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Some categorical aspects of the inverse limits in ditopological context

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Abstract

This paper considers some various categorical aspects of the inverse systems (projective spectrums) and inverse limits described in the category **ifPDitop**, *whose objects are ditopological plain texture spaces and morphisms are bicontinuous point functions satisfying a compatibility condition between those spaces. In this context, the category* $\mathbf{Inv}_{\mathbf{ifPDitop}}$ *consisting of the inverse systems constructed by the objects and morphisms of* **ifPDitop**, *besides the inverse systems of mappings, described between inverse systems, is introduced, and the related ideas are studied in a categorical - functorial setting. In conclusion, an identity natural transformation is obtained in the context of inverse systems - limits constructed in* **ifPDitop** *and the ditopological infinite products are characterized by the finite products via inverse limits.*

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1. Introduction and preliminaries

Just as the methods used to derive a new space from two or more spaces are the products, subtextures and quotients of that spaces, so the another effective method is the theory of *inverse systems (projective spectrums)* and *inverse limits (projective limits)*.

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The origins of the study of inverse limits date back to the 1920 's. Classical theory of inverse systems and inverse limits are important in the extension of homology and cohomology theory. An exhaustive discussion of inverse systems which are in the some classical categories such as **Set**, **Top**, **Grp** and **Rng** defined in [1], was presented by the paper [5] which is a milestone in the development of that theory.

As is the case with products, the inverse limit might not exist in any category in general whereas inverse systems exist in every category. Note from that [5] inverse limits exist in any category when that category has products of objects and the equalizers [1] of pairs of morphisms, in other words, the inverse limits exist in any category if the category is complete, in the sense of [1]. Additionally, an inverse system has at most one limit. That is, if an inverse limit of any inverse system exists in any category $\mathcal{C}$, this limit is unique up to $\mathcal{C}$-isomorphism. Incidentally, inverse limits always exist in the categories **Set**, **Top**, **Grp** and **Rng**. Note also that inverse limits are generally restricted to diagrams over directed sets.

Similarly, a suitable theory of inverse systems and inverse limits for the categories consisting of textures and ditopological spaces is handled first-time in [17] and [18].

Incidentally, let 's recall the notions of *texture and ditopology* introduced in 1993, by Lawrence M. Brown : For a nonempty set S, the family $\mathcal{S} \subseteq \mathcal{P}(S)$ is called a *texturing* on S if $(\mathcal{S}, \subseteq)$ is a point-separating, complete, completely distributive lattice containing S and $\varnothing$, with meet coinciding with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture.* If $\mathcal{S}$ is closed under arbitrary unions, it is called *plain texturing* and $(S, \mathcal{S})$ is called *plain texture.* Since a texturing $\mathcal{S}$ need not be closed under the operation of taking the set-complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ and the set of *closed sets* κ, satisfy the some dual conditions. Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. In addition, a ditopological texture space or shortly *ditopological space* with respect to a ditopology (τ, κ) on the texture $(S, \mathcal{S})$ is denoted by $(S, \mathcal{S}, \tau, \kappa)$.

There is now a considerable literature on the theory of ditopological spaces. An adequate introduction to this theory and the motivation for its study may be obtained from [2, 3, 4, 8, 9, 10, 13]. As will be clear from these general references, it is shown that ditopological spaces provide a unified setting for the study of topology, bitopology and fuzzy topology on Hutton algebras. Some of the links with Hutton spaces and fuzzy topologies are expressed in a categorical setting in [14]. In addition, there are close and deep relationships between the bitopological and ditopological spaces as shown in [11, 12] and [15, 16]. In this study, we will use those close relationships insofar as the inverse systems and their inverse limits are concerned in a categorical view.

As it is stated before, in [2, 3, 4] we have a few methods, such as product space, subtexture space and quotient space, to derive a new ditopological space from two or more ditopological spaces just like classical case. Recently, it is seen in [17, 18] that the another method used to construct a new ditopological space is the theory of *ditopological inverse systems* and their limit spaces under the name *ditopological inverse limits* as the subspaces of ditopological product spaces described in [3, 4, 18].

There are considerable difficulties involved in constructing a suitable theory of inverse systems for general ditopological spaces. Hence, in [17] we confined our attention to a special category whose objects are plain textures, and the basic properties of inverse systems and their inverse limits are investigated in the first-time for texture theory in the context of that category. Accordingly, the various aspects of the inverse systems - limits for texture theory are investigated for plain case and placed them in a categorical - functorial setting.

Later, in [18], the theory of inverse systems and inverse limits is handled first-time in the ditopological textural context and we gave a detailed analysis of the theory of ditopological inverse systems and inverse limits insofar as the category **ifPDitop** whose objects are the ditopological texture spaces which have plain texturing and morphisms are the bicontinuous, w-preserving point functions, is concerned. (For a detailed information and some basic facts about the point-functions between texture spaces, see [3, 10, 11]).

By the way, no attempt isn 't made at the direct systems of ditopological spaces even plain ones, and their (direct) limits as the dual notions of inverse limits.

Returning to work at the moment, our main aim in the present paper is to give some further results on the theory of inverse systems and their inverse limits in the context of category **ifPDitop**. Especially, this paper will present some intriguing connections between the bitopological inverse systems - limit spaces and their ditopological counterparts, in a categorical - functorial setting. Here we will continue to work within the same framework given in [17, 18] that are the major sources of the topic on which we study.

According to that, frequent reference will be made to the author 's papers [17] and [18] which present all details related to the subjects *inverse system* and *inverse limit* constructed in the textural context for the plain case, besides providing some useful historical information located in the literature about inverse systems. Otherwise, this paper is largely self-contained although the reader may wish to refer to the literature cited in these papers, for motivation and additional background material specific to the main topic of this paper. Especially, the significant reference in the general field of inverse system theory is [5] and in addition, the reader is referred to [6] for the information about the inverse systems consisting of topological spaces.

Specifically, the reader may consult [7] for terms from lattice theory not mentioned here. In addition, we follow the terminology of [1] for all the general concepts relating to category theory. Thus, if **A** is a category, Ob **A** will denote the class of objects and Mor **A** the class of morphisms of **A**.

In this paper, generally we have tried to give enough details of the proofs to make it clear where various of the conditions imposed are needed, but at the same time to avoid boring the reader with routine verifications.

Accordingly, this paper consists of six sections and the layout of paper is as follows:

After presenting some background information via the references mentioned in the first section, we introduce and study the category $\mathbf{Inv}_{\mathbf{ifPDitop}}$ in Section 2, mainly. For the paper, it will denote the category whose objects are the inverse systems constructed by the objects of **ifPDitop** and morphisms are the inverse systems of mappings in the sense of mappings defined between inverse systems. Following that, by describing another related categories and the required functors between the corresponding categories which have some useful properties, we continued to discuss various aspects of the inverse systems and their limits in **ifPDitop**. In addition, there is a close relationship between ditopological spaces restricted to plain textures and bitopological spaces, as exemplified by a special functor isomorphism given in that section. Hence, we are interested in the connections between bitopological and ditopological inverse systems together with their limits, via that isomorphism. In the end of this section, as one of the principal aims of paper, we obtained an identity natural transformation constructed between the related appropriate functors, described via those connections just mentioned. Specifically, this section contains some examples and other results that are important in their own right and which will also be needed later on.

In a similar way, in Section 3 we presented a few connections between the category of topological spaces and the category **ifPDitop** insofar as the inverse systems and their inverse limits are concerned in a categorical setting.

Besides these, in Section 4 we investigated the effect of closure operators on inverse systems and limits in **ifPDitop**, with respect to the joint topologies correspond to the ditopologies located on those inverse systems and limits.

A significant characterization theorem which says that by applying the inverse limit operation, any cartesian products of ditopological plain spaces which are the objects of **ifPDitop** can be expressed in terms of the finite cartesian products of those spaces, is proved in Section 5. Following that, this section ends with two principal corollaries of that characterization.

As the last part of paper, Section 6 gives a conclusion about the whole of this study.

2. Relationships between the inverse systems-limits in the categories of bitopological and ditopological spaces

In this section, firstly, let 's recall all the considerations presented in [12, Section 2] as follows:

Let **Bitop** be the category whose objects are bitopological spaces and morphisms are pairwise continuous functions, and the category **ifPDitop**, introduced in [18], is known from the previous section.

Accordingly, consider the mapping $\mathfrak{U}$ from **ifPDitop** to **Bitop** by

$$\mathfrak{U}((S,\mathcal{S},\tau_S,\kappa_S) \xrightarrow{\varphi} (T,\mathcal{T},\tau_T,\kappa_T)) = (S,\tau_S,\kappa_S^c) \xrightarrow{\varphi} (T,\tau_T,\kappa_T^c).$$

It is trivial to verify that this is indeed a functor and we omit the details.

When applied to many important ditopological spaces, such as the unit interval and real space, the corresponding *ditopological* T_0 *axiom* as a separation axiom is described as

$$Q_s \not\subseteq Q_t \implies \exists C \in \tau \cup \kappa \text{ with } P_s \not\subseteq C \not\subseteq Q_t$$

and it behaves more like the bitopological *weak pairwise* T_0 *axiom*,

$$x \in \overline{y}^u \cap \overline{y}^v \text{ and } y \in \overline{x}^u \cap \overline{x}^v \implies x = y.$$

Why this is so, at least in the case of plain textures, we now see by setting up a new functor in the opposite direction of $\mathfrak{U}$.

To define the suitable functor such that preserves T_0 axiom, we restrict ourselves to weakly pairwise T_0 bitopological spaces (X,u,v), and consider the smallest subset $\mathcal{K}_{uv}$ of $\mathcal{P}(X)$ which contains $u \cup v^c$ and is closed under arbitrary intersections and unions. Clearly the elements of $\mathcal{K}_{uv}$ have the form

$$A = \bigcap_{j\in J} A_j, \text{ where } A_j = U_j \cup \bigcup_{i\in I_j} \{(V_i^j)^c \mid V_i^j \in v\},\ U_j \in u,\ j \in J. \tag{2.1}$$

In summary, for a weakly pairwise T_0 bitopological space (X,u,v), the set $u \cup v^c$ generates a texturing, denoted by $\mathcal{K}_{uv}$ on X.

Moreover, it is easy to verify that $\mathcal{K}_{uv}$ is a plain texturing of X since it separates points, by using the property "weakly pairwise T_0" of the space (X,u,v). Finally, we have the plain ditopological space $(X,\mathcal{K}_{uv},u,v^c) \in \mathrm{Ob}\,$**ifPDitop** satisfying the ditopological T_0 separation axiom.

Specifically, for a space $(S,\mathcal{S},\tau,\kappa) \in \mathrm{Ob}\,$**ifPDitop** the equality $\mathcal{K}_{\tau\kappa^c} = \mathcal{S}$ is known from [12, Corollary 3.8].

With all these considerations, this process gives a mapping between the subcategory $\mathbf{Bitop_{w0}}$ of **Bitop**, consisting of weakly pairwise T_0 bitopological spaces - pairwise continuous functions and the subcategory $\mathbf{ifPDitop_0}$ of **ifPDitop**, consisting of T_0 ditopological spaces and bicontinuous, w-preserving point functions, as follows:

$$\mathfrak{H}((X,u_X,v_X) \xrightarrow{\varphi} (Y,u_Y,v_Y)) = (X,\mathcal{K}_{u_Xv_X},u_X,v_X^c) \xrightarrow{\varphi} (Y,\mathcal{K}_{u_Yv_Y},u_Y,v_Y^c)$$

Clearly, it defines a functor $\mathfrak{H} : \mathbf{Bitop_{w0}} \to \mathbf{ifPDitop_0}$ as mentioned in [9]. Note that this concrete functor is a variant of the functor with the same name considered in [12, 15] in connection with real dicompactness.

We are now in a position to give two examples denote the importance of the functor $\mathfrak{H}$.

Example 2.1.

(1) The unit interval ditopological space $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}}) \in \mathrm{Ob}\,\mathbf{ifPDitop_0}$ is the image of the bitopological space $(\mathbb{I}, u_{\mathbb{I}}, v_{\mathbb{I}}) \in \mathrm{Ob}\,\mathbf{Bitop_{w0}}$ under $\mathfrak{H}$, where $\tau_{\mathbb{I}} = u_{\mathbb{I}} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}$ and $\kappa^c_{\mathbb{I}} = v_{\mathbb{I}} = \{(r, 1] \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}$.

(2) The real ditopological space $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}}) \in \mathrm{Ob}\,\mathbf{ifPDitop_0}$ is the image of the bitopological space $(\mathbb{R}, u_{\mathbb{R}}, v_{\mathbb{R}}) \in \mathrm{Ob}\,\mathbf{Bitop_{w0}}$ under $\mathfrak{H}$, where $\tau_{\mathbb{R}} = u_{\mathbb{R}} = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$ and $\kappa^c_{\mathbb{R}} = v_{\mathbb{R}} = \{(r, \infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \varnothing\}$.

It may be verified that $\mathfrak{H}$ preserves the other basic ditopological separation axioms, besides T_0 axiom. Consequently, we have the following fact from [9, 12]:

Theorem 2.2. *$\mathfrak{H}$ is a concrete isomorphism between the constructs* $\mathbf{Bitop_{w0}}$ *and* $\mathbf{ifPDitop_0}$.

