $z_e$-filter.

Proof. The proofs of (i), (ii) and (iii) are clear. (iv) This is presented in the proof of Proposition 2.5.\]

Examples 2.2 and 2.3 below show that the inclusions in (i) of the above corollary may be strict even when $I$ is an ideal and $\mathcal{F}$ is a $z_e$-filter.
Example 2.2. Let $X$ be the discrete space $\mathbb{N} \times \mathbb{N}$, $f(m, n) = \frac{1}{mn}$ and $I$ the ideal in $C^*_c(X)(= C^*(X))$ generated by $f^2$. Obviously $f \notin I$. Since $\{x \in X : f(x) \leq \epsilon\} = \{x \in X : f^2(x) \leq \epsilon^2\}$, we have $E^*_e(f) \in E_e(I)$. So $I \nsubseteq E^*_e^{-1}(E_e(I))$.

Example 2.3. Let $X$ be the zero-dimensional space $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathcal{F} = \{Z \in Z_c(X) : (0, 0) \in Z\}$. Then $\mathcal{F}$ is a $\mathcal{Z}_c$-filter on $X$. Now, if we define $f(x, y) = \frac{|x| + |y|}{1 + |x| + |y|}$ then $f \in C^*_c(X)(= C^*(X))$ and $Z(f) = \{(0, 0)\}$. Given $\delta > 0$ and $g = f + \epsilon$, so we have $E^*_e(g) = \{(0, 0)\}$. If we take $0 < \delta < \epsilon$, then $E^*_e(g) = \emptyset$. Hence $E^*_e(g)$ is not contained in $E^*_e(E^*_e^{-1}(\mathcal{F}))$. Therefore the latter set is contained in $\mathcal{F}$ properly, which gives the result.

Definition 2.4. A $\mathcal{Z}_c$-filter $\mathcal{F}$ is called an $e_c$-filter if $\mathcal{F} = E_e(E^*_e^{-1}(\mathcal{F}))$, or equivalently, whenever $Z \in \mathcal{F}$ then there exist $f \in C^*_c(X)$ and $\epsilon > 0$ such that $Z = E^*_e(f)$ and $E^*_e(f) \in \mathcal{F}$, for each $\delta > 0$.

Proposition 2.5. If $I$ is a proper ideal in $C^*_c(X)$, then $E_e(I)$ is an $e_c$-filter.

Proof. First, we show that $E_e(I)$ is a $\mathcal{Z}_c$-filter, i.e., it satisfies the following conditions.

(i) $\emptyset \notin E_e(I)$.
(ii) $E^*_e(f), E^*_e(g) \in E_e(I)$, then $E^*_e(f) \cap E^*_e(g) \in E_e(I)$.
(iii) $E^*_e(f) \in E_e(I)$, $Z \in Z_c(X)$ with $Z \geq E^*_e(f)$, then $Z \in E_e(I)$.

(i). Suppose that for some $\epsilon > 0$ and $f \in I$, $E^*_e(f) = \emptyset$. So $\epsilon < |f|$, which yields $f$ is a bounded away from zero. Hence $I$ contains the unit $f$, which is impossible. (ii). This is equivalent to say that if $E^*_e(f), E^*_e(g) \in E_e(I)$, then $E^*_e(f) \cap E^*_e(g)$ contains a member of $E_e(I)$. Suppose that $f', g' \in I$ and $\epsilon', \delta' > 0$ such that $E^*_e(f) = E^*_e(f')$ and $E^*_e(g) = E^*_e(g')$. Without loss of generality, we may suppose that $\delta' < \epsilon' < 1$. Hence $f'^2 + g'^2 \in I$ and $E^*_e(f'^2 + g'^2) \subseteq E^*_e(f') \cap E^*_e(g')$, which gives the result. (iii). Assume that $E^*_e(f) \subseteq Z(f')$, where $f \in I$ and $f' \in C^*_c(X)$. Since $E^*_e(f) = E^*_e(f^2)$ and $Z(f') = Z(|f'|)$, we can suppose that $f \geq 0$ and $f' \geq 0$. Now, define

\[ g(x) = \begin{cases} 
1, & \text{if } x \in E^*_e(f) \\
 f'(x) + \frac{\epsilon}{f(x)}, & \text{if } x \in X \setminus \text{int} E^*_e(f).
\end{cases} \]

