

When is $X \times Y$ homeomorphic to $X \times_l Y$?

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ABSTRACT

We identify a class of linearly ordered topological spaces X that may satisfy the property that $X \times X$ is homeomorphic to $X \times_l X$ or can be embedded into a linearly ordered space with the stated property. We justify the conjectures by partial results.

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1. QUESTIONS

In this paper we provide a discussion that justifies our interest in the question of the title. We also identify more specific questions that may lead to affirmative resolutions. We back up our curiosity by some partial results and examples. The main result of this work is Theorem 2.7. To proceed further let us agree on some terminology. A linear order will also be called an order. An order $<$ on X is compatible with the topology of X , if the topology induced by $<$ is equal to the topology of X . A linearly ordered topological space (abbreviated as LOTS) is a pair $\langle X, < \rangle$ of a topological space X and a topology-compatible order $<$ on X . A topological space X is orderable if its topology can be induced by some order on X . When we consider the lexicographical product $X \times_l Y$ of two LOTS X and Y , we first take the lexicographical products of the ordered sets X and Y and then induce the topology as determined by the lexicographical order on $X \times_l Y$. For the purpose of readability we will assume an informal style when describing some folklore-type structures or arguments.

The operations of Cartesian product and lexicographical product produce (more often than not) completely different structures. The former results in a visually "more voluminous" structure, while the latter keeps "visual linearity" but introduces "stretches". In rare cases, however, both operations produce the same results from a topological point of view. For example, $\mathbb{Q} \times \mathbb{Q}$ is homeomorphic to $\mathbb{Q} \times_l \mathbb{Q}$. Also, $S \times S$ is homeomorphic to $S \times_l S$, where $S = \{\pm 1/n : n = 1, 2, \dots\}$. Note that S is homeomorphic to the space \mathbb{N} of natural numbers. However, $\mathbb{N} \times \mathbb{N}$ is not homeomorphic to $\mathbb{N} \times_l \mathbb{N}$. Indeed, the former is discrete while the latter has non-isolated points such as $\langle 2, 1 \rangle, \langle 3, 1 \rangle$, etc. Following this discussion, it is not hard to see that given any discrete space D , it is possible to find a topology-compatible order \prec on D such that $D^* = \langle D, \prec \rangle$ is discrete and $D^* \times D^*$ is homeomorphic to $D^* \times_l D^*$. Our discussion prompts the following general problem.

Problem 1.1. *What conditions on X guarantee that there exists a topology-compatible order \prec on X such that $X \times X$ is homeomorphic to $\langle X, \prec \rangle \times_l \langle X, \prec \rangle$?*

Note that homogeneity is not a necessary condition as follows from the following folklore fact.

Example 1.2 (Folklore). $(\omega + 1) \times_l (\omega + 1)$ is homeomorphic to $(\omega + 1) \times (\omega + 1)$.

Proof. First observe that $Y = [(\omega + 1) \times_l (\omega + 1)] \setminus \{\langle \omega, n \rangle : n = 1, 2, \dots\}$ is homeomorphic to $(\omega + 1) \times_l (\omega + 1)$. We will, therefore, provide a homeomorphism between $X = (\omega + 1) \times (\omega + 1)$ and Y . We define our homeomorphism in three stages as follows:

Stage 1: For every $n \in \omega$, fix a bijection f_n between $\{\langle n, k \rangle \in X : k = n, \dots, \omega\} \subset (\omega + 1) \times (\omega + 1)$ and $\{\langle 2n, m \rangle \in Y : m \in \omega + 1\} \subset (\omega + 1) \times_l (\omega + 1)$. Such a homeomorphism exists since both subspaces are homeomorphic to $\omega + 1$.

Stage 2: For every $n \in \omega$, fix a bijection g_n between $\{\langle k, n \rangle \in X : k = n + 1, \dots, \omega\}$ and $\{\langle 2n + 1, m \rangle \in Y : m \in \omega + 1\}$.

