\textbf{\textit{\tau}-metrizable spaces}

A. C. Megaritis

Technological Educational Institute of Peloponese, Department of Computer Engineering, 23100, Sparta, Greece (thanasismeg13@gmail.com)

Communicated by D. Georgiou

\textbf{Abstract}

In [1], A. A. Borubaev introduced the concept of \textit{\tau}-metric space, where \textit{\tau} is an arbitrary cardinal number. The class of \textit{\tau}-metric spaces as \textit{\tau} runs through the cardinal numbers contains all ordinary metric spaces (for \textit{\tau} = 1) and thus these spaces are a generalization of metric spaces. In this paper the notion of \textit{\tau}-metrizable space is considered.

2010 MSC: 54A05; 54E35.

Keywords: \textit{\tau}-metric space; \textit{\tau}-metrizable space; \textit{\tau}-metrization theorem.

\section{Preliminaries and notations}

Our notation and terminology is standard and generally follows [2]. The cardinality of a set \(X\) is denoted by \(|X|\). Throughout, we denote by \(\tau\) an arbitrary nonzero cardinal number. The cardinalities of the natural numbers and of the real numbers are denoted by \(\aleph_0\) and \(\mathfrak{c}\), respectively. The character, the weight and the density of a topological space \(X\) are denoted by \(\chi(X)\), \(w(X)\) and \(d(X)\), respectively. As usual \(I\) denotes the closed unit interval \([0, 1]\) with the Euclidean metric topology.

By \(\mathbb{R}_+^\tau\) we denote the topological product of \(\tau\) copies of the space \(\mathbb{R}_+ = [0, +\infty)\) (with the natural topology). On the space \(\mathbb{R}_+^\tau\), the operations of addition, multiplication, and multiplication by a scalar, as well as a partial ordering, are defined in a natural way (coordinatewise).

Now, we present the notion of \textit{\tau}-metric space [1]. Let \(X\) be a nonempty set. A mapping \(\rho_\tau : X \times X \to \mathbb{R}_+^\tau\) is called a \(\tau\)-metric on \(X\) if the following axioms hold:
(1) \( \rho_\tau(x, y) = \theta \) if and only if \( x = y \), where \( \theta \) is the point of the space \( \mathbb{R}_+^\tau \) whose all coordinates are zeros.

(2) \( \rho_\tau(x, y) = \rho_\tau(y, x) \) for all \( x, y \in X \).

(3) \( \rho_\tau(x, z) \leq \rho_\tau(x, y) + \rho_\tau(y, z) \) for all \( x, y, z \in X \).

The pair \((X, \rho_\tau)\) is called a \( \tau \)-metric space and the elements of \( X \) are called points.

Every \( \tau \)-metric space \((X, \rho_\tau)\) generates a Tychonoff (that is, completely regular and Hausdorff) topological space \((X, T_{\rho_\tau})\). The topology \( T_{\rho_\tau} \) on \( X \) defined by the local basis consisting of the sets of the form

\[ G(x) = \{ y \in X : \rho_\tau(x, y) \in O(\theta) \}, \]

where \( O(\theta) \) runs through all open neighbourhoods of the point \( \theta \) in the space \( \mathbb{R}_+^\tau \), of each point \( x \in X \) is called the topology induced by the \( \tau \)-metric \( \rho_\tau \).

In this paper the notion of \( \tau \)-metrizable space is introduced. The paper is organized as follows. Section 2 contains the basic concepts of \( \tau \)-metrizable spaces. Generally, \( \tau \)-metrizable spaces may be not metrizable. We prove that if \( \tau \leq \aleph_0 \), then every \( \tau \)-metrizable space is metrizable. In section 3 we obtain a generalization of the classical metrization theorem of Urysohn. More precisely, we prove that every Tychonoff space of weight \( \tau > \aleph_0 \) is \( \tau \)-metrizable. Finally, in section 4 we prove that every compact \( \tau \)-metrizable space has density less than or equal to \( \tau \).

2. Basic concepts

The notion of a \( \tau \)-metric space leads to the notion of a \( \tau \)-metrizable space which is inserted in the following definition.