So $g$ is continuous, since it is continuous on two closed sets whose union is $X$, in fact, $g \in C^*_c(X)$. Note that $fg \in I$ and

\[ (fg)(x) = \begin{cases} 
f(x), & \text{if } x \in E^*_e(f) \\
 (ff')(x) + \epsilon, & \text{if } x \in X \setminus \text{int} E^*_e(f).
\end{cases} \]

It is easily seen that $Z(f') = E^*_e(fg)$. So $Z(f') \in E_e(I)$. This shows that $E_e(I)$ is a $\mathcal{Z}_c$-filter. Now, apply (i) and (ii) of Corollary 2.1 for the ideal $I$ and the $\mathcal{Z}_c$-filter $E_e(I)$, to get the inclusions $E_e(I) \subseteq E_e(E^*_e^{-1}(E_e(I)))$ and $E_e(E^*_e^{-1}(E_e(I))) \subseteq E_e(I)$, which yields $E_e(I)$ is an $e_c$-filter.

Definition 2.6. An ideal $I$ in $C^*_c(X)$ is called $e_c$-ideal if $I = E^*_e^{-1}(E_e(I))$, or equivalently, if $f \in C^*_c(X)$ and $E^*_e(f) \in E_e(I)$ for all $\epsilon$, then $f \in I$.
Proposition 2.7. If $F$ is a $z_c$-filter, then $E_c^{-1}(F)$ is an $e_c$-ideal in $C^*_c(X)$.

Proof. Let $f, g \in E_c^{-1}(F)$, $h \in C^*_c(X)$ and let $\epsilon$ be an upper bound for $h$ and $\epsilon > 0$. Then $E_c^*(f)$, $E_c^*(g)$ and hence $E_c^*(f) \cap E_c^*(g)$ belong to $F$. Hence $E_c^*(f) \cap E_c^*(g) \subseteq E_c^*(f + g)$ implies that $E_c^*(f + g) \in F$, or equivalently, $f + g \in E_c^{-1}(F)$. Moreover, $E_c^*(f) \subseteq E_c^*(f h)$ implies $fh \in E_c^{-1}(F)$. Therefore $E_c^{-1}(F)$ is ideal. In view of Corollary 2.1, we have $E_c^{-1}(F) \subseteq E_c^{-1}(E_c(E_c^{-1}(F))) \subseteq E_c^{-1}(F)$ and so the equality holds, i.e., $E_c^{-1}(F)$ is an $e_c$-ideal.

$\blacksquare$

Corollary 2.8. Maximal ideals of $C^*_c(X)$ and an arbitrary intersection of them are $e_c$-ideals.

Proof. Let $M$ be a maximal ideal of $C^*_c(X)$. If $E_c^{-1}(E_c(M))$ is not a proper $e_c$-ideal, then it contains the constant function 1 and $E_c^*(1) = \emptyset \in E_c(M)$ $(0 < c \leq 1)$ which is impossible, see Propositions 2.5 and 2.7. Hence $M = E_c^{-1}(E_c(M))$, i.e., $M$ is an $e_c$-ideal. The second part is obtained by this fact, the fact that the intersection of a family of maximal ideals is an ideal contained in each of them and (ii) of Corollary 2.1. $\blacksquare$

The next corollary is an immediate result of Propositions 2.5 and 2.7.

Corollary 2.9. The correspondence $I \mapsto E_c(I)$ is one-one from the set of $e_c$-ideals in $C^*_c(X)$ onto the set of $e_c$-filters on $X$.