Remark 2.3. In view of the above statements, the equalities $\mathfrak{U} \circ \mathfrak{H} = \mathbf{1}_{\mathbf{Bitop_{w0}}}$ and $\mathfrak{H} \circ \mathfrak{U} = \mathbf{1}_{\mathbf{ifPDitop_0}}$ are trivial for the functor $\mathfrak{U} : \mathbf{ifPDitop_0} \to \mathbf{Bitop_{w0}}$ defined as above. Hence, $\mathfrak{U}$ is the inverse of $\mathfrak{H}$ as an isomorphism functor. Incidentally, it is concrete isomorphism since $\mathfrak{U}$ is identity carried, as well.

Now, we can turn our attention to the *inverse systems* and their *inverse limits* constructed in **ifPDitop**, in the light of [18]. Before everything, note that:

Remark 2.4. The inverse systems constructed by the objects and morphisms of the category **ifPDitop**, which are the bonding maps satisfying some conditions given in [18, Definition 3.1], have an *inverse limit space* described as in [18, Definition 4.1], since **ifPDitop** has products and equalizers as stated in [18, Corollary 2.6]. Also, the *uniqueness* of the limit space in the category **ifPDitop** was mentioned just before [18, Examples 4.5]. Hence, the operation $\lim\limits_{\leftarrow}$ will be meaningful for the inverse systems given in the context of that category.

Notation: According to the major theorem given as [18, Theorem 4.6], if take the inverse system $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha \geq \beta}$ constructed in **ifPDitop**, over a directed set Λ, then the notations $(\tau_\infty, \kappa_\infty)$ and $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)$ will be used as *inverse limit ditopology* and *(ditopological) inverse limit space*, respectively, where $S_\infty = \lim\limits_{\leftarrow}\{S_\alpha\}$, in the remainder of paper.

According to let 's take a glimpse of the *mappings between inverse systems*: Consider two inverse systems $\mathcal{A} = \{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha \geq \beta}$ and $\mathcal{B} = \{(T_\alpha, \mathcal{T}_\alpha, \tau'_\alpha, \kappa'_\alpha), \psi_{\alpha\beta}\}_{\alpha \geq \beta}$ over Λ described in **ifPDitop**, as in [18, Definition 3.1]. Take into consideration [17, Definition 3.9] which introduces the notion *inverse system of mappings* or *mapping of inverse systems* denoted by $\{t_\alpha\} : \mathcal{A} \to \mathcal{B}$, consisting of the components $t_\alpha \in \mathrm{Mor}\,\mathbf{ifPDitop}$, satisfying the equality $\psi_{\beta\alpha} \circ t_\beta = t_\alpha \circ \varphi_{\beta\alpha}$, that is, the commutativity of diagram

$$\begin{array}{ccc} S_\beta & \xrightarrow{t_\beta} & T_\beta \\ \downarrow{\scriptstyle \varphi_{\beta\alpha}} & & \downarrow{\scriptstyle \psi_{\beta\alpha}} \\ S_\alpha & \xrightarrow{t_\alpha} & T_\alpha \end{array}$$

which associates the bonding maps with the components t_α. Hence, by recalling the notion *inverse limit space* with the notation S_∞ defined as in [18, Definition 4.1] and the map $t_\infty = \varprojlim\{t_\alpha\}_{\alpha\in\Lambda}$ defined in [17, Theorem 4.14], called *inverse limit map* of the inverse system $\{t_\alpha\}$ of mappings, now let 's focus on the following crucial theorem proved in [18, Theorem 4.24]:

Theorem 2.5. *Let* $\{t_\alpha\} : \{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\beta\alpha}\}_{\beta\geq\alpha} \to \{(T_\alpha, \mathcal{T}_\alpha, \tau'_\alpha, \kappa'_\alpha), \psi_{\beta\alpha}\}_{\beta\geq\alpha}$ *be an inverse system of mappings in* **ifPDitop***, over a directed set* Λ*. Then there exists a unique map* $t_\infty \in \mathrm{Mor}\,$**ifPDitop** *between the spaces* $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)$ *and* $(T_\infty, \mathcal{T}_\infty, \tau'_\infty, \kappa'_\infty)$ *having the property that for each* $\alpha \in \Lambda$ *,the diagram*

$$\begin{array}{ccc} S_\infty & \xrightarrow{t_\infty} & T_\infty \\ \downarrow{\scriptstyle \mu_\alpha} & & \downarrow{\scriptstyle \eta_\alpha} \\ S_\alpha & \xrightarrow{t_\alpha} & T_\alpha \end{array}$$

is commutative, that is $t_\alpha \circ \mu_\alpha = \eta_\alpha \circ t_\infty$*.*
In this case,

i) *If each* t_α *is an* **ifPDitop***-isomorphism,* t_∞ *is an* **ifPDitop***-isomorphism.*
ii) *If each* $t_\alpha \circ \mu_\alpha$ *is surjective,* $t_\infty(S_\infty)$ *is jointly dense in* T_∞*.*

Notations: In this study, $\mathbf{Inv}_{\mathcal{C}}$ denotes the category whose objects are the inverse systems constructed by the objects of category $\mathcal{C}$ and morphisms are the mappings of inverse systems, described as just before Theorem 2.5, namely, the inverse systems of $\mathcal{C}$-morphisms defined between the objects of $\mathcal{C}$.

Particulary, the following notation will be required for the remainder of paper, mostly:

$\mathbf{Inv}_{\mathbf{ifPDitop_0}}$ will denote the category consisting of inverse systems constructed by T_0 ditopological plain texture spaces as objects of $\mathbf{ifPDitop}_0$, and by the mappings between inverse systems, namely, the inverse systems of mappings defined as in Theorem 2.5.

Incidentally, we have the following categorical fact about the inverse systems due to [18, Remark 3.2]:

Remark 2.6. An inverse system in any category admits an alternative description in terms of functors. A directed set Λ becomes a category if each relation $\alpha \leq \beta$ is regarded as a map $\alpha \to \beta$, that is the morphisms consist of arrows $\alpha \to \beta$ if and only if $\alpha \leq \beta$. Then, *Any inverse system in the category* **ifPDitop** *over the directed set* Λ *is actually a contravariant functor from* Λ *to* **ifPDitop**.

In the light of Remark 2.6, note that the objects and morphisms of $\mathbf{Inv_{ifPDitop}}$ may be regarded as the functors and natural transformations, respectively.

Example 2.7. If $\{(S_\alpha, u_\alpha, v_\alpha), f_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{Bitop_{w0}}}$ then the system $\{(S_\alpha, \mathcal{K}_{u_\alpha v_\alpha}, u_\alpha, v_\alpha^c), \varphi_{\alpha\beta}\}_{\alpha\geq\beta}$ consisting of the spaces $\mathfrak{H}(S_\alpha, u_\alpha, v_\alpha) = (S_\alpha, \mathcal{K}_{u_\alpha v_\alpha}, u_\alpha, v_\alpha^c) \in \mathrm{Ob}\,\mathbf{ifPDitop_0}$, corresponding to the bitopological spaces $(S_\alpha, u_\alpha, v_\alpha) \in \mathrm{Ob}\,\mathbf{Bitop_{w0}}$, describes an inverse system via the isomorphism functor $\mathfrak{H}$ given in Theorem 2.2 and all the above considerations. Trivially, this system is an object of $\mathbf{Inv_{ifPDitop_0}}$.

Now, by taking into account Example 2.7, immediately we have the following:

Example 2.8. If $(S_\infty, u_\infty, v_\infty) \in \mathrm{Ob}\,\mathbf{Bitop_{w0}}$ is the inverse limit of the inverse system $\{(S_\alpha, u_\alpha, v_\alpha), f_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{Bitop_{w0}}}$ then the corresponding plain space $(S_\infty, \mathcal{K}_{u_\infty v_\infty}, u_\infty, v_\infty^c) \in \mathrm{Ob}\,\mathbf{ifPDitop_0}$ is the inverse limit of corresponding inverse system $\{(S_\alpha, \mathcal{K}_{u_\alpha v_\alpha}, u_\alpha, v_\alpha^c), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{ifPDitop_0}}$, where $\varphi_{\alpha\beta} = f_{\alpha\beta}$ for $\alpha \geq \beta$. Let 's prove it:

First of all, recall the fact $S_\infty \subseteq \prod_\alpha S_\alpha$. Thus, it is clear that $u_\infty = (\prod_\alpha u_\alpha)|_{S_\infty} = (\bigotimes_\alpha u_\alpha)|_{S_\infty}$ since the textural and classical products of topologies are coincide by the plainness property. On the other hand, similar to the explanations given in [15, Section 3] we have $v_\infty^c = (\bigotimes_\alpha v_\alpha^c)|_{S_\infty}$ since $(\bigotimes_\alpha v_\alpha)|_{S_\infty} = (\prod_\alpha v_\alpha)|_{S_\infty} = v_\infty$ and by [3, Lemma 2.7] which is peculiar to the theory of product ditopologies.

Hence, it remains to prove the equality $\mathcal{K}_{u_\infty v_\infty} = (\bigotimes_\alpha \mathcal{K}_{u_\alpha v_\alpha})|_{S_\infty}$. For it, we can show that the types of elements of these two families are absolutely the same:

If $A \in \mathcal{K}_{u_\infty v_\infty}$ then let 's recall the form of A as follows:

$$A = \bigcap_{j\in J} A_j, \text{ where } A_j = U_j \cup \bigcup_{i\in I_j} \{(V_i^j)^c \mid V_i^j \in v_\infty\},\ U_j \in u_\infty,\ j \in J$$

Here, $V_i^j \in v_\infty = (\prod_\alpha v_\alpha)|_{S_\infty}$ and so $V_i^j = C_i^j \cap S_\infty$, where $C_i^j \in \prod_\alpha v_\alpha$. In this case, for $T_{\alpha_i}^j \in v_{\alpha_i}$ $C_i^j = \bigcup\bigcap \pi_{\alpha_i}^{-1}[T_{\alpha_i}^j]$ and so $(V_i^j)^c = \bigcap\bigcup(\pi_{\alpha_i}^{-1}[S_{\alpha_i} \setminus T_{\alpha_i}^j])$. Similarly, $U_j = B_j \cap S_\infty$ where $B_j \in \bigotimes_\alpha u_\alpha$, and so $U_j = (\bigcup\bigcap \pi_{\alpha_i}^{-1}[G_{\alpha_i}^j]) \cap S_\infty$ where $G_{\alpha_i}^j \in u_{\alpha_i}$, by the definition of product topology. Hence, $A_j = (\bigcup\bigcap(\pi_{\alpha_i}|_{S_\infty})^{-1}[G_{\alpha_i}^j]) \cup (\bigcup\bigcap(\pi_{\alpha_i}|_{S_\infty})^{-1}[S_{\alpha_i} \setminus T_{\alpha_i}^j])$ and finally, by the fact that $A = \bigcap_{j\in J} A_j$ we have $A = \bigcap_{j\in J}[(\bigcup\bigcap(\pi_{\alpha_i}|_{S_\infty})^{-1}[G_{\alpha_i}^j]) \cup (\bigcup\bigcap(\pi_{\alpha_i}|_{S_\infty})^{-1}[S_{\alpha_i} \setminus T_{\alpha_i}^j])]$.

On the other hand, if $B \in (\bigotimes_{\alpha} \mathcal{K}_{u_\alpha v_\alpha})|_{S_\infty}$ then $B = M \cap S_\infty$ where $M \in \bigotimes_{\alpha \in I} \mathcal{K}_{u_\alpha v_\alpha}$. In this case, $M = \bigcap \bigcup_{\alpha \in I} \pi_\alpha^{-1}[K_\alpha]$, where $K_\alpha \in \mathcal{K}_{u_\alpha v_\alpha}$. Thus, we have the form $K_\alpha = \bigcap_{j \in J} D_j^\alpha$, where $D_j^\alpha = W_j^\alpha \cup \bigcup_{i \in I_j} \{S_\alpha \setminus (Z_i^j)^\alpha \mid (Z_i^j)^\alpha \in v_\alpha\}$, ${W_j}^\alpha \in u_\alpha$, $j \in J$. Hence $K_\alpha = \bigcap ({U_j}^\alpha \cup (\bigcup S_\alpha \setminus (Z_i^j)^\alpha))$, and so $M = \bigcap (\bigcup (\pi_j^{-1}[\bigcap (W_j^\alpha \cup \bigcup S_\alpha \setminus (Z_i^j)^\alpha])$). In this case, with $B = M \cap S_\infty$ we have $B = \bigcap [\bigcup (((\pi_j|_{S_\infty})^{-1}[\bigcap W_j^\alpha] \cup (\pi_j|_{S_\infty})^{-1}[\bigcap \bigcup (S_\alpha \setminus (Z_i^j)^\alpha)])] = \bigcap [\bigcup \bigcap (\pi_j|_{S_\infty})^{-1}[W_j^\alpha] \cup \bigcap \bigcup (\pi_j|_{S_\infty})^{-1}[S_\alpha \setminus (Z_i^j)^\alpha]]$.

