On monotonous separately continuous functions

YAROSLAV I. GRUSHKA

Department of Nonlinear analysis, Institute of Mathematics NAS of Ukraine, Kyiv (grushka@imath.kiev.ua)

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Abstract

Let \( T = (T, \leq) \) and \( T_1 = (T_1, \leq_1) \) be linearly ordered sets and \( X \) be a topological space. The main result of the paper is the following:

If function \( f(t, x) : T \times X \to T_1 \) is continuous in each variable ("t" and "x") separately and function \( f_x(t) = f(t, x) \) is monotonous on \( T \) for every \( x \in X \), then \( f \) is continuous mapping from \( T \times X \) to \( T_1 \), where \( T \) and \( T_1 \) are considered as topological spaces under the order topology and \( T \times X \) is considered as topological space under the Tychonoff topology on the Cartesian product of topological spaces \( T \) and \( X \).

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1. Introduction

In 1910 W.H. Young had proved the following theorem.

**Theorem A** (see [9]). Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be separately continuous. If \( f(\cdot, y) \) is also monotonous for every \( y \), then \( f \) is continuous.

In 1969 this theorem was generalized for the case of separately continuous function \( f : \mathbb{R}^d \to \mathbb{R} \) (\( d \geq 2 \)):

**Theorem B** (see [5]). Let \( f : \mathbb{R}^{d+1} \to \mathbb{R} \) (\( d \in \mathbb{N} \)) be continuous in each variable separately. Suppose \( f(t_1, \ldots, t_d, \tau) \) is monotonous in each \( t_i \) separately (\( 1 \leq i \leq d \)). Then \( f \) is continuous on \( \mathbb{R}^{d+1} \).

Note that theorems A and B were also mentioned in the overview [2]. In the papers [6, 7] authors investigated functions of kind \( f : T \times X \to \mathbb{R} \), where
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\((T, \leq)\) is linearly ordered set equipped by the order topology, \((X, \tau_X)\) is any topological space and the function \(f\) is monotonous relatively to the first variable as well continuous (or quasi-continuous) relatively to the second variable. In particular in \([7]\) it was proven that each separately quasi-continuous and monotonous relatively to the first variable function \(f : \mathbb{R} \times X \to \mathbb{R}\) is quasi-continuous relatively to the set of variables. The last result may be considered as the abstract analog of Young’s theorem (Theorem A) for separately quasi-continuous functions.

However, we do not know any direct generalization of Theorem A (for separately continuous and monotonous relatively to the first variable function) in abstract topological spaces at the present time. In the present paper we prove the generalization of theorems A and B for the case of (separately continuous and monotonous relatively to the first variable) function \(f : T \times X \to T_1\), where \((T, \leq), (T_1, \leq_1)\) are linearly ordered sets equipped by the order topology and \(X\) is any topological space.

2. Preliminaries

Let \(T = (T, \leq)\) be any linearly (ie totally) ordered set (in the sense of \([1]\)). Then we can define the strict linear order relation on \(T\) such that for any \(t, \tau \in T\) the correlation \(t < \tau\) holds if and only if \(t \leq \tau\) and \(t \neq \tau\). Together with the linearly ordered set \(T\) we introduce the linearly ordered set \(T_{\pm\infty} = (T \cup \{-\infty, +\infty\}, \leq)\), where the order relation is extended on the set \(T \cup \{-\infty, +\infty\}\) by means of the following clear conventions:

- **(a):** \(-\infty < +\infty\);
- **(b):** \((\forall t \in T)\) \((-\infty < t < +\infty)\).

Recall \([1]\) that every such linearly ordered set \(T = (T, \leq)\) can be equipped by the natural “internal” order topology \(\tau_{pi}[T]\), generated by the base consisting of the open sets of kind:

\[
(\tau_1, \tau_2) = \{t \in T | \tau_1 < t < \tau_2\},
\]

where \(\tau_1, \tau_2 \in T \cup \{-\infty, +\infty\}, \tau_1 < \tau_2\).

Let \((X, \tau_X)\), \((Y, \tau_Y)\) and \((Z, \tau_Z)\) be topological spaces. By \(C(X, Y)\) we denote the collection of all continuous mappings from \(X\) to \(Y\). For a mapping \(f : X \times Y \to Z\) and a point \((x, y) \in X \times Y\) we write

\[
f^x(y) := f_y(x) := f(x, y).
\]

Recall \([3]\) that the mapping \(f : X \times Y \to Z\) is refereed to as separately continuous if and only if \(f^x \in C(Y, Z)\) and \(f_y \in C(X, Z)\) for every point \((x, y) \in X \times Y\) (see also \([6–8]\)). The set of all separately continuous mappings \(f : X \times Y \to Z\) is denoted by \(CC(X \times Y, Z)\) \([3, 6–8]\).