**Definition 2.1.** A topological space \((X, T)\) is called \( \tau \)-metrizable if there exists a \( \tau \)-metric \( \rho_\tau \) on the set \( X \) such that the topology \( T_{\rho_\tau} \) induced by the \( \tau \)-metric \( \rho_\tau \) coincides with the original topology \( T \) of \( X \). \( \tau \)-metrics on the set \( X \) which induce the original topology of \( X \) will be called \( \tau \)-metrics on the space \( X \).

Note that \( \tau \)-metrizable spaces are useful because only such spaces can be presented as limits of \( \tau \)-long projective systems of metric spaces [1, Theorem 3].

**Proposition 2.2.** A metric space is \( \tau \)-metrizable.

**Proof.** Let \((X, \rho)\) be a metric space, \( t_{\rho} \) be the topology induced by the metric \( \rho \), and let \( \tau \) be a cardinal number. Consider a set \( \Lambda \) such that \( |\Lambda| = \tau \) and set \( \rho_\lambda = \rho \) for each \( \lambda \in \Lambda \). The mapping \( \rho_\tau : X \times X \to \mathbb{R}_+^\tau \) defined by \( \rho_\tau(x, y) = \{ \rho_\lambda(x, y) \}_{\lambda \in \Lambda} \) for every \( x, y \in X \) is a \( \tau \)-metric on \( X \). It is easy to see that \( t_{\rho} = T_{\rho_\tau} \). \( \square \)

**Proposition 2.3.** A \( \tau \)-metrizable space is \( \tau' \)-metrizable for every cardinal number \( \tau' > \tau \).
\(\tau\)-metrizable spaces

**Proof.** Let \(X\) be a \(\tau\)-metrizable space, \(\rho_\tau\) be a \(\tau\)-metric on the space \(X\) and \(\tau'\) be a cardinal number such that \(\tau' > \tau\). Consider two sets \(K\) and \(\Lambda\) such that \(K \subset \Lambda\), \(|K| = \tau\) and \(|\Lambda| = \tau'\), and set \(\rho_{\tau'}(x, y) = \{\rho_{\tau}^k(x, y)\}_{k \in K}\) for every \(x, y \in X\). Let \(k_0\) be one fixed element of \(K\). The mapping \(\rho_{\tau'} : X \times X \rightarrow \mathbb{R}_+^{\tau'}\) defined by \(\rho_{\tau'}(x, y) = \{\rho_{\tau}^k(x, y)\}_{k \in \Lambda}\) for every \(x, y \in X\), where

\[
\rho_{\tau'}^k(x, y) = \begin{cases} 
\rho_{\tau}^k(x, y), & \text{if } \lambda \in K \\
\rho_{k_0}^k(x, y), & \text{if } \lambda \in \Lambda \setminus K,
\end{cases}
\]

is a \(\tau'\)-metric on \(X\) such that \(T_{\rho_{\tau'}} = T_{\rho_{\tau}}\). 

The following examples show that \(\tau\)-metrizable spaces may be not metrizable.

**Example 2.4.** The product \(\mathbb{R}^\alpha = \prod_{\lambda \in \Lambda} X_\lambda\), where \(X_\lambda = \mathbb{R}\) for every \(\lambda \in \Lambda\) and \(|\Lambda| = \mathfrak{c}\), of uncountably many copies of the real line \(\mathbb{R}\) is not metrizable, since it is not first-countable. However, the space \(\mathbb{R}^\alpha\) is \(c\)-metrizable. Assuming each copy \(X_\lambda\) of \(\mathbb{R}\) has its usual metric \(d_\lambda\), the mapping \(\rho_c : \mathbb{R}^\alpha \times \mathbb{R}^\alpha \rightarrow \mathbb{R}_+^\alpha\) defined by \(\rho_c(x, y) = \{d_\lambda(x_\lambda, y_\lambda)\}_{\lambda \in \Lambda}\) for every \(x = \{x_\lambda\}_{\lambda \in \Lambda} \in \mathbb{R}^\alpha\) and \(y = \{y_\lambda\}_{\lambda \in \Lambda} \in \mathbb{R}^\alpha\) is a \(c\)-metric on \(\mathbb{R}^\alpha\) and the topology induced by \(\rho_c\) coincides with the product topology.