Lemma 2.10. (i) Let $I$ and $J$ be ideals in $C^*_c(X)$ and $J$ an $e_c$-ideal. Then $I \subseteq J$ if and only if $E_c(I) \subseteq E_c(J)$.

(ii) Let $F_1$ and $F_2$ be $z_c$-filters on $X$ and $F_1$ an $e_c$-filter. Then $F_1 \subseteq F_2$ if and only if $E_c^{-1}(F_1) \subseteq E_c^{-1}(F_2)$.

Proof. It is straightforward. $\blacksquare$

Proposition 2.11. Let $I$ be an ideal in $C^*_c(X)$ and $F$ a $z_c$-filter on $X$. Then:

(i) $E_c^{-1}(E_c(I))$ is the smallest $e_c$-ideal containing $I$.

(ii) $E_c(E_c^{-1}(F))$ is the largest $e_c$-filter contained in $F$.

Proof. (i). Propositions 2.5 and 2.7 respectively show that $E_c(I)$ is an $e_c$-filter and $E_c^{-1}(E_c(I))$ is an $e_c$-ideal. Now, suppose that $K$ is an $e_c$-ideal containing $I$. So $E_c^{-1}(E_c(I)) \subseteq E_c^{-1}(E_c(K)) = K$. Hence we are done. (ii). This is proved similarly. $\blacksquare$

The next theorem plays an important role in many of the following results.

Theorem 2.12. Let $A$ be a $z_c$-ultrafilter. Then a zero set $Z$ meets every element of $E_c(E_c^{-1}(A))$ if and only if $Z \in A$.

Proof. Since $A$ is a filter and $E_c(E_c^{-1}(A)) \subseteq A$, the sufficient condition is evident. For the necessary condition, it is recalled at first that if $Z$ meets every element of $A$ then $Z \in A$, see (iv) of Proposition 1.4. Now, we claim that if $Z$ meets every element of $E_c(E_c^{-1}(A))$ as a particular subfamily of $A$, then also $Z \in A$. Otherwise, for some $Z' \in A$, $Z \cap Z' = \emptyset$. Since the closed sets $Z$ and
$Z'$ are $c$-completely separated, there is $f \in C^*_c(X)$ (in fact $0 \leq f \leq 1$) such that $f(Z) = 1$ and $f(Z') = 0$. Notice that $Z' \subseteq Z(f) \subseteq E^c_\epsilon(f)$, for all $\epsilon$, and; $E^c_\epsilon(f) \in \mathcal{A}$, since $Z' \in \mathcal{A}$. So $E^c_\epsilon(f) \in E_\epsilon(E^{-1}_c(A))$. Now, if $\epsilon$ is taken less than 1, then $Z \cap E^c_\epsilon(f) = \emptyset$ which contradicts with our assumption of $Z$. So $Z \in \mathcal{A}$ and the proof is complete. □

The following proposition shows that, as $Z^{-1}_c(A)$ is a maximal ideal in $C^*_c(X)$, $E^{-1}_c(A)$ is also a maximal ideal in $C^*_c(X)$, where $A$ is a $z_c$-ultrafilter on $X$.

**Proposition 2.13.** Let $A$ be a $z_c$-ultrafilter on $X$. Then:

(i) $E^{-1}_c(A)$ is a maximal ideal.

(ii) $E^{-1}_c(A)$ is an $e_c$-ideal.

(iii) $E^{-1}_c(A) = E^{-1}_c(E_\epsilon(E^{-1}_c(A)))$.