Consequently, it is easy to check that the sets A and B have the same type if consider $G_{\alpha_i}^j$ as W_j^α and $T_{\alpha_i}^j$ as Z_i^j by neglecting the details of indices, as well as by leaving the other details of required equality to the interested reader.

Now, let 's recall the notion of *inverse limit map* introduced in [17, Theorem 4.14] as a notion of peculiar to the texture theory, as well as mentioned in Section 1. Accordingly, in order to prove the next theorem, we need a special property of inverse limits maps, which is proved in the following:

Proposition 2.9. *Consider* $\{h_\alpha\} : \{(S_\alpha, \mathcal{S}_\alpha), \varphi_{\beta\alpha}\}_{\alpha \le \beta} \to \{(T_\alpha, \mathcal{T}_\alpha), \psi_{\beta\alpha}\}_{\alpha \le \beta}$ *and* $\{g_\alpha\} : \{(T_\alpha, \mathcal{T}_\alpha), \psi_{\beta\alpha}\}_{\alpha \le \beta} \to \{(Z_\alpha, \mathcal{Z}_\alpha), \phi_{\beta\alpha}\}_{\alpha \le \beta}$ *between the inverse systems of textures then* $\{g_\alpha \circ h_\alpha\} : \{(S_\alpha, \mathcal{S}_\alpha), \varphi_{\beta\alpha}\}_{\alpha \le \beta} \to \{(Z_\alpha, \mathcal{Z}_\alpha), \phi_{\beta\alpha}\}_{\alpha \le \beta}$ *is also a mapping of inverse system and*

$$\lim_{\leftarrow} \{g_\alpha \circ h_\alpha\}_{\alpha \in \Lambda} = \lim_{\leftarrow} \{g_\alpha\}_{\alpha \in \Lambda} \circ \lim_{\leftarrow} \{h_\alpha\}_{\alpha \in \Lambda}$$

Proof. At first, we define the composition operation for the mappings of inverse systems as follows :

$$\{g_\alpha\} \circ \{h_\alpha\} = \{g_\alpha \circ h_\alpha\}$$

by using the composition operation on the morphisms of **ifPDitop**.

On the other hand, because of the first inverse system, we have the equality $\psi_{\beta\alpha} \circ h_\beta = h_{a\alpha} \circ \varphi_{\beta\alpha}$ by the commutativity of related diagram constructed between the sets $S_\alpha, T_\alpha, S_\infty$ and T_∞. Similarly, from the second inverse system, we have the equality $\phi_{\beta\alpha} \circ g_\beta = g_{a\alpha} \circ \psi_{\beta\alpha}$ by the commutativity of related diagram constructed between the sets $T_\alpha, Z_\alpha, T_\infty$ and Z_∞.

Hence, by considering the above two equalities, we have the result:

$$\phi_{\beta\alpha} \circ (g_\beta \circ h_\beta) = (g_\alpha \circ h_\alpha) \circ \varphi_{\beta\alpha}$$

In fact, it says that $\{g_\alpha \circ h_\alpha\}$ becomes an inverse system of mappings by [17, Definition 3.9].

Therefore, now we can look at the commutativity of diagram. Firstly, recall $\mu_\alpha \circ h_\infty = h_\alpha \circ \lambda_\alpha$ and $\eta_\alpha \circ g_\infty = g_\alpha \circ \mu_\alpha$ by [17, Theorem 4.14]. Thus, due to these equalities, we have

$$\eta_\alpha \circ (g_\infty \circ h_\infty) = (\eta_\alpha \circ g_\infty) \circ h_\infty = (g_\alpha \circ \mu_\alpha) \circ h_\infty = g_\alpha \circ (\mu_\alpha \circ h_\infty) = (g_\alpha \circ h_\alpha) \circ \lambda_\alpha$$

and so the related diagram is commutative. Finally, from the uniqueness of inverse limit maps, mentioned in Theorem 2.5, the required result $\lim_{\leftarrow} \{g_\alpha \circ$

$h_\alpha\}_{\alpha\in\Lambda} = g_\infty \circ h_\infty$ is proved. That is, $\lim\limits_{\leftarrow}\{g_\alpha \circ h_\alpha\}_{\alpha\in\Lambda} = \lim\limits_{\leftarrow}\{g_\alpha\}_{\alpha\in\Lambda} \circ \lim\limits_{\leftarrow}\{h_\alpha\}_{\alpha\in\Lambda}$. □

Remark 2.10. For the remainder of paper, we will use the above final equality under the name *transitivity property* of inverse limit maps.

From Remark 2.4, the inverse systems which are the objects of $\mathbf{Inv_{ifPDitop}}$ have a *unique* inverse limit space as an object of **ifPDitop**. With the reference to this fact, we have the following immediately;

Theorem 2.11. *The limit operation* $\lim\limits_{\leftarrow}$ *of assigning an inverse limit in* **ifPDitop** *to each object in* $\mathbf{Inv_{ifPDitop}}$ *and an inverse limit map* $t_\infty \in \operatorname{Mor}\mathbf{ifPDitop}$ *to each inverse system* $\{t_\alpha\}_\alpha \in \operatorname{Mor}\mathbf{Inv_{ifPDitop}}$ *of maps* $t_\alpha \in \operatorname{Mor}\mathbf{ifPDitop}$*, forms the covariant functor* $\lim\limits_{\leftarrow} : \mathbf{Inv_{ifPDitop}} \to \mathbf{ifPDitop}$.

Proof. Let 's recall that for each inverse system which is an object of $\mathbf{Inv_{ifPDitop_0}}$ we can obtain an inverse limit space in **ifPDitop** and moreover, it is unique by Remark 2.4. Now, according to Theorem 2.5, if take the morphism $\{t_\alpha\}_\alpha : \{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\beta\alpha}\}_{\beta\geq\alpha} \to \{(T_\alpha, \mathcal{T}_\alpha, \tau'_\alpha, \kappa'_\alpha), \psi_{\beta\alpha}\}_{\beta\geq\alpha}$ in $\mathbf{Inv_{ifPDitop}}$ then there exists a unique map $t_\infty = \lim\limits_{\leftarrow}\{t_\alpha\}_{\alpha\in\Lambda} \in \operatorname{Mor}\mathbf{ifPDitop}$ between the corresponding inverse limit spaces $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)$ and $(T_\infty, \mathcal{T}_\infty, \tau'_\infty, \kappa'_\infty)$ which are the objects of **ifPDitop**, having the property that for each $\alpha \in \Lambda$ the diagram

$$\begin{array}{ccc} S_\infty & \xrightarrow{t_\infty} & T_\infty \\ \downarrow{\scriptstyle \mu_\alpha} & & \downarrow{\scriptstyle \eta_\alpha} \\ S_\alpha & \xrightarrow{t_\alpha} & T_\alpha \end{array}$$

is commutative, that is $t_\alpha \circ \mu_\alpha = \eta_\alpha \circ t_\infty$. Also, t_∞ is the identity $\mathrm{id}_{(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)}$ if suppose that the mapping $\{t_\alpha\}_\alpha$ of inverse systems is identity, that is each map $t_\alpha : S_\alpha \to T_\alpha$, $\alpha \in \Lambda$ is the identity $\mathrm{id}_{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha)}$ on S_α. Additionally, as it is stated in Proposition 2.9, the inverse limit maps have the transitivity property and so the limit operation $\lim\limits_{\leftarrow}$ satisfies the composition rule $\lim\limits_{\leftarrow}\{t_\alpha \circ h_\alpha\} = \lim\limits_{\leftarrow}\{t_\alpha\} \circ \lim\limits_{\leftarrow}\{h_\alpha\}$. Hence, the mapping $\lim\limits_{\leftarrow} : \mathbf{Inv_{ifPDitop}} \to \mathbf{ifPDitop}$ is a covariant functor. □

Notation: The covariant functor $\lim\limits_{\leftarrow}$ described in Theorem 2.11, as the limit operation in the context of **ifPDitop**, will be used under the notation $\mathfrak{C}$ for the remainder of paper.

Actually, note that we can always define a covariant functor between the categories $\mathcal{C}$ and $\mathbf{Inv}_{\mathcal{C}}$, for any category $\mathcal{C}$ which has the equalizers and products.

Remark 2.12.

(1) By virtue of the fact that any inverse system consisting of the objects of **Bitop** has an inverse limit since **Bitop** has equalizers and products, we can describe covariant functor, under the name $\mathfrak{B}$ between the categories **Bitop** and $\mathbf{Inv_{Bitop}}$.

(2) The above functor $\mathfrak{B}$ introduced in (1) may be considered as the restricted mapping between the full subcategory $\mathbf{Bitop_{w0}}$ of **Bitop** and the full subcategory $\mathbf{Inv_{Bitop_{w0}}}$ of $\mathbf{Inv_{Bitop}}$. Obviously, that restriction is a covariant functor, as well.

(3) Furthermore, if we recall that the categories $\mathbf{Bitop_{w0}}$ and $\mathbf{ifPDitop}_0$ are isomorphic via the functor $\mathfrak{H}$ constructed by using the fact that weakly pairwise T_0 bitopology generates the smallest plain texturing and T_0 ditopology, as mentioned in Theorem 2.2, then we may describe a functor between the categories $\mathbf{Inv_{Bitop_{w0}}}$ and $\mathbf{Inv_{ifPDitop_0}}$ in a natural way.

According to the statement (3), we are now in a position to give a next isomorphism functor as follows:

Theorem 2.13. *The categories* $\mathbf{Inv_{Bitop_{\mathit{w0}}}}$ *and* $\mathbf{Inv_{ifPDitop_0}}$ *are concretely isomorphic.*

Proof. First of all, if consider the isomorphism functor $\mathfrak{H}$ given in Theorem 2.2, between the categories $\mathbf{Bitop_{w0}}$ and $\mathbf{ifPDitop}_0$, clearly the mapping $\mathfrak{X} : \mathbf{Inv_{Bitop_{w0}}} \to \mathbf{Inv_{ifPDitop_0}}$ may be defined by using $\mathfrak{H}$:

Taking into account the ideas given in Example 2.7, then we may define the map $\mathfrak{X}(\{(S_\alpha, u_\alpha, v_\alpha), f_{\alpha\beta}\}_{\alpha\geq\beta}) = \{(S_\alpha, \mathcal{K}_{u_\alpha v_\alpha}, u_\alpha, v_\alpha^c), f_{\alpha\beta}\}_{\alpha\geq\beta}$ where $\mathfrak{H}(S_\alpha, u_\alpha, v_\alpha) = (S_\alpha, \mathcal{K}_{u_\alpha v_\alpha}, u_\alpha, v_\alpha^c)$, $\mathfrak{H}(f_{\alpha\beta}) = f_{\alpha\beta}$, and if take the inverse system $\{t_\alpha\}$ of mappings as the morphism between two inverse systems which are objects of $\mathbf{Inv_{Bitop_{\mathit{w0}}}}$ then it is easy to show that it is also a morphism in $\mathbf{Inv_{Bitop}}$. Indeed, if take $t_\alpha \in \mathrm{Mor}\,\mathbf{Bitop}$, for each α, that is, t_α is pairwise continuous then it is w-preserving and bicontinuous between the corresponding ditopological plain spaces and finally, the equality $\mathfrak{X}(\{t_\alpha\}) = \{t_\alpha\}$ is meaningful, as well. In this case, for the inverse system mappings, the equality $\mathfrak{X}(\{t_\alpha\} \circ \{h_\alpha\}) = \mathfrak{X}(\{t_\alpha \circ h_\alpha\}) = \{t_\alpha \circ h_\alpha\} = \{t_\alpha\} \circ \{h_\alpha\}$ is trivial. Also, from $\mathfrak{X}(\mathrm{id}_{\{(S_\alpha,u_\alpha,v_\alpha),f_{\alpha\beta}\}_{\alpha\geq\beta}}) = \mathrm{id}_{\mathfrak{X}(\{(S_\alpha,u_\alpha,v_\alpha),f_{\alpha\beta}\}_{\alpha\geq\beta})}$, the map $\mathfrak{X}$ describes a functor, naturally.

Now we will turn our attention to the isomorphism conditions for $\mathfrak{X}$. It is easy to show that $\mathfrak{X}$ is full and faithful, since it is bijective between hom-set restrictions by the fact that the functor $\mathfrak{H}$ given in Theorem 2.2 is full and faithful.

As the final step, it remains to prove that the bijectivity of $\mathfrak{X}$ on objects of $\mathbf{Inv_{Bitop_{w0}}}$ and $\mathbf{Inv_{ifPDitop_0}}$, and it is clear from the bijectivity of the functor $\mathfrak{H}$. □

In the light of considerations presented in Remark 2.12 and Theorem 2.13, now we can start to construct a major part in that theory, consisting of the useful implications and an identity natural transformation which arises from those implications:

A Natural Transformation in the Context of Inverse Systems and Limits Located Inside the Categories $\mathbf{Bitop_{w0}}$ and $\mathbf{ifPDitop_0}$:

As we promised in Section 1, firstly a natural transformation will be described between the corresponding functors, and later, that the natural transformation is identity will be proved, thoroughly.