**Example 2.5.** Let \(\mathbb{R}\) be the set of real numbers with the discrete topology \(\mathcal{D}\) and \((\mathbb{R}_\infty, \mathcal{D}_\infty)\) be the Alexandroff’s one-point compactification of the space \((\mathbb{R}, \mathcal{D})\), that is \(\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}\) and \(\mathcal{D}_\infty = \mathcal{D} \cup \{\mathbb{R}_\infty \setminus K : K\) is a finite subset of \(\mathbb{R}\}\). The space \((\mathbb{R}_\infty, \mathcal{D}_\infty)\) is not metrizable (because it is not separable). We prove that the space \((\mathbb{R}_\infty, \mathcal{D}_\infty)\) is \(c\)-metrizable. Let \(\text{Fin}(\mathbb{R})\) be the collection of all the nonempty finite subsets of \(\mathbb{R}\) with \(|\text{Fin}(\mathbb{R})| = \mathfrak{c}\). For every \(F \in \text{Fin}(\mathbb{R})\) we define:

1. \(\rho_F(x, x) = 0\) for each \(x \in \mathbb{R}_\infty\).
2. \(\rho_F(x, \infty) = \rho_F(\infty, x) = \begin{cases} 
0, & \text{if } x \notin F \\
1, & \text{otherwise}
\end{cases}\) for each \(x \in \mathbb{R}\).
3. \(\rho_F(x, y) = \begin{cases} 
0, & \text{if } x \notin F \text{ and } y \notin F \\
1, & \text{otherwise}
\end{cases}\) for each \(x, y \in \mathbb{R}\) with \(x \neq y\).

The mapping \(\rho_c : \mathbb{R}_\infty \times \mathbb{R}_\infty \rightarrow \mathbb{R}_+\) defined by \(\rho_c(x, y) = \{\rho_F(x, y)\}_{F \in \text{Fin}(\mathbb{R})}\) for every \(x, y \in \mathbb{R}_\infty\) is a \(c\)-metric on \(\mathbb{R}_\infty\). We prove that the topology \(T_{\rho_c}\) induced by the \(c\)-metric \(\rho_c\) coincides with the topology \(\mathcal{D}_\infty\).

Let \(x \in \mathbb{R}\). If \(G(x) = \{y \in \mathbb{R}_\infty : \rho_c(x, y) \in O(\theta)\}\), where \(O(\theta)\) is an open neighbourhood of the point \(\theta\) in the space \(\mathbb{R}_+\), then \(\{x\} \in \mathcal{D}_\infty\) and \(\{x\} \subseteq G(x)\). Moreover, for the open neighbourhood \(\prod_{F \in \text{Fin}(\mathbb{R})} W_F\) of the point \(\theta\), where

\[
W_F = \begin{cases} 
[0, \frac{1}{2}), & \text{if } F = \{x\} \\
\mathbb{R}_+ \setminus \{x\}, & \text{otherwise}
\end{cases}
\]

we have \(G(x) = \{y \in \mathbb{R}_\infty : \rho_c(x, y) \in \prod_{F \in \text{Fin}(\mathbb{R})} W_F\} \subseteq \{x\}\).

© ACT, UPV, 2018

*Appl. Gen. Topol.* 19, no. 2 | 255
A. C. Megaritis

Now, we consider the point \( \infty \) of \( \mathbb{R}_\infty \). If \( \{\infty\} \cup (\mathbb{R} \setminus K) \), where \( K \in \text{Fin}(\mathbb{R}) \) is an open neighbourhood of the point \( \infty \) in the space \( \mathbb{R}_\infty \), then for the open neighbourhood \( \prod_{F \in \text{Fin}(\mathbb{R})} W_F \) of the point \( \theta \), where
\[
W_F = \begin{cases} [0, \frac{1}{2}), & \text{if } F = K \\ \mathbb{R}_+, & \text{otherwise} \end{cases}
\]
we have \( G(\infty) = \{ y \in \mathbb{R}_\infty : \rho_\lambda(\infty, y) \in \prod_{F \in \text{Fin}(\mathbb{R})} W_F \} \subseteq \{\infty\} \cup (\mathbb{R} \setminus K) \).