**Proof.** (i). Let $M$ be a maximal ideal of $C^*_c(X)$ containing $E^{-1}_c(A)$. Hence $E_\epsilon(E^{-1}_c(A)) \subseteq E_\epsilon(M)$. Since every element of $E_\epsilon(M)$ meets every element of $E^{-1}_c(E^{-1}_c(A))$, Theorem 2.12 gives $E_\epsilon(M) \subseteq A$. So $M = E^{-1}_c(E_\epsilon(M)) \subseteq E^{-1}_c(A)$ and hence $M = E^{-1}_c(A)$. (ii). It follows by (i). (iii). Since the maximal ideal $E^{-1}_c(A)$ is contained in the proper ideal $E^{-1}_c(E_\epsilon(E^{-1}_c(A)))$, the result now holds. □

An $e_c$-ultrafilter on $X$ is meant a maximal $e_c$-filter, i.e., one not contained in any other $e_c$-filter. As usual, every $e_c$-filter $F$ is contained in an $e_c$-ultrafilter. This is obtained by considering the collection of all $e_c$-filters containing $F$ and the use of the Zorn’s lemma, where the partially ordered relation on $F$ is inclusion.

**Proposition 2.14.** Let $M$ be an ideal in $C^*_c(X)$ and $F$ a $z_c$-filter on $X$. Then:

(i) If $M$ is a maximal ideal then $E_\epsilon(M)$ is an $e_c$-ultrafilter.

(ii) If $F$ is an $e_c$-ultrafilter then $E^{-1}_c(F)$ is a maximal ideal.

(iii) If $M$ is an $e_c$-ideal, then $M$ is maximal if and only if $E_\epsilon(M)$ is an $e_c$-ultrafilter.

(iv) If $F$ is an $e_c$-filter, then $F$ is $e_c$-ultrafilter if and only if $E^{-1}_c(F)$ is a maximal ideal.

**Proof.** (i). Note that $M = E^{-1}_c(E_\epsilon(M))$. Let $F'$ be an $e_c$-ultrafilter containing $E_\epsilon(M)$, then $M \subseteq E^{-1}_c(F')$ and hence $M = E^{-1}_c(F')$. Therefore $E_\epsilon(M) = E_\epsilon(E^{-1}_c(F')) = F'$, which yields the result. (ii). Let $M$ be a maximal ideal of $C^*_c(X)$ containing $E^{-1}_c(F)$. Then $F \subseteq E_\epsilon(M)$. Hence $F = E_\epsilon(M)$ and so $E^{-1}_c(F) = M$. The proofs of (iii) and (iv) are similarly done and further details are omitted. □

**Corollary 2.15.** There is a one-to-one correspondence $\psi$ between the sets of maximal ideals of $C^*_c(X)$ and $e_c$-ultrafilters on $X$, defined by $\psi(M) = E_\epsilon(M)$.

**Proposition 2.16.** Let $A$ be a $z_c$-ultrafilter. Then it is the unique $z_c$-ultrafilter containing $E_\epsilon(E^{-1}_c(A))$, and also $E_\epsilon(E^{-1}_c(A))$ is the unique $e_c$-ultrafilter contained in $A$. Hence every $e_c$-ultrafilter is contained in unique $z_c$-ultrafilter.
Proof. Let $\mathcal{B}$ be a $z_c$-ultrafilter containing $E_c(E_c^{-1}(A))$ and $Z \in \mathcal{B}$. Since $Z$ meets every element of $E_c(E_c^{-1}(A))$, Theorem 2.12 gives $\mathcal{B} \subseteq A$ and hence $\mathcal{B} = A$. So the first part of the proposition holds. Now, let $\mathcal{K}$ be an $e_c$-ultrafilter contained in $A$. Then $\mathcal{K} = E_c(E_c^{-1}(\mathcal{K})) \subseteq E_c(E_c^{-1}(A))$. Since the latter set is an $e_c$-filter, the inclusion cannot be proper, i.e., $\mathcal{K} = E_c(E_c^{-1}(A))$. Hence the result is obtained.

\textbf{Corollary 2.17.} The $z_c$-ultrafilters are in one-to-one correspondence with the $e_c$-ultrafilters.