Let 's start by recalling the corresponding required functors as follows:

$$\mathbf{Inv_{Bitop_{w0}}} \xrightarrow{\mathfrak{B}} \mathbf{Bitop_{w0}} \xrightarrow{\mathfrak{H}} \mathbf{ifPDitop_0}$$

$$\{(X_\alpha, u_\alpha, v_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \mapsto (X_\infty, u_\infty, v_\infty) \mapsto (X_\infty, \mathcal{K}_{u_\infty v_\infty}, u_\infty, (v_\infty)^c)$$

$$\mathbf{Inv_{Bitop_{w0}}} \xrightarrow{\mathfrak{X}} \mathbf{Inv_{ifPDitop_0}} \xrightarrow{\mathfrak{E}} \mathbf{ifPDitop_0}$$

$$\{(X_\alpha, u_\alpha, v_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \mapsto \{(X_\alpha, \mathcal{K}_{u_\alpha v_\alpha}, u_\alpha, (v_\alpha)^c), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \mapsto (X_\infty, \mathcal{Z}, \mathcal{T}, \mathcal{K})$$

where $\mathcal{Z} = (\bigotimes_\alpha \mathcal{K}_{u_\alpha v_\alpha})|_{X_\infty}$, $\mathcal{T} = (\bigotimes_\alpha u_\alpha)|_{X_\infty}$ and $\mathcal{K} = (\bigotimes_\alpha v_\alpha^c)|_{X_\infty}$

Now, with the previous considerations, if take the equalities

$$F = \mathfrak{H} \circ \mathfrak{B} : \mathbf{Inv_{Bitop_{w0}}} \to \mathbf{ifPDitop_0}$$

$$G = \mathfrak{E} \circ \mathfrak{X} : \mathbf{Inv_{Bitop_{w0}}} \to \mathbf{ifPDitop_0}$$

then it is clear that F and G are functors as compositions of the functors $\mathfrak{H}, \mathfrak{B}$ and $\mathfrak{E}, \mathfrak{X}$, respectively.

Consider a mapping $\tau : F \to G$. In particular;

Theorem 2.14. *τ is an identity natural transformation between the functors F and G.*

Proof. Let the inverse system $\mathcal{A} = \{(X_\alpha, u_\alpha, v_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{Bitop_{w0}}}$ over Λ and the mapping $\tau_A : F\mathcal{A} \to G\mathcal{A}$. Firstly, it is easy to verify that $F\mathcal{A} = G\mathcal{A}$ by the considerations mentioned in Example 2.8 and thus, the mapping τ_A is an $\mathbf{ifPDitop_0}$-identity morphism.

On the other hand, for the inverse system $\mathcal{A}' = \{(X'_\alpha, u'_\alpha, v'_\alpha), \varphi'_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{Bitop_{w0}}}$ over Λ, take the inverse system $\{k_\alpha\} : \mathcal{A} \to \mathcal{A}' \in \mathrm{Mor}\,\mathbf{Inv_{Bitop_{w0}}}$ of mappings $k_\alpha : X_\alpha \to X'_\alpha$, $\alpha \in \Lambda$, as in described in Theorem 2.5. Also, assume that $\lim\limits_{\leftarrow} \mathcal{A} = \lim\limits_{\leftarrow}\{X_\alpha\}_{\alpha\in\Lambda} = X_\infty$ and $\lim\limits_{\leftarrow} \mathcal{A}' = \lim\limits_{\leftarrow}\{X'_\alpha\}_{\alpha\in\Lambda} = X'_\infty$.

Let $\xi : \{(X_\alpha, u_\alpha, v_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \to \{(X'_\alpha, u'_\alpha, v'_\alpha), \varphi'_{\alpha\beta}\}_{\alpha\geq\beta}$ be the mapping $\{k_\alpha\}_{\alpha\in\Lambda}$ of inverse systems, with the components $k_\alpha : X_\alpha \to X'_\alpha \in \mathrm{Mor}\,\mathbf{Bitop_{w0}}$, $\alpha \in \Lambda$.

With all the above notations, now we may construct the following diagram:

$$\begin{array}{ccc}
(X'_\infty, \mathcal{K}_{u'_\infty v'_\infty}, u'_\infty, (v'_\infty)^c) & \xrightarrow{\tau_{A'}} & (X'_\infty, (\bigotimes_\alpha \mathcal{K}_{u'_\alpha v'_\alpha})|_{X'_\infty}, (\bigotimes_\alpha u'_\alpha)|_{X'_\infty}, (\bigotimes_\alpha (v'_\alpha)^c)|_{X'_\infty}))) \\
\uparrow F\xi & & \uparrow G\xi \\
(X_\infty, \mathcal{K}_{u_\infty v_\infty}, u_\infty, (v_\infty)^c) & \xrightarrow[\tau_A]{} & (X_\infty, (\bigotimes_\alpha \mathcal{K}_{u_\alpha v_\alpha})|_{X_\infty}, (\bigotimes_\alpha u_\alpha)|_{X_\infty}, (\bigotimes_\alpha (v_\alpha)^c)|_{X_\infty}
\end{array}$$

In order to see that this diagram is commutative, we need to show the equality $F\xi = G\xi$ for all $\xi \in \operatorname{Mor} \mathbf{Inv}_{\mathbf{Bitop_{w0}}}$:

Clearly, each $k_\alpha : X_\alpha \to X'_\alpha$ is pairwise continuous and by $F = \mathfrak{H} \circ \mathfrak{B}$ we have $F(\{k_\alpha\}_{\alpha\in\Lambda}) = \mathfrak{H}(\mathfrak{B}\{k_\alpha\}_{\alpha\in\Lambda}) = \mathfrak{H}(k_\infty)$ where $k_\infty = \lim\limits_{\leftarrow}\{k_\alpha\}_{\alpha\in\Lambda} \in \operatorname{Mor} \mathbf{Bitop_{w0}}$ and by applying the isomorphism $\mathfrak{H} : \mathbf{Bitop_{w0}} \to \mathbf{ifPDitop_0}$ to the limit map $k_\infty \in \operatorname{Mor} \mathbf{Inv}_{\mathbf{Bitop_{w0}}}$, we obtained $\mathfrak{H}(k_\infty) = k_\infty$ since $\mathfrak{H}$ is identity on morphisms. Finally, $F\xi = F(\{k_\alpha\}_{\alpha\in\Lambda}) = k_\infty$.

On the other hand, now let 's turn our attention to $G(\xi)$ and recall the equality $G = \mathfrak{E}\circ\mathfrak{X}$. According to that, we have $G(\{k_\alpha\}_{\alpha\in\Lambda}) = \mathfrak{E}(\mathfrak{X}\{k_\alpha\}_{\alpha\in\Lambda}) = \mathfrak{E}(\{k_\alpha\}_{\alpha\in\Lambda})$ since the isomorphism $\mathfrak{X}$ described in Theorem 2.13 is the identity on morphisms of $\mathbf{Inv}_{\mathbf{Bitop_{w0}}}$ and $\mathbf{Inv}_{\mathbf{ifPDitop_0}}$. Hence, by applying the functor $\mathfrak{E} : \mathbf{Inv}_{\mathbf{ifPDitop_0}} \to \mathbf{ifPDitop_0}$ to the mapping $\{k_\alpha\}_{\alpha\in\Lambda}$, we describe the map $\mathfrak{E}(\{k_\alpha\}) = h_\infty$, where $h_\infty = \lim\limits_{\leftarrow}\{k_\alpha\}_{\alpha\in\Lambda} \in \operatorname{Mor} \mathbf{Bitop_{w0}}$. Hence $G\xi = G(\{k_\alpha\}_{\alpha\in\Lambda}) = h_\infty$.

Now, let 's see that $t_\infty = h_\infty$: the inverse systems considered above are exactly same since the spaces and bonding maps are the same. Also, the property of commutativity $\eta_\alpha \circ t_\infty = t_\alpha \circ \mu_\alpha$, $\alpha \in \Lambda$ is satisfied for the map h_∞, as well. In this case, by virtue of the fact that the inverse limit of the mappings of inverse systems is unique by [17, Theorem 4.14], we have $F\xi = G\xi$. Thus, the equality $F = G$ is verified and τ is identity. Moreover, we have $G\xi\circ\tau_A = \tau_{A'}\circ F\xi$ since $\tau_A, \tau_{A'}$ are identities and so, the diagram is commutative. □

As a result of the above considerations, $\tau : F \to G$ is the identity natural transformation.

In a similar way to the considerations given in Section 2, next section will discuss the relations between the topological inverse systems - limits and ditopological inverse systems - limits insofar as the theory of plain textures are concerned.

3. Relationships between the inverse systems-limits in the categories of topological and ditopological spaces

Now we will show that we may associate with the ditopology (τ, κ) on a plain texture $(S, \mathcal{S})$ a topology $\mathcal{J}_{\tau\kappa}$ on S, by adapting the notion of *appropriate joint topology for a ditopology* described in [11], to the plain case:

Definition 3.1. Let $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{ifPDitop}$. We define the joint topology on S in terms of its family $\mathcal{J}^c_{\tau\kappa}$ of closed sets by the condition

$$W \in \mathcal{J}^c_{\tau\kappa} \iff (s \in S,\ G \in \eta(s), K \in \mu(s) \implies G \cap W \nsubseteq K) \implies s \in W.$$

Here $\eta(s) = \{N \in \mathcal{S} \mid \quad P_s \subseteq G \subseteq N \nsubseteq Q_s \text{ for some } G \in \tau\}$ and $\mu(s) = \{M \in \mathcal{S} \mid \ P_s \nsubseteq M \subseteq K \subseteq Q_s \text{ for some } K \in \kappa\}$. For the details about filter $\eta(s)$ and cofilter $\mu(s)$ for $s \in S$, see [8, 11, 16].

The verification of that $\mathcal{J}^c_{\tau\kappa}$ satisfies the closed-set axioms is straightforward and on passing to the complement this reveals that

(i) $\{G \subseteq S \mid G \in \tau\} \cup \{S \setminus K \subseteq S \mid K \in \kappa\}$ is a subbase, and
(ii) $\{G \cap (S \setminus K) \subseteq S \mid G \in \tau, K \in \kappa\}$ a base

of open sets for the topology $\mathcal{J}_{\tau\kappa}$ on S.

In case (X, u, v) is an object of **Bitop**, we have the space $(X, \mathcal{P}(X), u, v^c) \in \mathrm{Ob}\,\mathbf{ifPDitop}$, and clearly obtain $\mathcal{J}_{\tau\kappa} = u \vee v$ as the joint topology of (u, v), where $\tau = u$ and $\kappa = v^c$. Hence we will refer to $\mathcal{J}_{\tau\kappa}$ as the *joint topology of* (τ, κ) *on* S.

Remark 3.2. (1) For $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{ifPDitop}$, it is trivial to see that $\kappa \subseteq \mathcal{J}^c_{\tau\kappa}$ and $\tau \subseteq \mathcal{J}_{\tau\kappa}$. In addition, the family $\tau \cup \kappa^c$ is the subbase for the joint topology $\mathcal{J}_{\tau\kappa}$.

(2) From now on, in this work we will use the terms *jointly closed (open, dense)* for the set which is closed (open, dense) with respect to the appropriate joint topology of the ditopology on space.

Note that the following statements are adapted forms of general cases given in [11] to the category **ifPDitop**. Here **Top** will denote the category of topological spaces and continuous functions.

Theorem 3.3. *The mapping* $\mathfrak{J} : \mathbf{ifPDitop} \to \mathbf{Top}$ *defined by*

$$\mathfrak{J} : ((S, \mathcal{S}, \tau_S, \kappa_S) \xrightarrow{\varphi} (T, \mathcal{T}, \tau_T, \kappa_T)) = (S, \mathcal{J}_{\tau_S\kappa_S}) \xrightarrow{\varphi} (T, \mathcal{J}_{\tau_T\kappa_T})$$

is an adjoint functor.

It is clear that $\mathfrak{J}$ is full, faithful and isomorphism-dense functor although it is not a functor isomorphism since it is not one-to-one on the objects.

Corollary 3.4. *The functor* $\mathfrak{T} : \mathbf{Top} \to \mathbf{ifPDitop}$ *given by*

$$\mathfrak{T}(X, \mathcal{T}) = (X, \mathcal{P}(X), \mathcal{T}, \mathcal{T}^c),\ \mathfrak{T}(\varphi) = \varphi$$

is the co-adjoint of $\mathfrak{J}$.

Here note also that $\mathfrak{T}$ is not a functor isomorphism.

In this section, we will be interested in the category $\mathbf{Inv_{Top}}$ whose objects are the inverse systems constructed by the objects of **Top** and morphisms are the inverse systems constructed by the morphisms of **Top**, as well as the mappings between the inverse systems constructed in **Top**. Naturally, a covariant functor may be established between the categories **Top** and $\mathbf{Inv_{Top}}$ since any inverse system constructed in **Top** has an inverse limit by the fact that **Top** has equalizers and products as mentioned in [5].