Let \(T = (T, \leq)\) and \(T_1 = (T_1, \leq_1)\) be linearly ordered sets. We say that a function \(f : T \to T_1\) is non-decreasing (non-increasing) on \(T\) if and only if for every \(t, \tau \in T\) the inequality \(t \leq \tau\) leads to the inequality \(f(t) \leq_1 f(\tau)\)
(f(τ) ≤_1 f(t)) correspondingly. Function f : T → T_1, which is non-decreasing or non-increasing on T is called by **monotonous**.

3. Main Results

Let (X_1, τ_{X_1}), . . . , (X_d, τ_{X_d}) (d ∈ N) be topological spaces. Further we consider X_1 × · · · × X_d as a topological space under the Tychonoff topology τ_{X_1×···×X_d} on the Cartesian product of topological spaces X_1, . . . , X_d. Recall [4, Chapter 3] that topology τ_{X_1×···×X_d} is generated by the base of kind:

\[ \{ U_1 × · · · × U_d | (∀ j ∈ \{1, . . . , d\}) (U_j ∈ τ_{X_j}) \}. \]

**Theorem 3.1.** Let T = (T, ≤) and T_1 = (T_1, ≤_1) be linearly ordered sets and (X, τ_X) be a topological space.

If f ∈ CC(T × X, T_1) and function f_x(t) = f(t, x) is monotonous on T for every x ∈ X, then f is continuous mapping from the topological space (T × X, τ_{T×X}) to the topological space (T_1, τ_{pi}[T_1]).

**Proof.** Consider any ordered pair (t_0, x_0) ∈ T × X. Take any open set V ⊆ T_1 such that f(t_0, x_0) ∈ V. Since the sets of kind (2.1) form the base of topology τ_{pi}[T_1], there exist τ_1, τ_2 ∈ T_1 ∪ {−∞, +∞} such that τ_1 <_1 f(t_0, x_0) <_1 τ_2 and (τ_1, τ_2) ⊆ V, where <_1 is the strict linear order, generated by (non-strict) order ≤_1 (on T_1 ∪ {−∞, +∞}). The function f is separately continuous.

So, since the sets of kind (2.1) form the base of topology τ_{pi}[T], there exist t_1, t_2 ∈ T ∪ {−∞, +∞} such that

\[ t_1 < t_0 < t_2 \quad \text{and} \]

\[ f[(t_1, t_2) × \{x_0\}] ⊆ (τ_1, τ_2). \]

Further we need the some additional denotations.

- In the case, where (t_1, t_0) ≠ ∅ we choose any element α_1 ∈ T such that t_1 < α_1 < t_0 and denote α_1 := α_1. In the opposite case we denote α_1 := t_0, α_1 := t_1.
- In the case (t_0, t_2) ≠ ∅ we choose any element α_2 ∈ T such that t_0 < α_2 < t_2 and denote α_2 := α_2. In the opposite case we denote α_2 := t_0, α_2 := t_2.

It is not hard to verify, that in the all cases the following conditions are performed:

\[ α_1, α_2 ∈ T, \quad \tilde{α}_1, \tilde{α}_2 ∈ T ∪ \{-∞, +∞\}; \]

\[ α_1 ≤ α_2; \]

\[ \tilde{α}_1 < \tilde{α}_2; \]

\[ [α_1, α_2] ⊆ (t_1, t_2), \quad \text{where} \ [α_1, α_2] = \{t ∈ T | α_1 ≤ t ≤ α_2\}; \]

\[ t_0 ∈ (\tilde{α}_1, \tilde{α}_2) ⊆ [α_1, α_2]. \]

According to (3.3), α_1, α_2 ∈ (t_1, t_2). Hence, according to (3.2), interval (τ_1, τ_2) is an open neighborhood of the both points f(α_1, x_0) and f(α_2, x_0).
Since the function \( f \) is separately continuous on \( T \times X \), then there exist an open neighborhood \( U \in \tau_X \) of the point \( x_0 \) (in the space \( X \)) such that:

\[
(3.5) \quad f[\{(\alpha_1) \times U\}] \subseteq (\tau_1, \tau_2);
\]

\[
(3.6) \quad f[\{(\alpha_2) \times U\}] \subseteq (\tau_1, \tau_2).
\]

The set \((\tilde{\alpha}_1, \tilde{\alpha}_2) \times U \) is an open neighborhood of the point \((t_0, x_0)\) in the topology \( \tau_{T \times X} \) of the space \( T \times X \). Now our aim is to prove that

\[
(3.7) \quad \forall (t, x) \in (\tilde{\alpha}_1, \tilde{\alpha}_2) \times U \ (f(t, x) \in (\tau_1, \tau_2) \subseteq V).
\]