Finally, let \( \prod_{\in F \in \text{Fin}(\mathbb{R})} U_F \) be an open neighbourhood of the point \( \theta \) in the space \( \mathbb{R}_\mathbb{Z}^+ \) and suppose that \( \{ F \in \text{Fin}(\mathbb{R}) : U_F \neq \mathbb{R}_+ \} = \{K_1, \ldots, K_m\} \). Then,
\[
\{\infty\} \cup (\mathbb{R} \setminus (K_1 \cup \ldots \cup K_m)) \subseteq G(\infty) = \{ y \in \mathbb{R}_\infty : \rho_\lambda(\infty, y) \in \prod_{\in F \in \text{Fin}(\mathbb{R})} U_F \}.
\]

However, a \( \tau \)-metrizable space may be metrizable considering addition conditions as the following assertions show.

**Proposition 2.6.** A \( n \)-metric space is metrizable for every finite cardinal number \( n \).

**Proof.** Let \( (X, \rho_n) \) be a \( n \)-metric space and \( T_{\rho_n} \) be the topology induced by \( \rho_n \).

Consider a vector expression of the form \( \rho_n(x, y) = (\rho_n^1(x, y), \ldots, \rho_n^n(x, y)) \) for every \( x, y \in X \). The mapping \( \rho : X \times X \to \mathbb{R}_+^n \) defined by
\[
\rho(x, y) = \max\{\rho_n^1(x, y), \ldots, \rho_n^n(x, y)\}
\]
for every \( x, y \in X \) is a metric on \( X \). It is easy to see that the metric topology is the same as \( T_{\rho_n} \). \( \square \)

**Definition 2.7.** Two \( \tau \)-metrics \( \rho_{1, \tau} \) and \( \rho_{2, \tau} \) on a set \( X \) are called equivalent if they induce the same topology on \( X \), that is \( T_{\rho_{1, \tau}} = T_{\rho_{2, \tau}} \).

**Example 2.8.** Let \( \rho_\tau \) be a \( \tau \)-metric on \( X \). Consider a set \( \Lambda \) such that \( |\Lambda| = \tau \) and let us set \( \rho_\tau(x, y) = \{\rho_\lambda^1(x, y), \ldots, \rho_\lambda^n(x, y)\} \) for every \( x, y \in X \). The mapping \( \rho_\tau^\Lambda : X \times X \to \mathbb{R}_+^\tau \) defined by \( \rho_\tau^\Lambda(x, y) = \{\min\{1, \rho_\lambda(x, y)\}\} \) for every \( x, y \in X \) is a \( \tau \)-metric on \( X \) equivalent to \( \rho_\tau \).

**Proposition 2.9.** An \( \aleph_0 \)-metric space is metrizable.

**Proof.** Let \( (X, \rho_{\aleph_0}) \) be an \( \aleph_0 \)-metric space. Consider the equivalent \( \aleph_0 \)-metric \( \rho_{\aleph_0}^\Lambda \) to \( \rho_{\aleph_0} \) of Example 2.8. Let \( \rho_{\aleph_0}^\Lambda(x, y) = (\rho_{\aleph_0}^1(x, y), \rho_{\aleph_0}^2(x, y), \ldots) \) for every \( x, y \in X \). The mapping \( \rho : X \times X \to \mathbb{R}_+ \) defined by
\[
\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_{\aleph_0}^i(x, y)
\]
for every \( x, y \in X \) is a metric on \( X \). The process of proving that the topology induced by the metric \( \rho \) coincides with the topology \( T_{\rho_{\aleph_0}} \) is similar to the proof of the Theorem 4.2.2 of [2]. \( \square \)

**Corollary 2.10.** If \( \tau \leq \aleph_0 \), then every \( \tau \)-metrizable space is metrizable.

© ACT UPV, 2018  
Appl. Gen. Topol. 19, no. 2  |  256
For each \( \tau > 2018 \) and every \( \tau \)-metrizable space \( X \), we have \( \chi(X) \leq \tau \).