Obviously, we can't expect to find an isomorphism between the categories $\mathbf{Inv_{Top}}$ and $\mathbf{Inv_{ifPDitop}}$ and now, we may turn our attention to the relationships between the objects of categories $\mathbf{Inv_{Top}}$ and $\mathbf{Inv_{ifPDitop}}$:

It is known that an object of $\mathbf{Inv_{ifPDitop}}$ can be obtained as the natural counterpart of an object of $\mathbf{Inv_{Top}}$ by [18, Example 3.4]. Thus, by applying the similar considerations to Corollary 3.4 we can describe a co-adjoint functor from $\mathbf{Inv_{Top}}$ to $\mathbf{Inv_{ifPDitop}}$.

Conversely, in order to construct an opposite functor from $\mathbf{Inv_{ifPDitop}}$ to $\mathbf{Inv_{Top}}$, let's consider the reciprocal objects, and the adjoint functor $\mathfrak{J}$ firstly. That is, take $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{ifPDitop}}$, and construct the image $\mathfrak{J}(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha) = (S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}) \in \mathrm{Ob}\,\mathbf{Top}$. In this case, for the bonding map $\varphi_{\alpha\beta} : S_\alpha \to S_\beta \in \mathrm{Mor}\,\mathbf{ifPDitop}$, we have $\mathfrak{J}(\varphi_{\alpha\beta}) = \varphi_{\alpha\beta} : (S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}) \to (S_\beta, \mathcal{J}_{\tau_\beta\kappa_\beta})$ as a morphism of **Top** since $\mathfrak{J}$ is a functor. In fact, $\mathfrak{J}$ is the identity on morphisms. Hence, we construct the inverse system $\{(S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}), \varphi_{\beta\alpha}\}_{\beta\geq\alpha} \in \mathrm{Ob}\,\mathbf{Inv_{Top}}$ and so a mapping which is described as follows :

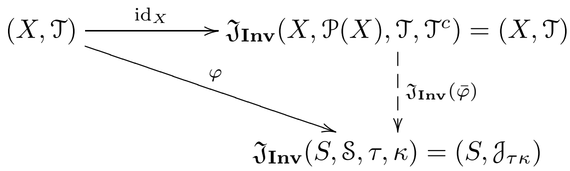
Theorem 3.5. *The mapping* $\mathfrak{J}_{\mathbf{Inv}} : \mathbf{Inv_{ifPDitop}} \to \mathbf{Inv_{Top}}$ *defined by*

$$\mathfrak{J}_{\mathbf{Inv}} : (\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \xrightarrow{\{t_\alpha\}} \{(T_\alpha, \mathcal{T}_\alpha, \tau'_\alpha, \kappa'_\alpha), \psi_{\alpha\beta}\}_{\alpha\geq\beta}) =$$

$$\{(S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \xrightarrow{\{t_\alpha\}} \{(T_\alpha, \mathcal{J}_{\tau'_\alpha\kappa'_\alpha}), \psi_{\alpha\beta}\}_{\alpha\geq\beta} \text{ is an adjoint functor.}$$

Proof. Firstly, we need to check that $\mathfrak{J}_{\mathbf{Inv}}$ is a functor. Assume that $\{t_\alpha\}_\alpha \in \mathrm{Mor}\,\mathbf{Inv_{ifPDitop}}$. In this case, the maps $t_\alpha : S_\alpha \to T_\alpha$ for each α, are bicontinuous and w-preserving as the morphisms in **ifPDitop**. By the definition of joint topology, it is easy to show that t_α is continuous for each α, as the morphism of **Top** between the joint topological spaces $(S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha})$ and $(T_\alpha, \mathcal{J}'_{\tau'_\alpha\kappa'_\alpha})$.

To show $\mathfrak{J}_{\mathbf{Inv}}$ is an adjoint, now take $\{(X_\alpha, \mathcal{T}_\alpha), \phi_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{Top}}$. Then $(\{\mathrm{id}_{X_\alpha}\}, \{(X_\alpha, \mathcal{P}(X_\alpha), \mathcal{T}_\alpha, \mathcal{T}^c_\alpha), \phi_{\alpha\beta}\}_{\alpha\geq\beta})$ is a $\mathfrak{J}_{\mathbf{Inv}}$-structured arrow by $\mathfrak{J}_{\mathbf{Inv}}(\{(X_\alpha, \mathcal{P}(X_\alpha), \mathcal{T}_\alpha, \mathcal{T}^c_\alpha), \phi_{\alpha\beta}\}_{\alpha\geq\beta}) = \{(X_\alpha, \mathcal{J}_{\mathcal{T}_\alpha\mathcal{T}^c_\alpha}), \phi_{\alpha\beta}\}_{\alpha\geq\beta})$ and by the fact that $\{\mathrm{id}_{X_\alpha}\} : \{(X_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}), \phi_{\alpha\beta}\}_{\alpha\geq\beta} \to \{(X_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}), \phi_{\alpha\beta}\}_{\alpha\geq\beta}$ is an $\mathbf{Inv_{Top}}$-morphism. To show $(\{\mathrm{id}_{X_\alpha}\}, \{(X_\alpha, \mathcal{P}(X_\alpha), \mathcal{T}_\alpha, \mathcal{T}^c_\alpha), \phi_{\alpha\beta}\}_{\alpha\geq\beta})$ has the universal property, take $\{(S_\alpha, \mathcal{S}_\alpha, \tau^*_\alpha, \kappa^*_\alpha), \theta_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{ifPDitop}}$ and let $\{\varphi_\alpha\} : \{(X_\alpha, \mathcal{T}_\alpha), \phi_{\alpha\beta}\}_{\alpha\geq\beta} \to \mathfrak{J}_{\mathbf{Inv}}\{(S_\alpha, \mathcal{S}_\alpha, \tau^*_\alpha, \kappa^*_\alpha), \theta_{\alpha\beta}\}_{\alpha\geq\beta} = \{(S_\alpha, \mathcal{J}_{\tau^*_\alpha\kappa^*_\alpha}), \theta_{\alpha\beta}\}_{\alpha\geq\beta}$ be an $\mathbf{Inv_{Top}}$-morphism.



Since φ maps into S the only point function $\bar{\varphi} : X \to S$ making the above diagram commutative is φ, so it remains only to verify that $\varphi : (X, \mathcal{P}(X), \mathcal{T}, \mathcal{T}^c) \to$

$(S, \mathcal{S}, \tau, \kappa)$ is a morphism in **ifPDitop**. Certainly φ is ω-preserving, due to the fact that $\varphi(x)\,\omega_S\,\varphi(x)$ for all $x \in X$. Moreover, φ is bicontinuous. To see this, note that we have $\varphi^{\leftarrow}A = \varphi^{-1}A = \varphi^{-1}(A \cap S_p)$ for all $A \in \mathcal{S}$. Hence, $G \in \tau \implies G \cap S_p \in \mathcal{J}_{\tau\kappa} \implies \varphi^{\leftarrow}G = \varphi^{-1}(G \cap S_p) \in \mathfrak{T}$, and $K \in \kappa \implies \varphi^{\leftarrow}K \in \mathfrak{T}^c$ likewise. □

Particularly, by virtue of Theorem 3.3 and Theorem 3.5 we have the following:

Remark 3.6. Let 's take an inverse system $\mathcal{A} = \{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \operatorname{Ob}\mathbf{Inv}_{\mathbf{ifPDitop}}$. In this case, we construct the system $\mathfrak{J}_{\mathbf{Inv}}(\mathcal{A}) = \{(S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \operatorname{Ob}\mathbf{Inv}_{\mathbf{Top}}$, and have the inverse limit space $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty) \in \operatorname{Ob}\mathbf{ifPDitop}$ by Theorem 2.11. Thus, $\mathfrak{J}(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty) = (S_\infty, \mathcal{J}_{\tau_\infty\kappa_\infty}) \in \operatorname{Ob}\mathbf{Top}$.

In addition, we have an inverse limit $\lim\limits_{\leftarrow}\mathfrak{J}_{\mathbf{Inv}}(\mathcal{A}) = (Y, \mathcal{V}) \in \operatorname{Ob}\mathbf{Top}$ due to the fact that **Top** has equalizers and products as mentioned in [5]. Now let 's turn our attention to the main question; Is the space $(Y, \mathcal{V})$ same with the space $(S_\infty, \mathcal{J}_{\tau_\infty\kappa_\infty})$ in **Top** ?

Firstly note that the systems $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta}$ and $\{(S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha}), \varphi_{\alpha\beta}\}_{\alpha\geq\beta}$ have the same bonding maps, so $Y = S_\infty$ as a subset of $\prod\limits_\alpha S_\alpha$, trivially.

As a next step, in order to prove the equality $\mathcal{J}_{\tau_\infty\kappa_\infty} = \mathcal{V}$, note that the facts $(\prod\limits_\alpha \mathcal{J}_{\tau_\alpha\kappa_\alpha})|_{S_\infty} = \mathcal{V}$, $(\bigotimes\limits_\alpha \mathcal{S}_\alpha)|_{S_\infty} = \mathcal{S}_\infty$, $(\bigotimes\limits_\alpha \tau_\alpha)|_{S_\infty} = \tau_\infty$ and $(\bigotimes\limits_\alpha \kappa_\alpha)|_{S_\infty} = \kappa_\infty$. Accordingly, let $A \in \mathcal{J}_{\tau_\infty\kappa_\infty}$, so $A = \bigcup\limits_\delta (G_\delta \cap (S_\infty \setminus K_\delta))$ where $G_\delta \in (\bigotimes\limits_\alpha \tau_\alpha)|_{S_\infty}$, $K_\delta \in (\bigotimes\limits_\alpha \kappa_\alpha)|_{S_\infty}$ for each δ. Note here that $G_\delta = (\bigcup(\bigcap\limits_{\text{finite}} \pi_{\alpha_i}^{-1}[G_{\alpha_i}^\delta])) \cap S_\infty$ and similarly, $K_\delta = (\bigcap(\bigcup\limits_{\text{finite}} \pi_{\alpha_i}^{-1}[K_{\alpha_i}^\delta])) \cap S_\infty$, where $G_{\alpha_i}^\delta \in \tau_{\alpha_i}$ and $K_{\alpha_i}^\delta \in \kappa_{\alpha_i}$. Hence, $A = (\bigcup \bigcap\limits_{\text{finite}} \pi_{\alpha_i}^{-1}[\bigcup G_{\alpha_i}^\delta]) \cap (\bigcup \bigcap\limits_{\text{finite}} \pi_{\alpha_i}^{-1}[\bigcup (S_{\alpha_i} \setminus K_{\alpha_i}^\delta)]) \cap S_\infty$.

On the other hand, let the set $B \in (\prod\limits_\alpha \mathcal{J}_{\tau_\alpha\kappa_\alpha})|_{S_\infty} = \mathcal{V}$, where $\mathcal{V}$ denotes the product topology on S_∞. In this case, B can be written as $(\bigcup \bigcap\limits_{\text{finite}} \pi_{\alpha_j}^{-1}[G_{\alpha_j}]) \cap S_\infty$, $G_{\alpha_j} \in \mathcal{J}_{\tau_\alpha\kappa_\alpha}$. Thus, $B = (\bigcup \bigcap\limits_{\text{finite}} \pi_{\alpha_j}^{-1}[\bigcup (C_{\alpha_j} \cap (S_{\alpha_j} \setminus D_{\alpha_j}))]) \cap S_\infty$ where $C_{\alpha_j} \in \tau_{\alpha_j}$, $D_{\alpha_j} \in \kappa_{\alpha_j}$ and so, $B = (\bigcup \bigcap\limits_{\text{finite}} \pi_{\alpha_j}^{-1}[\bigcup C_{\alpha_j}]) \cap (\bigcup \bigcap\limits_{\text{finite}} \pi_{\alpha_j}^{-1}[\bigcup (S_{\alpha_j} \setminus D_{\alpha_j})]) \cap S_\infty$. Consequently, it is easy to check that the types of sets A and B are the same. It means that the topologies $\mathcal{V}$ and $\mathcal{J}_{\tau_\infty\kappa_\infty}$ coincides.