Stage 3: Define the promised homomorphism from X to Y as follows:

$$f(x) = \begin{cases} f_n(x) & x \in \{\langle n, k \rangle \in X : k = n, \dots, \omega\} \\ g_n(x) & x \in \{\langle k, n \rangle \in X : k = n + 1, \dots, \omega\} \\ \langle \omega, 0 \rangle & x = \langle \omega, \omega \rangle \end{cases}$$

Visually, f maps the n -th vertical at or above the diagonal in $(\omega + 1) \times (\omega + 1)$ onto the $(2n)$ -th copy of $(\omega + 1)$ in $(\omega + 1) \times_l (\omega + 1)$. Also f maps the n -th horizontal under the diagonal in $(\omega + 1) \times (\omega + 1)$ onto the $(2n + 1)$ -st copy of $(\omega + 1)$ in $(\omega + 1) \times_l (\omega + 1)$. Finally, f maps the upper right corner point of the Cartesian product to $\langle \omega, 0 \rangle$ of the lexicographical product, which is the only point that is the limit of a sequence of non-isolated points.

Clearly, f is a bijection. Let us show that f and f^{-1} are continuous. Since the domains and images of f_n and g_n are clopen in the respective superspaces, it

remains to show that f is continuous at $\langle \omega, \omega \rangle$ and f^{-1} is continuous at $\langle \omega, 0 \rangle$. For this let $U_n = [n, \omega] \times [n, \omega]$. Then $f(U) = \{\langle (a, b) \in Y : a \geq 2n \rangle\}$, which is an open neighborhood of $\langle \omega, 0 \rangle$ in Y . We have $\{U_n\}_n$ is a basis at $\langle \omega, \omega \rangle$ in X and $\{f(U_n)\}_n$ is a basis at $\langle \omega, 0 \rangle$ in Y . Since f is bijective, f^{-1} is continuous at $\langle \omega, 0 \rangle$. We proved that $(\omega + 1) \times (\omega + 1)$ is homeomorphic to Y , and therefore, to $(\omega + 1) \times_l (\omega + 1)$. \square

Even though $(\omega + 1)$ is not homogeneous, it is homogeneous at all non-isolated points (since there is only such point). But even this property is not necessary for the two types of products to be homeomorphic. A similar argument can be used to verify the presence of the studied phenomenon in the following example.

Example 1.3. $X \times_l X$ is homeomorphic to $X \times X$, where $X = (\omega \times_l \omega) + 1$.

The limit points in this example have different natures. The leftmost point cannot be carried by a homeomorphism to any internal limit point. We omit the proof of the statement of Example 1.3 since we will prove a more general one later (Lemma 2.6). Following Example 1.2 and the fact that any discrete space has the property under discussion, one may wonder if any linearly ordered space with a single non-isolated point has the property. The following example shows that the answer is negative and opens another direction for our study.

Example 1.4. Let $X = (\omega + 1) \oplus D$, where D is an ω_1 -sized discrete space. Then the following hold:

- (1) X is orderable.
- (2) $X \times X$ is not homeomorphic to $\langle X, \prec \rangle \times_l \langle X, \prec \rangle$ for any topology-compatible order \prec on X .

Proof. To see why X is orderable, first observe that we can think of X as the subspace of ω_1 that contains only all isolated ordinals of ω_1 and the ordinal ω . To order X , simply reverse the order of every sequence in form $\{\alpha + 1, \alpha + 2, \dots\}$, for each limit ordinal greater than ω .

To prove part (2), fix an arbitrary topology-compatible order \prec on X . The space $X_\prec = \langle X, \prec \rangle$ has at least one of extreme points or neither. Let us consider all possibilities.

Case (X_\prec has neither minimum nor maximum): Then $\{x\} \times_l X_\prec$ is clopen in $X_\prec \times_l X_\prec$ for each x . Therefore, $X_\prec \times_l X_\prec$ is the free sum of ω_1 many topological copies of X . Hence, $X_\prec \times_l X_\prec$ is not homeomorphic to $X \times X$.

Case (X_\prec has minimum but not maximum): Assume first that X_\prec has a strictly increasing sequence $\{a_n\}_n$ converging to ω . Then any neighborhood of $\langle \omega, \min X_\prec \rangle$ contains $\{a_n\} \times_l X_\prec$. Therefore, any neighborhood of $\langle \omega, \min X_\prec \rangle$ has size ω_1 , while no point in $X \times X$ has such populous base neighborhoods.

We now assume that X_\prec has no strictly increasing sequences converging to ω . This and the absence of a maximum imply that $X_\prec \times X_\prec$

does not have a topological copy of $\omega \times_l \omega + 1$. However, $X \times X$ does, which is $\langle \omega, \omega \rangle$. In other words, the second derived set of the lexicographical product is empty but $(X \times X)'' = \{\langle \omega, \omega \rangle\}$.