**Proof.** Let \( X \) be a \( \tau \)-metrizable space and \( \rho \) be a \( \tau \)-metric on the space \( X \) with \( \tau \geq 2018 \). Consider a set \( \Lambda \) such that \( |\Lambda| = \tau \). The family \( \mathcal{B}_0 \) of all products \( \prod_{\lambda \in \Lambda} W_\lambda \), where finitely many \( W_\lambda \) are intervals of the form \([0, b)\) with rational \( b \) and the remaining \( W_\lambda = \mathbb{R}^+ \), form a local basis of the point \( \theta \) in the space \( \mathbb{R}^\tau_+ \). Hence, for every \( x \in X \), the family

\[
\mathcal{B}(x) = \{ G(x) = \{ y \in X : \rho(x, y) \in B \} : B \in \mathcal{B}_0 \}
\]

is a local basis of the point \( x \) in the space \( X \). Since \( |\mathcal{B}_0| = \tau \), we have \( |\mathcal{B}(x)| \leq \tau \).


### 3. A \( \tau \)-METRIZATION THEOREM

Metrization theorems are theorems that give sufficient conditions for a topological space to be metrizable (see [2,5]). In this section we obtain a generalization of the classical metrization theorem of Urysohn.

**Lemma 3.1.** If \((X, \rho)\) is a \( \tau \)-metric space and \( A \) is a subspace of \( X \), then the topology induced by the restriction of the \( \tau \)-metric \( \rho \) to \( A \times A \) is the same as the subspace topology of \( A \) in \( X \).

**Theorem 3.2.** Every Tychonoff space of weight \( \tau > 2018 \) is \( \tau \)-metrizable.

**Proof.** Let \( X \) be a Tychonoff space such that \( \omega(X) = \tau > 2018 \). The space \( I^\tau = \prod_{\lambda \in \Lambda} X_\lambda \), where \( X_\lambda = I \) for every \( \lambda \in \Lambda \) and \( |\Lambda| = \tau \) is \( \tau \)-metrizable (see Example 2.4). Assuming each copy \( X_\lambda \) of \( I \) has its usual metric \( d_\lambda \), the mapping \( d_\tau : I^\tau \times I^\tau \to \mathbb{R}^\tau_+ \), defined by \( d_\tau(x, y) = \{ d_\lambda(x_\lambda, y_\lambda) \}_{\lambda \in \Lambda} \) for every \( x = \{ x_\lambda \}_{\lambda \in \Lambda} \in I^\tau \) and \( y = \{ y_\lambda \}_{\lambda \in \Lambda} \in I^\tau \) is a \( \tau \)-metric on \( I^\tau \). We shall prove that \( X \) is \( \tau \)-metrizable by imbedding \( X \) into the \( \tau \)-metrizable space \( I^\tau \), i.e., by showing that \( X \) is homeomorphic with a subspace of \( I^\tau \). But this follows immediately from the fact that the Tychonoff cube \( I^\tau \) is universal for all Tychonoff spaces of weight \( \tau \) (see [2, Theorem 2.3.23]). By Lemma 3.1, the space \( X \) is \( \tau \)-metrizable.

As every \( \tau \)-metrizable space is Tychonoff (see [1]), we get the following result.

**Corollary 3.3.** A space of weight \( \tau > 2018 \) is \( \tau \)-metrizable if and only if it is Tychonoff.
Remark 3.4. We can use Theorem 3.2 to find $\tau$-metrizable spaces, where $\tau > \aleph_0$, that are not metrizable. Below we consider some examples. Example 3.5 is a $c$-metrizable space which is not second-countable, Example 3.6 is a $c$-metrizable space which is not normal and Example 3.7 is a $2^\tau$-metrizable space, where $\tau \geq c$, which is not metrizable.

Example 3.5. Let $S$ be the Sorgenfrey line, that is the real line with the topology in which local basis of $x$ are the sets $[x, y)$ for $y > x$. Since $X$ is separable but not second-countable, it cannot be metrizable. Furthermore, $S$ is Tychonoff and $w(S) = c$. From Theorem 3.2 it follows that the Sorgenfrey line is a $c$-metrizable space.