4. Effect of the closure operator on inverse systems and limits in the category **ifPDitop**

By recalling the notion of *appropriate joint topology* described for a ditopology, as presented in the previous section, we have the following significant theorem, immediately:

Theorem 4.1. *Let Λ be a directed set. For subspace $(U, \mathcal{S}_U, \tau_U, \kappa_U) \in \operatorname{Ob}\mathbf{ifPDitop}$ of the inverse limit space $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty) \in \operatorname{Ob}\mathbf{ifPDitop}$ of the inverse system $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \operatorname{Ob}\mathbf{Inv_{ifPDitop}}$, the families $\{(U_\alpha, \mathcal{S}_\alpha|_{U_\alpha}, \tau_\alpha|_{U_\alpha}, \kappa_\alpha|_{U_\alpha}), \varphi_{\alpha\beta}|_{U_\alpha}\}_{\alpha\geq\beta}$ and $\{(\overline{U}_\alpha, \mathcal{S}_\alpha|_{\overline{U}_\alpha}, \tau_\alpha|_{\overline{U}_\alpha}, \kappa_\alpha|_{\overline{U}_\alpha}), \overline{\varphi}_{\alpha\beta}\}_{\alpha\geq\beta}$ describe two objects in $\mathbf{Inv_{ifPDitop}}$ as the inverse systems, where $U_\alpha = \mu_\alpha(U) = \pi_\alpha|_{S_\infty}(U)$, $\overline{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta}|_{\overline{U}_\alpha}$ and $\overline{U}_\alpha$ denotes the closure in S_α of the subset $U_\alpha \subseteq S_\alpha$ with respect to the joint topology of the ditopology $(\tau_\alpha, \kappa_\alpha)$, $\alpha \in \Lambda$.*

Proof. Firstly, let us prove that $\{(\overline{U}_\alpha, \mathcal{S}_\alpha|_{\overline{U}_\alpha}, \tau_\alpha|_{\overline{U}_\alpha}, \kappa_\alpha|_{\overline{U}_\alpha}), \overline{\varphi}_{\alpha\beta}\}_{\alpha\geq\beta}$ is an object of $\mathbf{Inv_{ifPDitop}}$: Note that we have $\mu_\beta(s) = \overline{\varphi}_{\alpha\beta}(\mu_\alpha(s))$ for $s \in U$ and $\beta \leq \alpha$. Indeed, if $s \in U$ then $\mu_\alpha(s) \in \mu_\alpha(U)$ and so $\mu_\alpha(s) \in \overline{U}_\alpha$. In this case, $\overline{\varphi}_{\alpha\beta}(\mu_\alpha(s)) = \varphi_{\alpha\beta}(\mu_\alpha(s))$. Also the equality $\varphi_{\alpha\beta}(\mu_\alpha(s)) = \mu_\beta(s)$ for $\alpha \geq \beta$ is known by [17, Lemma 4.3], thus we have $\overline{\varphi}_{\alpha\beta}(\mu_\alpha(s)) = \mu_\beta(s)$ for $s \in U$ and $\alpha \geq \beta$, as required.

On the other hand, with the continuity of bonding map $\overline{\varphi}_{\alpha\beta}$ we have $\overline{\varphi}_{\alpha\beta}(\overline{U}_\alpha) = \overline{\varphi}_{\alpha\beta}(\overline{\mu_\alpha(U)}) \subseteq \overline{\overline{\varphi}_{\alpha\beta}(\mu_\alpha(U))} = \overline{\mu_\beta(U)} = \overline{U}_\beta$ and then, it is clear that the point function $\overline{\varphi}_{\alpha\beta}$ is defined from $\overline{U}_\alpha$ onto $\overline{U}_\beta$. Following that, $\overline{\varphi}_{\alpha\beta}$ is a morphism of **ifPDitop** since it is a restriction of $\varphi_{\alpha\beta} \in \operatorname{Mor}\mathbf{ifPDitop}$ to the subset $\overline{U}_\alpha \subseteq S_\alpha$.

Incidentally, the equality $\overline{\varphi}_{\beta\gamma} \circ \overline{\varphi}_{\alpha\beta} = \overline{\varphi}_{\alpha\gamma}$ may be easily proved for the elements of $\overline{U}_\alpha$ via the equality $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$.

As a next step, we have the equality $\overline{\varphi}_{\alpha\alpha}(s) = \varphi_{\alpha\alpha}(s) = s$ for $s \in \overline{U}_\alpha$, as $\varphi_{\alpha\alpha}$ is the identity id_{S_α} on S_α. That is, $\overline{\varphi}_{\alpha\alpha} = id_{\overline{U}_\alpha} = id_{S_\alpha}|_{\overline{U}_\alpha}$.

Consequently, the family $\{(\overline{U}_\alpha, \mathcal{S}_\alpha|_{\overline{U}_\alpha}, \tau_\alpha|_{\overline{U}_\alpha}, \kappa_\alpha|_{\overline{U}_\alpha}), \overline{\varphi}_{\alpha\beta}\}_{\alpha\geq\beta}$ forms an object in $\mathbf{Inv_{ifPDitop}}$.

Furthermore, in a similar way to the above proof, it is easy to check that the family $\{(U_\alpha, \mathcal{S}_\alpha|_{U_\alpha}, \tau_\alpha|_{U_\alpha}, \kappa_\alpha|_{U_\alpha}), \varphi_{\alpha\beta}|_{U_\alpha}\}_{\alpha\geq\beta}$ describes an inverse system in **ifPDitop**, and so an object in $\mathbf{Inv_{ifPDitop}}$. □

According to Remark 2.4, we have the following, right away.

Proposition 4.2. $U_\infty = \varprojlim \{U_\alpha\} \subseteq \varprojlim \{S_\alpha\} = S_\infty$

Proof. Conversely, assume that $U_\infty = \varprojlim \{U_\alpha\} \nsubseteq \varprojlim \{S_\alpha\} = S_\infty$, so there exists $s = \{s_\alpha\} \in \prod_{\alpha\in\Lambda} S_\alpha$ such that $U_\infty \nsubseteq Q_s$ and $P_s \nsubseteq S_\infty$. In this case, $s \in \prod_{\alpha\in\Lambda} U_\alpha$ and $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = s_\beta$ for every $s_\alpha \in U_\alpha$, $\alpha, \beta \in \Lambda$ such that $\alpha \geq \beta$. Moreover, we have the equality $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in U_\alpha$. Thus, because of the facts $s_\alpha \in S_\alpha$, $\alpha \in \Lambda$ and $\varphi_{\alpha\beta}(s_\alpha) = s_\beta$ for $\alpha \geq \beta$, the point $s = \{s_\alpha\}$ becomes an element of S_∞, obviously and this gives a contradiction. □

Proposition 4.3. *Let $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \operatorname{Ob}\mathbf{Inv_{ifPDitop}}$ be an inverse system over a directed set Λ and $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty) \in \operatorname{Ob}\mathbf{ifPDitop}$ be the* *inverse limit of that system. If* $U_\alpha \in \mathfrak{J}^c_{\tau_\alpha\kappa_\alpha}$, $\alpha \in \Lambda$ *and* $\lim\limits_{\leftarrow} \{U_\alpha\} = U_\infty$ *for the inverse subsystem* $\{(U_\alpha, \mathcal{S}_\alpha|_{U_\alpha}, \tau_\alpha|_{U_\alpha}, \kappa_\alpha|_{U_\alpha}), \varphi_{\alpha\beta}|_{U_\alpha}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv_{ifPDitop}}$, *then* $U_\infty \in \mathfrak{J}^c_{\tau_\infty\kappa_\infty}$.

Proof. By the definition of inverse limit and the equality $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in U_\alpha$, $\alpha \geq \beta$, the inclusion $U_\infty = \lim\limits_{\leftarrow} \{U_\alpha\} \subseteq \lim\limits_{\leftarrow} \{S_\alpha\} = S_\infty$ is immediate, as mentioned in Proposition 4.2 as well.

Now, let us prove $U_\infty \in \mathfrak{J}^c_{\tau_\infty\kappa_\infty}$: If $P_a \not\subseteq U_\infty$, that is $a \notin U_\infty$ for $a = \{a_\alpha\} \in S_\infty$, then $a \notin \prod\limits_{\alpha\in\Lambda} U_\alpha$ due to the equality $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in U_\alpha$, $\alpha \geq \beta$. In this case, there exists $\alpha_0 \in \Lambda$ such that $a_{\alpha_0} \notin U_{\alpha_0}$, that is $P_{a_{\alpha_0}} \not\subseteq U_{\alpha_0}$. Additionally, the subset $\mu^{-1}_{\alpha_0}[U_{\alpha_0}] \subseteq S_\infty$ is an element of $\mathfrak{J}^c_{\tau_\infty\kappa_\infty}$ since the limiting projection map $\mu_{\alpha_0} : S_\infty \to S_{\alpha_0}$ is continuous between the corresponding joint topological spaces and $U_{\alpha_0} \in \mathfrak{J}^c_{\tau_{\alpha_0}\kappa_{\alpha_0}}$.

On the other hand, the statements $P_a \not\subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$ and $U_\infty \subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$ may be showed as follows:

Conversely, if $P_a \subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$ then we have $\mu_{\alpha_0}(a) = a_{\alpha_0} \in U_{\alpha_0}$ which is a contradiction.

Also, assume that $U_\infty \not\subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$. Thus there exists a point $z \in S_\infty$ such that $U_\infty \not\subseteq Q_z$ and $P_z \not\subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$. Hence, $\mu_{\alpha_0}(z) = z_{\alpha_0} \notin U_{\alpha_0}$ and $z = \{z_\alpha\} \notin \prod\limits_{\alpha\in\Lambda} U_\alpha$ gives the fact that $z \notin U_\infty$ which is a contradiction. □

From now on, in the remainder of this Section we will use all of the above notations, in exactly the same form. By virtue of Theorem 4.1 and the last proposition, now we have the next:

Theorem 4.4. *If* $\overline{U}$ *denotes the closure of the subset* $U \subseteq S_\infty$ *with respect to the joint topology of the limit ditopology* $(\tau_\infty, \kappa_\infty)$ *then*

(1) $\lim\limits_{\leftarrow}\{\overline{U}_\alpha\}$ *is jointly closed subspace of* S_∞
(2) $\lim\limits_{\leftarrow} \{\overline{U}_\alpha\} = \overline{U} \subseteq S$
(3) $\overline{U} = \bigcap\limits_{\alpha\in\Lambda} \mu^{-1}_\alpha[\overline{U}_\alpha]$

Proof. (1) Before everything, let 's see that $\lim\limits_{\leftarrow}\{\overline{U}_\alpha\} \subseteq S_\infty$, where $S_\infty = \lim\limits_{\leftarrow}\{S_\alpha\}$:

Conversely, if the inclusion is not true, then there exists a point $s = \{s_\alpha\} \in \prod\limits_{\alpha\in\Lambda} S_\alpha$ such that $\lim\limits_{\leftarrow}\{\overline{U}_\alpha\} \not\subseteq Q_s$ and $P_s \not\subseteq S_\infty$. Hence, by the facts $\overline{U}_\sigma \not\subseteq Q_{s_\sigma}$ and $s_\sigma \in \overline{U}_\sigma$ for every $\sigma \in \Lambda$, we have $\overline{\varphi}_{\alpha\beta}(s_\alpha) = s_\beta$ for $\alpha \geq \beta$.

On the other hand, it is easy to see that $P_{s_\sigma} \subseteq S_\sigma$ since the set $\overline{U}_\sigma$ is a subset of S_σ for every $\sigma \in \Lambda$, and so $P_s = \prod\limits_{\sigma\in\Lambda} P_{s_\sigma} \subseteq \prod\limits_{\sigma\in\Lambda} S_\sigma$. Also, if recall the equality $\overline{\varphi}_{\alpha\beta}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in \overline{U}_\alpha$ and $\alpha \geq \beta$, then we have $\varphi_{\alpha\beta}(s_\alpha) = s_\beta$ due to the fact that $\overline{\varphi}_{\alpha\beta}(s_\alpha) = s_\beta$ for $s_\alpha \in \overline{U}_\alpha$ and $\alpha \geq \beta$. Thus, by the definition of inverse limit, $s = \{s_\alpha\} \in S_\infty$ and it is a contradiction.

Accordingly, now let us show that $\underleftarrow{\lim}\{\overline{U}_\alpha\}$ is a jointly closed subspace of S_∞: Take a point $s = \{s_\alpha\} \in S_\infty$ such that $s \notin \underleftarrow{\lim}\{\overline{U}_\alpha\}$. In this case, because of the fact that $s \notin \prod_{\alpha\in\Lambda} \overline{U}_\alpha$ there exists an element $\sigma \in \Lambda$ such that $s_\sigma \notin \overline{U}_\sigma$. Thus, $s \notin \mu^{-1}_{\sigma\in\Lambda}[\overline{U}_\sigma]$ by the equality $\mu_\sigma(s) = s_\sigma$ and in view of the fact that $\overline{U}_\sigma$ is jointly closed in S_σ, the subset $\mu_\sigma^{-1}[\overline{U}_\sigma] \subseteq S_\infty$ is jointly closed in S_∞ due to the continuity of limiting projection $\mu_\sigma : S_\infty \to S_\sigma$ as given in [18, Proposition 4.4]. Now, we can prove that $\underleftarrow{\lim}\{\overline{U}_\sigma\} \subseteq \mu_\sigma^{-1}[\overline{U}_\sigma]$: If there exists a point $a = \{a_\alpha\} \in S_\infty$ such that $\underleftarrow{\lim}\{\overline{U}_\sigma\} \not\subseteq Q_a$ and $P_a \not\subseteq \mu_\sigma^{-1}[\overline{U}_\sigma]$ then $a \in \underleftarrow{\lim}\{\overline{U}_\sigma\}$ and so $a \in \prod_{\sigma\in\Lambda} \overline{U}_\sigma$. But also, the fact $a_\sigma = \mu_\sigma(a) \notin \overline{U}_\sigma$ gives a contradiction. As a result of the above considerations $\underleftarrow{\lim}\{\overline{U}_\alpha\}$ is a jointly closed subspace of S_∞.