Proof. Consider the mapping $\psi$ from the set of $z_c$-ultrafilters into the set of $e_c$-ultrafilters defined by $\psi(A) = E_c(E_c^{-1}(A))$. If $\psi(A) = \psi(\mathcal{B})$, then we have that $E_c(E_c^{-1}(A)) = E_c(E_c^{-1}(\mathcal{B}))$ and it is contained in both $A$ and $\mathcal{B}$. So each element of $\mathcal{B}$ meets each element of $E_c(E_c^{-1}(A))$. Now, Theorem 2.12 gives $\mathcal{B} \subseteq A$. Similarly, $A \subseteq \mathcal{B}$. Therefore $\psi$ is one-one. Let $\mathcal{K}$ be an $e_c$-ultrafilter and $A$ the unique $z_c$-ultrafilter containing it (Proposition 2.16). Then $\mathcal{K} = E_c(E_c^{-1}(A))$ and hence $\psi(A) = \mathcal{K}$. Therefore $\psi$ is onto.

Our next two theorems are applications that are based on the concepts of $e_c$-filters and $e_c$-ideals. In the first result (Theorem 2.18) we show that the maximal ideals of $C_c(X)$ are in one-to-one correspondence with those ones of $C^*_c(X)$ and the second result (Theorem 2.20) involves characterization of maximal ideals of $C^*_c(X)$.

\textbf{Theorem 2.18.} Let $\mathcal{M}$ (resp. $\mathcal{M}^*$) be the set of maximal ideals of $C_c(X)$ (resp. $C^*_c(X)$). Then $\mathcal{M}$ and $\mathcal{M}^*$ have the same cardinality.

Proof. If $M \in \mathcal{M}$ then $Z_c[M]$ is a $z_c$-ultrafilter and hence $E_c^{-1}(Z_c[M]) \in \mathcal{M}^*$, see Propositions 1.4 and 2.13. Define

$$
\varphi : \mathcal{M} \to \mathcal{M}^* \text{ which } M \mapsto E_c^{-1}(Z_c[M]).
$$

If $\varphi(M) = \varphi(M')$ then $E_c(E_c^{-1}(Z_c[M])) = E_c(E_c^{-1}(Z_c[M']))$ and it is contained in both $Z_c[M]$ and $Z_c[M']$. Since each element of $Z_c[M]$ meets each element of $E_c(E_c^{-1}(Z_c[M]))$, Theorem 2.12 yields $Z_c[M'] \subseteq Z_c[M]$. Similarly, $Z_c[M] \subseteq Z_c[M']$. Therefore $M = M'$. This verifies $\varphi$ is one-one. To show that $\varphi$ is onto, suppose that $M^* \in \mathcal{M}^*$. Hence $E_c(M^*)$ is an $e_c$-ultrafilter (Proposition 2.14). Now, let $A$ be the unique $z_c$-ultrafilter containing $E_c(M^*)$ (Proposition 2.16), then $Z_c^{-1}[A]$ is a maximal ideal in $C_c(X)$ (Proposition 1.4) and $M^* = E_c^{-1}(A)$. Recall that if $F$ is a $z_c$-filter, then we always have $Z_c[Z_c^{-1}[F]] = F$ and $E_c(E_c^{-1}(F)) \subseteq F$, but the equality occurs if $F$ is an $e_c$-filter. Now, if we let $M = Z_c^{-1}[A]$ then $\varphi(M) = E_c^{-1}(Z_c[M]) = E_c^{-1}(A) = M^*$. Hence $\varphi$ is onto, which it completes the proof.

\textbf{Remark 2.19.} Combining Corollaries 1.5, 2.15 and 2.17 gives another proof of the above theorem.

\textbf{Theorem 2.20.} Let $M$ be an ideal in $C^*_c(X)$. Then $M$ is maximal if and only if whenever $f \in C^*_c(X)$ and each $E_c^c(f)$ meets every element of $E_c(M)$, then $f \in M$. 