Case $(\langle X, \prec \rangle \text{ has maximum but not minimum})$: Similar to Case 2.

Case $(\langle X, \prec \rangle \text{ has both maximum and minimum})$: Similar to the first part of Case 2.

Since we have exhausted all cases, the proof is complete. \square

It is known (see, for example [2]) that given a subspace X of an ordinal, the square of X is homeomorphic to a subspace of a linearly ordered space if and only if X has no stationary subsets and is character homogeneous at all non-isolated points. This statement and the preceding discussion lead to the following question.

Question 1.5. *Let X be a subset of an ordinal, character homogeneous at non-isolated points, and have no stationary subsets. Can X be embedded in a linearly ordered space L for which $L \times L$ and $L \times_l L$ are homeomorphic?*

Note that even though spaces in examples 1.2 and 1.3 are not homogeneous, each point has a basis of mutually homeomorphic neighborhoods. This observation prompts the following question.

Question 1.6. *Let X be a subspace of an ordinal and every point of X has a basis of mutually homeomorphic neighborhoods. Can X be embedded in a linearly ordered space L for which $L \times L$ and $L \times_l L$ are homeomorphic?*

In the next section we will justify the discussed questions by proving a statement that generalizes Example 1.2. Namely, we will show that if X is a subspace of an ordinal and is homogeneous on its derived set X' , then X is embeddable in a linearly ordered space L that has homeomorphic Cartesian and lexicographical products (Theorem 2.7). To prove this we will first identify a special class of spaces for which the two types of products are homeomorphic (Lemma 2.6). The structure of these spaces is similar to that of the space in Example 1.3. We, therefore, generalize Example 1.3 too.

In notations and terminology we will follow [3]. If X is a linearly-ordered set, by $[a, b]_X$ we denote the closed interval in X . If it is clear that the interval is considered in X but not in some larger ordered set, we simply write $[a, b]$. The same concerns other types of intervals. By X' we will denote the set of all non-isolated points of X , that is, *the derived set of X* . We also say that X is *homogeneous on its subset A* if for every $x, y \in A$ there exists a homeomorphism $f : X \rightarrow X$ such that $f(x) = y$ and $f(y) = x$.

2. PARTIAL RESULTS

In what follows, by \mathcal{L} we denote *the class of all subspaces of ordinals that are homogeneous on their derived sets*.

To prove our main statement (Theorem 2.7), we start with two technical lemmas about the key properties of the members of \mathcal{L} that will be used in further arguments.

Lemma 2.1. *Let $X \in \mathcal{L}$. Then, for any $x \in X'$ there exists $\alpha_x < x$ such that x is the single non-isolated point of $[\alpha_x, x]_X$.*

Proof. By homogeneity of X on X' it suffices to show that the conclusion holds for some element of X' . We may assume that X' is not empty. Then $z = \min X'$ is defined. Then $\alpha_z = \min X$ is as desired for z . \square

Lemma 2.2. *Let $X \in \mathcal{L}$. Then, X can be written as $(\bigoplus_{x \in X'} I_x) \oplus D$ so that the following hold:*

- (1) D is clopen and discrete,
- (2) I_x and I_y are homeomorphic for any $x, y \in X'$,
- (3) x is the only non-isolated point of I_x for each $x \in X'$.

Proof. For any $x \in X'$, let α_x be as in Lemma 2.1. We can find β_x between α_x and x such that $I_x = (\beta_x, x]_X$ has the same cardinality as any smaller neighborhood of x . Then $D = X \setminus \bigcup \{I_x : x \in X'\}$ is a clopen discrete subset of X and $X = (\bigoplus_{x \in X'} I_x) \oplus D$ is a desired representation. \square

To prove our target statement, first for each infinite cardinal γ , we identify a linearly ordered topological space $\langle L_\gamma, \prec \rangle$ for which $\langle L_\gamma, \prec \rangle \times \langle L_\gamma, \prec \rangle$ is homeomorphic to $\langle L_\gamma, \prec \rangle \times_l \langle L_\gamma, \prec \rangle$. Next, we will direct our efforts on the task of embedding the members of \mathcal{L} into such spaces.

Construction of $\langle L_\gamma, \prec \rangle$ for an infinite cardinal γ .