In addition, now we will show that $\overline{U} = \underleftarrow{\lim}\{\overline{U}_\alpha\}$:

(2) First of all, let 's prove the inclusion $U \subseteq \underleftarrow{\lim}\{\overline{U}_\alpha\}$. Conversely, if $U \not\subseteq \underleftarrow{\lim}\{\overline{U}_\alpha\}$, then there exists $b \in S_\infty = \underleftarrow{\lim}\{S_\alpha\}$ such that $U \not\subseteq Q_b$ and $P_b \not\subseteq \underleftarrow{\lim}\{\overline{U}_\alpha\}$. In this case, $P_{b_\alpha} \subseteq \overline{U}_\alpha$ because of $\mu_\alpha(b) \in \mu_\alpha(U)$. Thus, $P_b = \prod_{\alpha\in\Lambda} P_{b_\alpha} \subseteq \prod_\alpha \overline{U}_{\alpha\in\Lambda}$.

On the other hand, $b \in \prod_\alpha S_{\alpha\in\Lambda}$ and $\varphi_{\alpha\beta}(b_\alpha) = b_\beta$ for $\alpha \geq \beta$, $\alpha, \beta \in \Lambda$. Also, by the definition of $\overline{\varphi}_{\alpha\beta}$ for $\alpha \geq \beta$ and the fact $b_\alpha \in U_\alpha$ for every $\alpha \in \Lambda$, the equality $\overline{\varphi}_{\alpha\beta}(b_\alpha) = \varphi_{\alpha\beta}(b_\alpha)$ is satisfied. Hence, $\overline{\varphi}_{\alpha\beta}(b_\alpha) = b_\beta$ for $\alpha \geq \beta$. That is, we obtained $b \in \underleftarrow{\lim}\{\overline{U}_\alpha\}$ which is a contradiction.

Therefore, from (1) if recall the fact that $\underleftarrow{\lim}\{\overline{U}_\alpha\}$ is jointly closed with respect to the limit ditopology $(\tau_\infty, \kappa_\infty)$ on $(S_\infty, \mathcal{S}_\infty)$, then the inclusion $\overline{U} \subseteq \underleftarrow{\lim}\{\overline{U}_\alpha\}$ is immediate.

For the other direction, assume $\underleftarrow{\lim}\{\overline{U}_\alpha\} \not\subseteq \overline{U}$. Thus, there exists a point $a = \{a_\alpha\} \in S_\infty$ such that $\underleftarrow{\lim}\{\overline{U}_\alpha\} \not\subseteq Q_a$ and $P_a \not\subseteq \overline{U}$. By the definition of joint topology, there exist $M \in \mu(a)$ and $N \in \eta(a)$ such that $\overline{U} \subseteq N \cap (S_\infty \setminus M)$ and so we have the sets $G \in \tau_\infty$ and $K \in \kappa_\infty$ such that $P_a \subseteq G \subseteq M$, $N \subseteq K \subseteq Q_a$ and $\overline{U} \subseteq K \cap (S_\infty \setminus G)$. Hence, by [18, Theorem 4.6], there exist $\alpha_0, \alpha_1 \in \Lambda$ and $A_{\alpha_0} \in \tau_{\alpha_0}$, $B_{\alpha_1} \in \kappa_{\alpha_1}$ such that the conditions $P_a \subseteq \mu^{-1}_{\alpha_0}[A_{\alpha_0}] \subseteq G$ and $K \subseteq \mu^{-1}_{\alpha_1}[B_{\alpha_1}] \subseteq Q_a$ are satisfied. In this case, the inclusion $\overline{U} \subseteq (S_\infty \setminus \mu^{-1}_{\alpha_0}[A_{\alpha_0}]) \cap \mu^{-1}_{\alpha_1}[B_{\alpha_1}]$ is trivial. Finally, we obtained $\alpha_1 \in \Lambda$ satisfying the conditions $U \subseteq \overline{U} \subseteq \mu^{-1}_{\alpha_1}[B_{\alpha_1}]$ and $P_a \not\subseteq \mu^{-1}_{\alpha_1}[B_{\alpha_1}]$. Thus $U_{\alpha_1} \subseteq B_{\alpha_1}$ for $\alpha_1 \in \Lambda$, because of the inclusions $\mu_{\alpha_1}(U) \subseteq \mu_{\alpha_1}({\mu_{\alpha_1}}^{-1}[B_{\alpha_1}]) \subseteq B_{\alpha_1}$. If we consider the closure operator on these sets, it is clear that $\overline{U}_{\alpha_1} \subseteq B_{\alpha_1}$ and so $\mu_{\alpha_1}(P_a) \not\subseteq \overline{U}_{\alpha_1}$ by $\mu_{\alpha_1}(a) \notin B_{\alpha_1}$. Moreover, it is easy to verify that $\mu_{\alpha_1}(P_a) = P_{a_{\alpha_1}}$:

$\mu_{\alpha_1}(P_a) = \{\mu_{\alpha_1}(x) \mid x \in P_a\} = \{x_{\alpha_1} \mid x \in P_a\} = \{x_{\alpha_1} \mid x \in \prod_{\alpha\in\Lambda} P_{a_\alpha}\} = \{x_{\alpha_1} \mid x_\alpha \in P_{a_\alpha}, \forall\alpha\} = P_{a_{\alpha_1}}$. As a result of these facts, we have $P_{a_{\alpha_1}} \not\subseteq \overline{U}_{\alpha_1}$ and so $a_{\alpha_1} \notin \overline{U}_{\alpha_1}$ for $\alpha_1 \in \Lambda$. This argument gives $a \notin \prod_{\alpha} \overline{U}_{\alpha\in\Lambda}$, clearly. It means that $a \notin \varprojlim\{\overline{U}_\alpha\}$ and so, a contradiction.

(3) Note that the closure set $\overline{U}_\alpha$ is jointly closed in the space S_α for each α. Thus, the sets $\mu_\alpha^{-1}[\overline{U}_\alpha]$, $\alpha \in \Lambda$ are jointly closed in the limit space S_∞ since the limiting projection μ_α is continuous for $\alpha \in \Lambda$, between the corresponding joint topological spaces $(S_\infty, \mathcal{J}_{\tau_\infty\kappa_\infty})$, $(S_\alpha, \mathcal{J}_{\tau_\alpha\kappa_\alpha})$ of the spaces $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty), (S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha) \in \mathrm{Ob}\,\mathbf{ifPDitop}$, respectively. In addition, with the equality $\mu_\alpha(U) = U_\alpha$, $\alpha \in \Lambda$ given in the hypothesis, it is clear that $U \subseteq \mu_\alpha^{-1}[\overline{U}_\alpha]$ and so $U \subseteq \bigcap_{\alpha\in\Lambda} \mu_\alpha^{-1}[\overline{U}_\alpha]$. Hence we have $\overline{U} \subseteq \bigcap_{\alpha\in\Lambda} \mu_\alpha^{-1}[\overline{U}_\alpha]$ since $\bigcap_{\alpha\in\Lambda} \mu_\alpha^{-1}[\overline{U}_\alpha]$ is jointly closed in S_∞.

For the converse, suppose that $\bigcap_{\alpha\in\Lambda} \mu_\alpha^{-1}[\overline{U}_\alpha] \not\subseteq \overline{U}$. In this case, there exists a point $a = \{a_\alpha\} \in S_\infty$ such that $\bigcap_{\alpha\in\Lambda} \mu_\alpha{}^{-1}[\overline{U}_\alpha] \not\subseteq Q_a$ and $P_a \not\subseteq \overline{U}$. Thus, $a \in \mu_\alpha{}^{-1}[\overline{U}_\alpha]$ and $\mu_\alpha(a) = a_\alpha \in \overline{U}_\alpha$ for every $\alpha \in \Lambda$.

On the other hand, if $P_a \not\subseteq \overline{U}$ and $\overline{U}$ is closed in S_∞ with respect to the joint topology of the ditopology on $(S_\infty, \mathcal{S}_\infty) \in$Ob **ifPTex**, then there exist $M \in \mu(a)$ and $N \in \eta(a)$ such that $\overline{U} \subseteq N \cap (S_\infty \backslash M)$. So we have the sets $G \in \tau_\infty$, $K \in \kappa_\infty$ such that $G \subseteq M$, $N \subseteq K$ and $\overline{U} \subseteq K \cap (S_\infty \backslash G)$. Therefore, by [18, Theorem 4.6] there exist $\alpha_0, \alpha_1 \in \Lambda$ and $A_{\alpha_0} \in \tau_{\alpha_0}$, $B_{\alpha_1} \in \kappa_{\alpha_1}$ satisfying the conditions $\mu_{\alpha_0}^{-1}[A_{\alpha_0}] \subseteq G$, $\mu_{\alpha_0}^{-1}[A_{\alpha_0}] \not\subseteq Q_a$ and $K \subseteq \mu_{\alpha_1}^{-1}[B_{\alpha_1}]$, $P_a \not\subseteq \mu_{\alpha_1}^{-1}[B_{\alpha_1}]$. In this case, the inclusion $\overline{U} \subseteq (S_\infty \backslash \mu_{\alpha_0}^{-1}[A_{\alpha_0}]) \cap \mu_{\alpha_1}^{-1}[B_{\alpha_1}]$ is trivial and so, we have $U_{\alpha_1} \subseteq B_{\alpha_1}$ for $\alpha_1 \in \Lambda$ by $U \subseteq \mu_{\alpha_1}^{-1}[B_{\alpha_1}]$. Consequently, $\overline{U}_{\alpha_1} \subseteq B_{\alpha_1}$ and the fact that $\mu_{\alpha_1}(a) = a_{\alpha_1} \notin B_{\alpha_1}$ means that $a_{\alpha_1} \notin \overline{U}_{\alpha_1}$ which is a contradiction. □

With the above notations, we have also the next result:

Corollary 4.5.

i) $U \subseteq \varprojlim\{U_\alpha\}$

ii) $\varprojlim\{U_\alpha\} \subseteq \varprojlim\{\overline{U}_\alpha\}$

Proof. i) If the inclusion is not true, there exists a point $a = \{a_\alpha\} \in S_\infty$ such that $U \not\subseteq Q_a$ and $P_a \not\subseteq \varprojlim\{U_\alpha\}$. In this case, by the fact $\mu_\alpha(a) \in \mu_\alpha(U) = U_\alpha$ we have $a_\alpha \in U_\alpha$ for every $\alpha \in \Lambda$ and so $a \in \prod_{\alpha\in\Lambda} U_\alpha$, obviously. Also, we have $\varphi_{\alpha\beta}|_{U_\alpha}(a_\alpha) = \varphi_{\alpha\beta}(a_\alpha) = a_\beta$ since $a_\alpha \in U_\alpha$, $\alpha \in \Lambda$. As a result of these considerations, we get $a = \{a_\alpha\} \in \lim\limits_{\leftarrow}\{U_\alpha\}$ which contradicts with $P_a \not\subseteq \lim\limits_{\leftarrow}\{U_\alpha\}$.

ii) Firstly, note that the limit sets $\lim\limits_{\leftarrow}\{U_\alpha\}$ and $\lim\limits_{\leftarrow}\{\overline{U}_\alpha\}$ are subsets of S_∞, due to Proposition 4.2 and Theorem 4.4. Now assume the converse of required inclusion. Thus, there exists a point $s = \{s_\alpha\} \in S_\infty$ such that $\lim\limits_{\leftarrow}\{U_\alpha\} \not\subseteq Q_s$ and $P_s \not\subseteq \lim\limits_{\leftarrow}\{\overline{U}_\alpha\}$. In this case, $s = \{s_\alpha\} \in \prod\limits_{\alpha\in\Lambda} U_\alpha$ and so $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = s_\beta$ for $\alpha \geq \beta$, $\alpha, \beta \in \Lambda$ because of $s_\alpha \in U_\alpha$. Hence, $s = \{s_\alpha\} \in \prod\limits_{\alpha\in\Lambda} \overline{U}_\alpha$ by $U_\alpha \subseteq \overline{U}_\alpha$. Also, for $\alpha \geq \beta$, we have the equalities $\overline{\varphi}_{\alpha\beta}(s_\alpha) = \varphi_{\alpha\beta}|_{\overline{U}_\alpha}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ and $\varphi_{\alpha\beta}(s_\alpha) = \varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha)$ due to $s_\alpha \in U_\alpha$. Consequently, the point $s = \{s_\alpha\} \in \prod\limits_{\alpha\in\Lambda} \overline{U}_\alpha$ is also an element of the inverse limit set $\lim\limits_{\leftarrow}\{\overline{U}_\alpha\}$ since we have the equality $\overline{\varphi}_{\alpha\beta}(s_\alpha) = \varphi_{\alpha\beta}|_{\overline{U}_\alpha}(s_\alpha) = s_\beta$ for $\alpha \geq \beta$, and it is a contradiction. □

According to all considerations presented above, we can mention a further result as the final stage of this section, besides the fact that it will be considered as the converse of Proposition 4.3.