Example 3.6. Let $P = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta > 0\}$ be the open upper half-plane with the Euclidean topology and $L = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta = 0\}$. We set $X = P \cup L$. For every $x \in P$ let $B(x)$ be the family of all open discs in $P$ centered at $x$. For every $x \in L$ let $B(x)$ be the family of all sets of the form $\{x\} \cup D$, where $D$ is an open disc in $P$ which is tangent to $L$ at the point $x$. The family $T$ of all subsets of $X$ that are unions of subfamilies of $\bigcup \{B(x) : x \in X\}$ is a topology on $X$ and the family $\{B(x) : x \in X\}$ is a neighbourhood system for the topological space $(X, T)$. The space $X$ is called the Niemytzki plane (see, for example, [2, 4]). $X$ is a Tychonoff space with $w(X) = c$, which is not normal. Therefore, by Theorem 3.2, $X$ is a $c$-metrizable space, but not metrizable.

Example 3.7. Let $\beta D(\tau)$ be the Čech-Stone compactification of the discrete space $D(\tau)$ of cardinality $\tau \geq c$. Then, $w(\beta D(\tau)) = 2^\tau$ (see [2, Theorem 3.6.11]). Since $\beta D(\tau)$ is zero-dimensional (see [2, Theorem 3.6.13]), it is Tychonoff. The space $D(\tau)$ is not compact. Therefore, $\beta D(\tau)$ is not metrizable (see [3, Exercise 9, §38, Ch.5]). From Theorem 3.2 it follows that $\beta D(\tau)$ is $2^\tau$-metrizable. Particularly, if one assumes the continuum hypothesis, the Čech-Stone compactification $\beta \omega$ of the discrete space of the non-negative integers $\omega = \{0, 1, 2, \ldots\}$ is $c$-metrizable.

Remark 3.8. A space $X$ may be $\tau$-metrizable for some infinite cardinal number $\tau < w(X)$, as shown in the following example.

Example 3.9. Let $\Lambda$ be a set of cardinality $\tau > \aleph_0$, $D(\kappa)$ the discrete space of cardinality $\kappa > \tau$, and $F = \prod_{\lambda \in \Lambda} X_{\lambda}$, where $X_{\lambda} = D(\kappa)$ for every $\lambda \in \Lambda$, with the Tychonoff product topology. We note that the points of $F$ are functions from $\Lambda$ to $D(\kappa)$. The space $F$ is not metrizable for $\chi(F) = \tau$ (see [2, Exercise 2.3.F(b)]). Moreover, $w(F) = \kappa$ (see [2, Exercise 2.3.F(a)]). We prove that the space $F$ is $\tau$-metrizable. For every $\lambda \in \Lambda$ we define:

1. $\rho_{\lambda}(f, f) = 0$ for each $f \in F$.
2. $\rho_{\lambda}(f, g) = \begin{cases} 0, & \text{if } f(\lambda) = g(\lambda) \\ 1, & \text{otherwise} \end{cases}$ for each $f, g \in F$ with $f \neq g$.

The mapping $\rho_{\tau} : F \times F \to \mathbb{R}_+^\tau$, defined by $\rho_{\tau}(f, g) = \{\rho_{\lambda}(f, g)\}_{\lambda \in \Lambda}$ for every $f, g \in F$ is a $\tau$-metric on $F$. We prove that the topology $T_{\rho_{\tau}}$ induced by the $\tau$-metric $\rho_{\tau}$ coincides with the Tychonoff product topology.
τ-metrizable spaces

Let $f \in F$, $\prod_{\lambda \in A} U_{\lambda}$ be an open neighbourhood of the point $\theta$ in the space $\mathbb{R}^+_\tau$, and suppose that $\{ \lambda \in A : U_{\lambda} \neq \mathbb{R}^+ \} = \{ \lambda_1, \ldots, \lambda_m \}$. For the open neighbourhood $\prod_{\lambda \in A} W_{\lambda}$ of the point $f$, where

$$W_{\lambda} = \begin{cases} \{ f(\lambda) \}, & \text{if } \lambda \in \{ \lambda_1, \ldots, \lambda_m \} \\ D(\kappa), & \text{otherwise} \end{cases}$$

we have $\prod_{\lambda \in A} W_{\lambda} \subseteq G(f) = \{ g \in F : \rho_\tau(f, g) \in \prod_{\lambda \in A} U_{\lambda} \}$.