Definition of L_γ . Denote by λ_γ the ordinal $(\gamma \times_l \gamma) + 1$. Define L_γ as the subspace of λ_γ that consists of all points α that fall into one of the following three categories:

- (1) $\alpha = \max \lambda_\gamma$
- (2) $[\alpha_0, \alpha]$ is order-isomorphic to $\gamma + 1$ for some $\alpha_0 < \alpha$.
- (3) α is isolated.

Remark. To help visualize L_γ , put $I = \{\alpha < \gamma : \alpha \text{ is isolated}\} \cup \{\gamma\}$. Then L_γ can be thought of as a γ -long sequence of γ -many clopen copies of I converging to $\max \lambda_\gamma$.

Definition of \prec . If $\gamma = \omega$, then $L_\gamma = \lambda_\gamma$ and we let \prec be equal to the existing ordering $<$. For $\gamma > \omega$, we will define \prec using a folklore ordering procedure. We first define the order formally and then follow up with a simple demonstration. For each $\alpha \in \lambda_\gamma \setminus L_\gamma$, put $R_\alpha = \{\alpha + 1, \alpha + 2, \dots\}$. By the definition of L_γ and the fact that $\gamma > \omega$, we conclude that R_α is a closed subset of L_γ . Define \prec_α on R_α as follows: $\dots \alpha + 5 \prec \alpha + 3 \prec \alpha + 1 < \alpha + 2 \prec \alpha + 4 \dots$. Define \prec as follows:

- (1) $x \prec y$ if $x, y \in R_\alpha$ and $x \prec_\alpha y$.
- (2) $x \prec y$ if $\{x, y\}$ is not a subset of R_α for any α and $x < y$.

Construction of $\langle L_\gamma, \prec \rangle$ is complete. \square

To convince a reader that the above definition is legal without going into painful details, let us demonstrate a folklore construction of a topology-compatible order for the space $X = \{-1/n : n = 1, 2, 3, \dots\} \cup \{5 - 1/n : n = 1, 2, 3, \dots\}$. The space X is not a linearly ordered space but there are many simple topology-compatible orders on X . The one that mimics the above construction is defined as follows. First, reverse the order on $\{5 - 1/n : n = 1, 2, 3, \dots\}$. The resulting set becomes order isomorphic to $\{\pm 1/n : n = 1, 2, 3, \dots\}$ and is homeomorphic to X . This short construction is formalized in the above definition in which we top every "missing limit point" by the reversed sequence "converging to the next missing limit point".

Note that in our definition of \prec for L_γ we do not change the order position of limit points of L_γ , which means that the new order coincides with the natural order when one of the compared elements is in L'_γ . In a sense, the new order \prec on L_γ is almost indistinguishable from the standard order $<$ if "observed from far away". Also note that if X is a subspace of an ordinal that is homogeneous on the derived set, then by Lemma 2.2, X can be embedded into L_γ for some γ . Let us record these observations for future reference.

Lemma 2.3. *The following hold:*

- (1) Every $X \in \mathcal{L}$ embeds in L_γ for some cardinal γ .
- (2) If $x \in L'_\gamma, y \in L_\gamma$, and $x < y$, then $x \prec y$.
- (3) If $x \in L'_\gamma, y \in L_\gamma$, and $y < x$, then $y \prec x$.

We will often use the facts in this summary lemma without explicit referencing. Our next goal is to show that lexicographical and Cartesian product operations produce topologically equivalent results when applied to an L_γ . We start by considering the two operations on smaller pieces of L_γ 's. In the following three statements the arguments will be very similar to each other. For clarity, we will also use similar wording.

Lemma 2.4. *Let γ be an infinite cardinal. Then $[0, \gamma]_{L_\gamma} \times [0, \gamma]_{L_\gamma}$ is homeomorphic to L_γ .*

Proof. To prove the statement we will visualize $[0, \gamma]_{L_\gamma}$ and L_γ as described in the remark after the definition of L_γ . Namely, $[0, \gamma]_{L_\gamma} = I = \{\alpha < \gamma : \alpha \text{ is isolated}\} \cup \{\gamma\}$ and L_γ is a γ -long sequence of γ -many clopen copies of I converging to $\infty = \max L_\gamma$. We can write then $L_\gamma = (\oplus\{I_\alpha = I : \alpha < \gamma, \alpha \text{ is isolated}\}) \cup \{\infty\}$, where every neighborhood of ∞ contains all I_α 's starting from some moment. Having these visuals in mind we will construct a desired homeomorphism in three stages as follows:

Stage 1: Partition the set of isolated ordinals of γ into pairs $\{\{a_\alpha, b_\alpha\} : \alpha < \gamma, \alpha \text{ is isolated}\}$ so that $b_\alpha = a_\alpha + 1$ and indexing agrees with the natural well-ordering $<$ of the partitioned set.