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Suppose that \( f \notin M \). So \((M, f) = C^*_c(X)\). Hence \( h + fg = 1 \), for some \( h \in M \) and \( g \in C^*_c(X) \). Let \( u \) be an upper bound for \( g \) and \( 0 < \epsilon < 1 \). Then
\[
\emptyset = E^*_c(1) = E^*_c(h + fg) \supseteq E^*_c(h) \cap E^*_c(fg) \supseteq E^*_c(h) \cap E^*_c(f),
\]
which contradicts with our assumption, since \( E^*_c(h) \cap E^*_c(f) \neq \emptyset \). So we are done.

**Sufficiency:** Let \( M' \) be a maximal ideal of \( C^*_c(X) \) containing \( M \) and \( f \in M' \). Then \( E_c(M) \subseteq E_c(M') \) and \( E^*_c(f) \in E_c(M') \), for all \( \epsilon \). Since \( E_c(M') \) is an \( e_c \)-filter, \( E^*_c(f) \) meets every element of \( E_c(M') \). Hence it also meets each element of \( E_c(M) \). Now, by hypothesis \( f \in M \). Therefore \( M = M' \), which gives the result. \( \square \)

### 3. Uniform Norm Topology on \( C^*_c(X) \) and Related Closed Ideals

Consider the supremum-norm on \( C^*_c(X) \), i.e., \( \|f\| = \sup_{x \in X} |f(x)| \), where \( f \in C^*_c(X) \). So its restriction on \( C^*_c(X) \) is also the supremum-norm. This defines a metric \( d \) as usual, \( d(f, g) = \|f - g\| \). The resulting metric topology is called the uniform norm topology on \( C^*_c(X) \). Convergence in this topology is uniform convergence of the functions. A base for the neighborhood system at \( g \) consists of all sets of the form
\[
\{f : \|f - g\| \leq \epsilon\} \quad (\epsilon > 0).
\]
Equivalently, a base at \( g \) is given by all sets
\[
\{f : |f(x) - g(x)| \leq u(x) \text{ for every } x \in X\},
\]
where \( u \) is a positive unit of \( C^*_c(X) \).

If \( I \) is an ideal in \( C^*_c(X) \) then its closure in \( C^*_c(X) \) is denoted by \( \text{cl}I \).

**Proposition 3.1.** Let \( I \) be an ideal in \( C^*_c(X) \). Then:

(i) \( \text{cl}I \) is ideal.

(ii) If \( I \) is a proper ideal then \( \text{cl}I \) is also a proper ideal.

(iii) If \( I \) is an \( e_c \)-ideal then it is a closed ideal.

**Proof.**

(i). Let \( f, g \in \text{cl}I, h \in C^*_c(X) \) and let \( u \) be an upper bound for \( h \) and \( \epsilon > 0 \) is fixed. Then for some \( f' \in N_{\frac{1}{2}}(f) \cap I \) and \( g' \in N_{\frac{1}{2}}(g) \cap I \) we have \( f' + g' \in N_{\frac{1}{2}}(f + g) \cap I \). Moreover, there exists \( f_1 \in N_{\frac{1}{2}}(f) \cap I \) and hence \( f_1 h \in N_{\frac{1}{2}}(fh) \cap I \). So \( \text{cl}I \) contains \( f + g \) and \( fh \). Hence \( \text{cl}I \) is ideal. (ii). If \( \text{cl}I \) is not a proper ideal then \( 1 \in \text{cl}I \) and hence \( N_\epsilon(1) \cap I \) contains a unit element of \( C^*_c(X) \) such as \( f \), since \( 1 - \epsilon < f < 1 + \epsilon \) gives \( f \) is bounded away from zero (of course, when \( 0 < \epsilon < 1 \)). But this is impossible since \( f \in I \). Thus \( \text{cl}I \) is a proper ideal.