So, chose any point \((t, x) \in (\tilde{\alpha}_1, \tilde{\alpha}_2) \times U \). According to the condition (3.4), we have \((t, x) \in [\alpha_1, \alpha_2] \times U \), that is \( \alpha_1 \leq t \leq \alpha_2 \) and \( x \in U \). In accordance with (3.5), (3.6), we have \( f(\alpha_1, x) \in (\tau_1, \tau_2) \) and \( f(\alpha_2, x) \in (\tau_1, \tau_2) \). Hence, since the function \( f(\cdot, x) \) is monotonous on \( T \) and \( \alpha_1 \leq t \leq \alpha_2 \), we deduce \( f(t, x) \in (\tau_1, \tau_2) \subseteq V \). Thus, the correlation (3.7) is proven. Hence, the function \( f \) is continuous in (every) point \((t_0, x_0) \in T \times X \).

Theorem A is a consequence of Theorem 3.1 in the case \( T = X = \mathbb{R} \), where \( \mathbb{R} \) is considered together with the usual linear order on the field of real numbers and usual topology.

**Corollary 3.2.** Let \( T_0 = (T_0, \leq_0) \), \( T_1 = (T_1, \leq_1) \), \ldots, \( T_d = (T_d, \leq_d) \) (\( d \in \mathbb{N} \)) be linearly ordered sets, and \( (X, \tau_X) \) be a topological space.

If the function \( f : T_1 \times \cdots \times T_d \times X \rightarrow T_0 \) is continuous in each variable separately and \( f(t_1, \ldots, t_d, \tau) \) is monotonous in each \( t_i \) separately (\( 1 \leq i \leq d \)) then \( f \) is a continuous mapping from the topological space \( (T_1 \times \cdots \times T_d \times X, \tau_{T_1 \times \cdots \times T_d \times X}) \) to the topological space \( (T_0, \Sigma_p \pi [T_0]) \).

**Proof.** We will prove this corollary by induction. For \( d = 1 \) the corollary is true by Theorem 3.1. Assume, that the corollary is true for the number \( d - 1 \), where \( d \in \mathbb{N} \), \( d \geq 2 \). Suppose, that function \( f : T_1 \times \cdots \times T_d \times X \rightarrow T_0 \) is satisfying the conditions of the corollary. Then we may consider this function as a mapping from \( T_1 \times X(d) \) to \( T_0 \), where \( X(d) = T_2 \times \cdots \times T_d \times X \). According to inductive hypothesis, function \( f(t_1, \cdot) \) is continuous on \( X(d) \) for every fixed \( t_1 \in T_1 \). So \( f \) is a separately continuous mapping from \( T_1 \times X(d) \) to \( T_0 \). Moreover, \( f \) is monotonous relatively to the first variable (by conditions of the corollary). Hence, by Theorem 3.1, \( f \) is continuous on \( T_1 \times X(d) \).

Theorem B is a consequence of Corollary 3.2 in the case \( T_0 = T_1 = \cdots = T_d = X = \mathbb{R} \), where \( \mathbb{R} \) is considered together with the usual linear order on the field of real numbers and usual topology. In the case \( T_0 = \mathbb{R} \), \( T_j = (a_j, b_j) \), \( X = (a_{d+1}, b_{d+1}) \) where \( a_j, b_j \in \mathbb{R} \) and \( a_j < b_j \) \((j \in \{1, \ldots, d + 1\})\) and intervals \((a_j, b_j)\) are considered together with the usual linear order and topology, induced from the field of real numbers, we obtain the following corollary.

**Corollary 3.3.** If the function \( f : (a_1, b_1) \times \cdots \times (a_d, b_d) \times (a_{d+1}, b_{d+1}) \rightarrow \mathbb{R} \) (\( d \in \mathbb{N} \)) is continuous in each variable separately and \( f(t_1, \ldots, t_d, \tau) \) is monotonous in each \( t_i \) separately (\( 1 \leq i \leq d \)) then \( f \) is a continuous mapping from \((a_1, b_1) \times \cdots \times (a_{d+1}, b_{d+1})\) to \( \mathbb{R} \).
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Remark 3.4. In fact in the paper [5] the more general result was formulated, in comparison with Theorem B. Namely the author of [5] had considered the real valued function \( f(t_1, \ldots, t_d, \tau) \) defined on an open set \( G \subseteq \mathbb{R}^{d+1}, d \in \mathbb{N} \) such that \( f \) is continuous in each variable separately and monotonous in each \( t_i \) separately (\( 1 \leq i \leq d \)). But this result of [5] can be delivered from Corollary 3.3, because for each point \( t = (t_1, \ldots, t_d, \tau) \in G \) in the open set \( G \) there exists the set of intervals \((a_1, b_1), \ldots, (a_{d+1}, b_{d+1})\) such that \( t \in (a_1, b_1) \times \cdots \times (a_{d+1}, b_{d+1}) \subseteq G \).

References