Stage 2: Since γ is an infinite cardinal, $[\alpha, \gamma]_{L_\gamma}$ is homeomorphic to I for any $\alpha < \gamma$. Therefore, for each isolated $\alpha < \gamma$ we can fix homeomorphisms $g_\alpha : \{\alpha\} \times [\alpha, \gamma]_{L_\gamma} \rightarrow I_{b_\alpha}$ and $h_\alpha : (\alpha, \gamma]_{L_\gamma} \times \{\alpha\} \rightarrow I_{a_\alpha}$. That

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is, g_α maps the α 's vertical of $[0, \gamma]_{L_\gamma} \times [0, \gamma]_{L_\gamma}$ at or above the diagonal onto b_α 's copy of I in L_γ and h_α maps the α 's horizontal strictly below the diagonal onto a_α 's copy of I .

Stage 3: Define a homomorphism f from $[0, \gamma]_{L_\gamma} \times [0, \gamma]_{L_\gamma}$ to L_γ as follows:

$$f(p) = \begin{cases} g_\alpha(p) & \text{if } p \in \{\alpha\} \times [\alpha, \gamma]_{L_\gamma} \\ h_\alpha(p) & \text{if } p \in [\alpha, \gamma]_{L_\gamma} \times \{\alpha\} \\ \infty & \text{if } p = \langle \gamma, \gamma \rangle \end{cases}$$

The argument similar to that in Example 1.2 shows that f is a homeomorphism. \square

Lemma 2.5. *Let γ be an infinite cardinal. Then $[0, \gamma]_{L_\gamma} \times L_\gamma$ is homeomorphic to $[0, \gamma]_{\langle L_\gamma, \prec \rangle} \times_l \langle L_\gamma, \prec \rangle$.*

Proof. Denote the spaces in the statement by X and Y , respectively. Since Y is homeomorphic to $Z = Y \setminus (\{\gamma\} \times_l [1, \gamma]_{\langle L_\gamma, \prec \rangle})$, it suffices to construct an isomorphism from X to Z , which we will do next.

When treating $[0, \gamma]_{\langle L_\gamma, \prec \rangle}$ and $\langle L_\gamma, \prec \rangle$ as topological spaces with regard to order, we will visualize them as described in Lemma 2.4. For convenience, let us copy our notation from Lemma 2.4 next:

$$\begin{aligned} [0, \gamma]_{L_\gamma} &= I = \{\alpha < \gamma : \alpha \text{ is isolated}\} \cup \{\gamma\} \\ L_\gamma &= (\oplus \{I_\alpha = I : \alpha < \gamma, \alpha \text{ is isolated}\}) \cup \{\infty\}, \end{aligned}$$

where ∞ is the maximum element of L_γ in either of the two orders. We are now ready to construct a desired homeomorphism in three stages as follows:

Stage 1: Partition the set of isolated ordinals of γ into pairs $\{\{a_\alpha, b_\alpha\} : \alpha < \gamma, \alpha \text{ is isolated}\}$ so that $b_\alpha = a_\alpha + 1$ and indexing agrees with the natural well-ordering $<$ of the partitioned set.

Stage 2: By Lemma 2.4, for each isolated ordinal $\alpha < \gamma$ there exists a homeomorphism h_α of $[\alpha, \gamma]_{L_\gamma} \times I_\alpha$ onto $\{a_\alpha\} \times_l \langle L_\gamma, \prec \rangle$. Since γ is an infinite cardinal, $V_\alpha = L_\gamma \setminus \bigcup_{\beta \leq \alpha} I_\beta$ is homeomorphic to L_γ . Hence, we can find a homeomorphism g_α from $\{\alpha\} \times V_\alpha$ onto $\{b_\alpha\} \times_l \langle L_\gamma, \prec \rangle$.