(iii). Let \( g \in \text{cl}I \) and \( \epsilon > 0 \) arbitrary. Then for some \( f \in N_{\frac{1}{2}}(g) \cap I \) and all \( x \in E^*_c(f) \), we have
\[
|g(x)| = |g(x) - f(x) + f(x)| \leq |g(x) - f(x)| + |f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Hence \( E^c_\epsilon(f) \subseteq E^c_\epsilon(g) \). Since the \( \epsilon \)-filter \( E_\epsilon(I) \) contains \( E^c_\epsilon(f) \), it also contains \( E^c_\epsilon(g) \), for all \( \epsilon \). So \( g \in E^{-1}_c(E_\epsilon(I)) = I \) and therefore \( \epsilon I \subseteq I \). This proves that \( I \) is closed and hence the proof is complete. \( \square \)

Immediately, we find there is no proper dense ideal in \( C^*_c(X) \), and further maximal ideals of \( C^*_c(X) \) and hence every intersection of them are closed, see Corollary 2.8 and (iii) of the above proposition.

We recall that \([6, 1D(1)]\) plays a useful role in the context of \( C(X) \). The following is the counterpart for \( C_c(X) \).

**Proposition 3.2.** If \( f, g \in C_c(X) \) and \( Z(f) \) is a neighborhood of \( Z(g) \), then \( f = gh \) for some \( h \in C_c(X) \).

In the remainder of this section, the zero-dimensional topological space \( X \) will be assumed to be compact. Hence it is \( c \)-pseudo-compact, i.e, \( C_c(X) = C^*_c(X) \).

**Lemma 3.3.** Let \( X \) be a compact space, \( I \) an ideal in \( C_c(X) \), \( f \in C_c(X) \) and \( Z(f) \) a neighborhood of \( \cap Z_c[I] \). Then \( f \in I \).

**Proof.** First, we recall that \( X \) is compact if and only if the intersection of members of any collection consisting of nonempty closed subsets of \( X \) with the finite intersection property (i.e., the intersection of each of a finite number of them is nonempty) is nonempty. The lemma is obvious when \( I = C_c(X) \). Now, if \( I \) is a proper ideal in \( C_c(X) \) then \( Z_c[I] \) satisfies the finite intersection property and hence \( \cap Z_c[I] \neq \emptyset \). By assumption \( \cap Z_c[I] \subseteq \text{int}Z(f) \). Hence \( X \setminus \text{int}Z(f) \subseteq \bigcup_{g \in I} \text{coz}(g) \) and so \( X = \bigcup_{f \in I} \text{coz}(g) \cup \text{int}Z(f) \). By compactness of \( X \), there are a finite number of elements of \( I \), say \( g_1, g_2, \ldots, g_n \), such that

\[
X = \bigcup_{i=1}^n \text{coz}(g_i) \cup \text{int}Z(f).
\]

Now, if we let \( g = \sum_{i=1}^n g_i^2 \) then \( g \in I \) and \( \emptyset \neq Z(g) = \bigcap_{i=1}^n Z(g_i) \subseteq \text{int}Z(f) \). In view of Proposition 3.2, \( f \) is a multiple of \( g \) and hence it is contained in \( I \). So the proof is complete. \( \square \)

**Proposition 3.4.** If \( g \in C_c(X) \) and \( \epsilon > 0 \) is fixed, then there exists \( f \in C_c(X) \) such that \( \| g - f \| \leq \epsilon \) and \( Z(f) \) is a neighborhood of \( Z(g) \).

**Proof.** The trivial solution is \( f = g \), of course when \( Z(g) \) is open. In general, it suffices to define

\[
f(x) = \begin{cases} 
g(x) - \epsilon, & \text{if } x \in g^{-1}((\epsilon, +\infty)) \\
0, & \text{if } x \in E_c^\epsilon(g) \\
g(x) + \epsilon, & \text{if } x \in g^{-1}((-\infty, -\epsilon]).
\end{cases}
\]

We note that \( X \) is the union of three closed sets \( g^{-1}((\epsilon, +\infty)) \), \( E_c^\epsilon(g) \) and \( g^{-1}((-\infty, -\epsilon]) \) and further \( f \) is continuous on each of them. Therefore \( f \) is continuous on \( X \), i.e., \( f \in C(X) \). Notice that the definition of \( f \) makes the
A maximal ideal in \( N \in Z \) is a neighborhood of \( Z \). If \( g / I \) means that \( g / I \in \epsilon \), then \( g / I \in \epsilon \) is a neighborhood of \( Z \).