Now, let $f \in F$ and $\prod_{\lambda \in A} W_{\lambda}$ be an open neighbourhood of the point $f$ in the space $F$, and suppose that $\{ \lambda \in A : W_{\lambda} \neq D(\kappa) \} = \{ \lambda_1, \ldots, \lambda_m \}$. For the open neighbourhood $\prod_{\lambda \in A} U_{\lambda}$ of the point $\theta$, where

$$U_{\lambda} = \begin{cases} [0, \frac{1}{\lambda}], & \text{if } \lambda \in \{ \lambda_1, \ldots, \lambda_m \} \\ \mathbb{R}^+, & \text{otherwise} \end{cases}$$

we have $G(f) = \{ g \in F : \rho_\tau(f, g) \in \prod_{\lambda \in A} U_{\lambda} \} \subseteq \prod_{\lambda \in A} W_{\lambda}$.

4. Compact τ-metrizable spaces

It is well known that every compact metrizable space is separable. An analogous result for τ-metrizable spaces is stated in this section.

Let us consider a set $\Lambda$ such that $|\Lambda| = \tau \geq \aleph_0$ and let $B_e$ be the family of all open subsets $\prod_{\lambda \in A} W_{\lambda}$ of the product $\mathbb{R}^+_\tau$, where finitely many $W_{\lambda}$ are intervals of the form $[0, \varepsilon)$ and the remaining $W_{\lambda} = \mathbb{R}^+$.

**Definition 4.1.** Let $(X, \rho_\tau)$ be a τ-metric space. A subset $A$ of $X$ is called $O_e$-dense in $(X, \rho_\tau)$, where $O_e \in B_e$, if for every $x \in X$ there exists $a \in A$ such that $\rho_\tau(x, a) < \varepsilon$.

**Definition 4.2.** A τ-metric space $(X, \rho_\tau)$ is called $\varepsilon$-totally bounded if for every $O_e \in B_e$ there exists a finite subset $A$ of $X$ which is $O_e$-dense in $(X, \rho_\tau)$. The τ-metric space $(X, \rho_\tau)$ is called totally bounded if it is $\varepsilon$-totally bounded for every $\varepsilon > 0$.

Recall that the density $d(X)$ of a topological space $X$, is defined to be $d(X) = \min\{|D| : D$ is a dense subset of $X\}$.

**Proposition 4.3.** For every totally bounded τ-metric space $X$, the inequality $d(X) \leq \tau$ holds.

**Proof.** Let $n \in \{1, 2, \ldots\}$. For each $O_{1/n} \in B_{1/n}$, let $A(O_{1/n})$ be a finite $O_{1/n}$-dense subset of $X$ and consider the subset $A_n = \cup\{A(O_{1/n}) : O_{1/n} \in B_{1/n}\}$ of $X$ with $|A_n| \leq \tau$. The subset $A = \cup_{n=1}^\infty A_n$ of $X$ is dense and $|A| \leq \tau$. \hfill $\square$

**Proposition 4.4.** Every compact τ-metric space $X$ is totally bounded.

**Proof.** Let $\varepsilon > 0$. For every $O_e \in B_e$ the family

$$\{G(x) = \{ y \in X : \rho_\tau(x, y) \in O_e \} : x \in X\}$$

forms an open cover of $X$. By compactness of $X$, there exists a finite subset $A$ of $X$ such that $\bigcup_{a \in A} G(a) = X$. For every $x \in X$ there exists $a \in A$ with
A. C. Megaritis

\( x \in G(a) \). Therefore, \( \rho_\tau(x, a) \in O_\varepsilon \) and the subset \( A \) of \( X \) is \( O_\varepsilon \)-dense in \( (X, \rho_\tau) \). □

**Theorem 4.5.** For every compact \( \tau \)-metrizable space \( X \) we have \( d(X) \leq \tau \).

**Proof.** Let \( X \) be a compact \( \tau \)-metrizable space. According to Proposition 4.4, the space \( X \) is totally bounded. Therefore, by virtue of Proposition 4.3, \( d(X) \leq \tau \). □

**Acknowledgements.** The author would like to thank both referees for their valuable comments and suggestions.

**References**