Stage 3: Define a homomorphism f from X to Z as follows:

$$f(p) = \begin{cases} g_\alpha(p) & \text{if } p \in \{\alpha\} \times V_\alpha \\ h_\alpha(p) & \text{if } p \in [\alpha, \gamma]_{L_\gamma} \times I_\alpha \\ \langle \gamma, \infty \rangle & \text{if } p = \langle \gamma, \infty \rangle \end{cases}$$

In words, f maps most of the α 's horizontal strip corresponding to I_α onto the a_α 's copy of $\langle L_\gamma, \prec \rangle$, most of the α 's vertical onto b_α 's copy of $\langle L_\gamma, \prec \rangle$, and the corner point of the Cartesian product to the maximum of Z . The argument similar to that in Example 1.2 shows that f is a homeomorphism. \square

We are now ready to prove a generalization of the statement of Example 1.3.

Lemma 2.6. *For every infinite cardinal γ , the space $L_\gamma \times L_\gamma$ is homeomorphic to $\langle L_\gamma, \prec \rangle \times_l \langle L_\gamma, \prec \rangle$.*

Proof. Denote by X and Y the two spaces in the statement. As in Lemma 2.5, it suffices to construct a homeomorphism from X to $Z = Y \setminus (\{\infty\} \times_l [1, \infty]_{\langle L_\gamma, \prec \rangle})$. As in the previous two lemmas, we visualize $[0, \gamma]_{L_\gamma}$ and L_γ as follows:

$$[0, \gamma]_{L_\gamma} = I = \{\alpha < \gamma : \alpha \text{ is isolated}\} \cup \{\gamma\}$$

$$L_\gamma = (\oplus \{I_\alpha = I : \alpha < \gamma, \alpha \text{ is isolated}\}) \cup \{\infty\},$$

where ∞ is the maximum element of L_γ in either of the two orders. We will closely follow our constructions in the previous two lemmas and construct the promised homeomorphism in three stages as follows:

Stage 1: Partition the set of isolated ordinals of γ into pairs $\{\{a_\alpha, b_\alpha\} : \alpha < \gamma, \alpha \text{ is isolated}\}$ so that $b_\alpha = a_\alpha + 1$ and indexing agrees with the natural well-ordering $<$ of the partitioned set.

Stage 2: By Lemma 2.5, for each isolated $\alpha < \gamma$, we can fix two homeomorphisms:

$$h_\alpha : L_\gamma \times I_\alpha \rightarrow \langle I_{a_\alpha}, \prec \rangle \times_l \langle L_\gamma, \prec \rangle$$

$$g_\alpha : I_\alpha \times \left(L_\gamma \setminus \bigcup_{\beta \leq \alpha} I_\beta \right) \rightarrow \langle I_{b_\alpha}, \prec \rangle \times_l \langle L_\gamma, \prec \rangle$$

Stage 3: Define a homomorphism f from X to Z as follows:

$$f(p) = \begin{cases} g_\alpha(p) & \text{if } p \in I_\alpha \times \left(L_\gamma \setminus \bigcup_{\beta \leq \alpha} I_\beta \right) \\ h_\alpha(p) & \text{if } p \in L_\gamma \times I_\alpha \\ \langle \infty, \infty \rangle & \text{if } p = \langle \infty, \infty \rangle \end{cases}$$

In words, f maps most of the α 's horizontal strip onto the a_α 's copy of $I \times_l L_\gamma$, most of the α 's vertical strip onto the b_α 's copy of $I \times_l L_\gamma$, and the corner point of the Cartesian product to the maximum of Z . An argument similar to one of Example 1.2 shows that f is a homeomorphism. \square

Lemmas 2.6 and 2.3 imply the following main statement of our discussion.

Theorem 2.7. *Let X be a subspace of an ordinal that is homogeneous on the derived set. Then X can be embedded into a LOT Z such that $Z \times_l Z$ is homeomorphic to $Z \times Z$.*

In search for candidates with the discussed phenomenon, it is clear that we should immediately eliminate any ordered spaces with stationary subsets. Indeed, the square of such a space is not orderable as follows from a standard generalization of Katetov's example [5]. Therefore, by the characterization of hereditary paracompactness for GO-spaces due to Engelking and Lutzer ([1] or [4]), we should consider only hereditary paracompact ordered spaces. It is clear that if X has no stationary subset, then X^2 does not have such either. Thus, we

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need to concentrate on spaces with orderable hereditary paracompact squares. While the Engelking-Lutzer characterization is incredibly handy for testing an ordered space for hereditary paracompactness, the author is not aware of any criterion for the square of a LOTS to be hereditary paracompact. Is there such a criterion? If not, let us find one!

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