**Theorem 3.5.** Let \( I \) be a proper ideal in \( C_c(X) \), \( \mathcal{T} = \cap \{ M_p^c : M_p^c \supseteq I \} \) and \( J = \{ g \in C_c(X) : Z(g) \supseteq \cap Z_c[I] \} \). Then:

(i) \( \mathcal{T} = J \).

(ii) \( \cap Z_c[I] = \cap Z_c[J] \).

**Proof.** (i). Let \( g \in J \) and \( M_p^c \) be a fixed maximal ideal of \( C_c(X) \) containing \( I \). Then \( Z(g) \supseteq \cap Z_c[I] \supseteq \cap Z_c[M_p^c] = \{ p \} \). So \( g(p) = 0 \) and hence \( g \in M_p^c \). Therefore \( g \in \mathcal{T} \). For the reverse inclusion, we show that if \( g \notin J \) then \( g \notin \mathcal{T} \).

If \( g \notin J \) then there exists \( x \in \cap Z_c[I] \setminus Z(g) \). So \( I \subseteq M_x^c \) but \( g \notin M_x^c \). This means that \( g \notin \mathcal{T} \). The proof of (i) is now complete.

(ii). By (i), we have \( \cap Z_c[\mathcal{T}] = \cap Z_c[J] \supseteq \cap Z_c[I] \). On the other hand, \( I \subseteq \mathcal{T} \) implies \( Z_c[I] \subseteq Z_c[\mathcal{T}] \) and therefore \( \cap Z_c[I] \supseteq \cap Z_c[\mathcal{T}] \). So it gives the result.

**Corollary 3.6.** Let \( I \) be a proper ideal in \( C_c(X) \) and \( \mathcal{T} \) as defined in Theorem 3.5. Then \( \mathcal{T} = \text{cl} I \).

**Proof.** Since maximal ideals are closed, \( \cap I \subseteq M \) is also closed, where \( M \) is a maximal ideal in \( C_c(X) \). Therefore \( \text{cl} I \subseteq \cap I \subseteq \mathcal{T} \). Let \( g \in \mathcal{T} \) and \( N_c(g) \) is a neighborhood of \( g \). By Proposition 3.4, there is \( f \) such that \( Z(f) \) is a neighborhood of \( Z(g) \) and \( ||g - f|| \leq \epsilon \). Hence, by Theorem 3.5, \( \cap Z_c[I] \subseteq Z(g) \subseteq \text{int} Z(f) \) and therefore Lemma 3.3 implies \( f \in I \). Now, since \( f \in N_c(g) \cap I \), it gives \( g \in \text{cl} I \). So \( \mathcal{T} \subseteq \text{cl} I \) and we are done.

We conclude the article with the following results for the proper ideals of \( C_c(X) \). Corollary 3.7 is a consequence of Corollary 2.8, Proposition 3.1 (iii) and Corollary 3.6; by the same results, plus Corollary 3.7 we obtain Corollary 3.8; finally Corollary 3.9 is the combination of Corollaries 3.7 and 3.8.

**Corollary 3.7.** An ideal \( I \) of \( C_c(X) \) is closed in \( C_c(X) \) if and only if it is an intersection of maximal ideals of \( C_c(X) \).

**Corollary 3.8.** An ideal \( I \) of \( C_c(X) \) is an \( e_c \)-ideal if and only if it is closed in \( C_c(X) \).

**Corollary 3.9.** An ideal \( I \) of \( C_c(X) \) is an \( e_c \)-ideal if and only if it is an intersection of maximal ideals of \( C_c(X) \).

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$e_c$-Filters and $e_c$-ideals in the functionally countable subalgebra of $C^*(X)$

References