Corollary 4.6. *Let the system* $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv}_{\mathbf{ifPDitop}}$ *over a directed set* Λ. *If take the ditopological subtexture space* $(U, \mathcal{S}_U, \tau_U, \kappa_U) \in \mathrm{Ob}\,\mathbf{ifPDitop}$ *of the inverse limit space* $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)$ *where* $U \in \mathcal{J}^c_{\tau_\infty\kappa_\infty}$, *then* $(U, \mathcal{S}_U, \tau_U, \kappa_U) \in \mathrm{Ob}\,\mathbf{ifPDitop}$ *is the inverse limit space of the inverse system* $\{(\overline{U}_\alpha, \mathcal{S}_{\overline{U}_\alpha}, \tau_{\overline{U}_\alpha}, \kappa_{\overline{U}_\alpha}), \overline{\varphi}_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv}_{\mathbf{ifPDitop}}$ *consisting of jointly closed subspaces* $(\overline{U}_\alpha, \mathcal{S}_{\overline{U}_\alpha}, \tau_{\overline{U}_\alpha}, \kappa_{\overline{U}_\alpha})$ *of the spaces* $(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha) \in \mathrm{Ob}\,\mathbf{ifPDitop}$, *where* $\pi_\alpha|_{S_\infty}(U) = \mu_\alpha(U) = U_\alpha$, $\mathcal{S}_{\overline{U}_\alpha} = \mathcal{S}_\alpha|_{\overline{U}_\alpha}$, $\tau_{\overline{U}_\alpha} = \tau_\alpha|_{\overline{U}_\alpha}$, $\kappa_{\overline{U}_\alpha} = \kappa_\alpha|_{\overline{U}_\alpha}$, $\alpha \in \Lambda$ *and* $\overline{\varphi}_{\alpha\beta} = \varphi_{\alpha\beta}|_{\overline{U}_\alpha}$, *for* $\alpha, \beta \in \Lambda$ *such that* $\alpha \geq \beta$.

In other words, if $U \in \mathcal{J}^c_{\tau_\infty\kappa_\infty}$ *then we have the equality*
$$U = \lim_{\leftarrow}\{U_\alpha\} = \lim_{\leftarrow}\{\overline{U}_\alpha\}.$$

Proof. If choose the set U as an element of $\mathcal{J}^c_{\tau_\infty\kappa_\infty}$, that is a closed set with respect to the joint topology of the limit ditopology $(\tau_\infty, \kappa_\infty)$ defined on the inverse limit texture, then by Theorem 4.4 (2) and the two inclusions presented in Corollary 4.5, the required equalities are straightforward. □

5. Identification of the ditopological products as an inverse limit in **ifPDitop**

Take into account all the previous considerations, it can be mentioned that the notion of inverse limit as an object of **ifPDitop** for any inverse system which is the object of $\mathbf{Inv}_{\mathbf{ifPDitop}}$ is derived from the products as the objects of **ifPDitop**.

Conversely, by applying the limit operation $\lim\limits_{\leftarrow}$ located in the theory of inverse systems, to the objects of $\mathbf{Inv}_{\mathbf{ifPDitop}}$, one can express infinite ditopological cartesian products [3, 4, 18] of the spaces which are the objects of

ifPDitop in terms of the finite cartesian products of those spaces belong to Ob **ifPDitop**.

Now, let 's mention and prove this significant characterization as a theorem:

Theorem 5.1. *For a directed set* Λ *and any family* $\{(X_s, \mathcal{S}_s, \tau_s, \kappa_s)\}_{s\in\Lambda}$ *of the objects in* **ifPDitop***, the product space* $(\prod_{s\in\Lambda} X_s, \bigotimes_{s\in\Lambda} \mathcal{S}_s, \bigotimes_{s\in\Lambda} \tau_s, \bigotimes_{s\in\Lambda} \kappa_s) \in$ Ob **ifPDitop** *may be expressed as the inverse limit of an inverse system over* Γ*, which is the object of* $\mathbf{Inv_{ifPDitop}}$ *and constructed by the finite cartesian product spaces*
$(\prod_{s\in I} X_s, \bigotimes_{s\in I} \mathcal{S}_s, \bigotimes_{s\in I} \tau_s, \bigotimes_{s\in I} \kappa_s) \in$ Ob **ifPDitop** *for* $I \in \Gamma$*, where the set* $\Gamma = \{I \subseteq \Lambda \mid \ I \ \text{ is finite}\}$ *is directed by the set inclusion. In other words,*

Any arbitrary textural product of the objects in **ifPDitop** *is exactly the inverse limit space of the inverse system consisting of finite products of those objects.*

Proof. Let $(X_s, \mathcal{S}_s, \tau_s, \kappa_s) \in$ Ob **ifPDitop**, $s \in \Lambda$ and Γ be directed by the set inclusion, that is $J \leq I \iff J \subseteq I$ for every $I, J \in \Gamma$. Now assume $J \leq I$ for any $J \in \Gamma$. If $x = \{x_s\}_{s\in I} \in \prod_{s\in I} X_s = X_I$ then $x_s \in X_s$ for all $s \in I$. In this case, $\{x_s\}_{s\in J} \in \prod_{s\in J} X_s = X_J$ by the facts that if $s \in J$ then $s \in I$ and $x_s \in X_s$ for all $s \in I$. Therefore, for $J \leq I$, describe the mapping

$$\varphi_{IJ} : X_I \to X_J$$

$$\{x_s\}_{s\in I} \mapsto \{x_s\}_{s\in J}.$$

Now let us prove that φ_{IJ} is ω-preserving and bicontinuous for $J \leq I$: Assume that $P_{\{x_s\}_{s\in I}} \not\subseteq Q_{\{x'_s\}_{s\in I}}$ for $\{x_s\}, \{x'_s\} \in X_I$. If $s_0 \in J$, then $s_0 \in I$ by $J \leq I$. Thus, $P_{x_{s_0}} \not\subseteq Q_{x'_{s_0}}$ and $P_{\{x_s\}_{s\in J}} \not\subseteq Q_{\{x'_s\}_{s\in J}}$ by [18, Corollary 1.2], since $P_{x_s} \not\subseteq Q_{x'_s}$ for all $s \in J$. Hence $P_{\varphi_{IJ}(x)} \not\subseteq Q_{\varphi_{IJ}(x')}$ and φ_{IJ} is ω-preserving.

For the second part, we prove that φ_{IJ} is bicontinuous between the product ditopological spaces $(X_I, \mathcal{S}_I, \tau_I, \kappa_I)$ and $(X_J, \mathcal{S}_J, \tau_J, \kappa_J)$ as follows:

Suppose that $J = \{1, 2, ..., m\}$, $I = \{1, 2, ..., t\}$ and $J \subseteq I$. In this case, $m < t$.

Now let $G \in \bigotimes_{s\in J} \tau_s = \tau_J$ and $\varphi_{IJ}^{-1}[G] \not\subseteq Q_x$ for $x = \{x_s\}_{s\in I} \in X_I$. In this case, $\varphi_{IJ}(x) = \{x_s\}_{s\in J} \in G$, that is $G \not\subseteq Q_{\{x_s\}_{s\in J}}$. Thus, there exists $B \in \mathcal{B}_{\tau_J}$, where $\mathcal{B}_{\tau_J}$ denotes the base for τ_J, such that $B \not\subseteq Q_{\{x_s\}_{s\in J}}$ and $B \subseteq G$, so there exists finite set $J_0 = \{1, 2, ..., n\} \leq J \ \ (n < m)$ such that $B = \bigcap_{j\in J_0} (\pi_j^J)^{-1}[G_j]$, where $G_j \in \tau_j$, $j \in J_0$. Thus, $x_j \in G_j$ for $j \in J_0$.

Also, $\varphi_{IJ}^{-1}[B] = \varphi_{IJ}^{-1}(\bigcap_{j\in J_0} ({\pi_j}^J)^{-1}[G_j]) \subseteq \varphi_{IJ}^{-1}[G]$ because of $B \subseteq G$. Thus, with all the arrangements, we have $B' = \bigcap_{j\in J_0} (\pi_j^J \circ \varphi_{IJ})^{-1}[G_j] = \bigcap_{j\in J_0} (\pi_j^I)^{-1}[G_j] \subseteq \varphi_{IJ}^{-1}[G]$ and so $B' \in \mathcal{B}_{\tau_I}$ where $\mathcal{B}_{\tau_I}$ denotes the base for τ_I.

On the other hand, we have $\{x_s\}_{s\in J} \in (\pi_j^J)^{-1}[G_j]$ since $\{x_s\}_{s\in J} \in B$ and $\pi_j^J(\{x_s\}_{s\in J}) \in G_j$ for every $j \in J_0$. Thus $x_1 \in G_1$, $x_2 \in G_2$,...,$x_n \in G_n$. In this case, by the fact $n < t$, $\pi_j^I(\{x_s\}) \in G$ for $j \in J_0$ and so $\{x_s\}_{s\in I} \in B'$.
Since, $\varphi_{IJ}^{-1}[G] \in \tau_I$ for $G \in \tau_J$, φ_{IJ} is continuous.

Dually, by using closed sets as the elements of κ_J, it is proved that φ_{IJ} is cocontinuous and so bicontinuous.

Furthermore, note that the mappings φ_{IJ} for $J \leq I$ are the bonding maps: Indeed, for the mapping $\varphi_{II} : X_I \to X_I$, the equality $\varphi_{II}(\{x_s\}_{s\in I}) = \{x_s\}_{s\in I}$ is clear and so φ_{II} is the identity id_{X_I}. In addition, for $K \geq I \geq J$, let 's prove $\varphi_{IJ} \circ \varphi_{KI} = \varphi_{KJ}$. If $\{x_s\}_{s\in K} \in X_K$ then $(\varphi_{IJ} \circ \varphi_{KI})(\{x_s\}_{s\in K}) = \varphi_{IJ}(\varphi_{KI}(\{x_s\}_{s\in K})) = \varphi_{IJ}(\{x_s\}_{s\in I}) = \{x_s\}_{s\in J} = \varphi_{KJ}(\{x_s\}_{s\in K})$.

Consequently, thanks to the above expressions, the fact $\{(X_I, \mathcal{S}_I, \tau_I, \kappa_I), \varphi_{IJ}\}_{I\geq J} \in \mathrm{Ob}\,\mathbf{Inv_{ifPDitop}}$ is trivial.

Now let us turn to our main aim: *The inverse limit space of inverse system* $\{(X_I, \mathcal{S}_I, \tau_I, \kappa_I), \varphi_{IJ}\}_{I\geq J} \in \mathrm{Ob}\,\mathbf{Inv_{ifPDitop}}$ *over* Γ *is* **ifPDitop**-*isomorphic to the arbitrary ditopological product space constructed on the set* $\prod_{s\in\Lambda} X_s$.

For proof, first of all we define a mapping between $\lim\limits_{\leftarrow}\{X_I\}_{I\in\Gamma}$ and $\prod_{s\in\Lambda} X_s$:

If $\{x_I\} \in \lim\limits_{\leftarrow}\{X_I\}_{I\in\Gamma}$ then $\{x_I\} \in \prod_{I\in\Gamma} X_I$ and so $x_I \in X_I$ for every $I \in \Gamma$. Now, for any $s \in \Lambda$ let $I_s = \{s\} \in \Gamma$, so by the fact $X_{I_s} = \prod_{z\in I_s} X_z = \prod_{z\in\{s\}} X_z = X_s$, we have $x_{\{s\}} = x_{I_s} \in X_s$, $s \in \Lambda$. Thus $\{x_{I_s}\} \in \prod_{s\in\Lambda} X_s$ and finally, we can define the mapping

$$\psi : \lim_{\leftarrow}\{X_I\}_{I\in\Gamma} \to \prod_{s\in\Lambda} X_s$$
$$\{x_I\}_{I\in\Gamma} \mapsto \{x_{I_s}\}_{s\in\Lambda}$$

It is easy to verify that ψ is well-defined. Now let us show that ψ is an **ifPDitop**-isomorphism:

ψ is ω-preserving: Let $\{x_I\}, \{x'_I\} \in \lim\limits_{\leftarrow}\{X_I\}_{I\in\Gamma}$ such that $P_{\{x_I\}} \not\subseteq Q_{\{x'_I\}}$. In this case, $P_{x_I} \not\subseteq Q_{x'_I}$ for all $I \in \Gamma$, by [18, Corollary 1.2]. Take $s \in \Lambda$, so $I_s = \{s\} \subseteq \Lambda$, that is $I_s \in \Gamma$. Thus, $P_{x_{I_s}} \not\subseteq Q_{x'_{I_s}}$ by the fact that $P_{x_I} \not\subseteq Q_{x'_I}$ for all $I \in \Gamma$. It means that $P_{x_{I_s}} \not\subseteq Q_{x'_{I_s}}$ for all $s \in \Lambda$. Hence $P_{\{x_{I_s}\}_{s\in\Lambda}} \not\subseteq Q_{\{x'_{I_s}\}_{s\in\Lambda}}$, that is $P_{\psi\{x_I\}} \not\subseteq Q_{\psi\{x'_I\}}$.
In addition, the bijectivity of ψ is straightforward.

Now, if consider the product ditopological spaces $(X_I, \mathcal{S}_I, \tau_I, \kappa_I)$ for $I \in \Gamma$, with the plain texturings then the product texturing $\bigotimes_{I\in\Gamma} \mathcal{S}_I$ and product ditopology $(\bigotimes_{I\in\Gamma} \tau_I, \bigotimes_{I\in\Gamma} \kappa_I)$ can be constructed over the product set $\prod_{I\in\Gamma} X_I$ in a suitable way. Therefore, the restricted texturing and ditopology will be taken over the subset $\lim\limits_{\leftarrow}\{X_I\}_{I\in\Gamma}$ of $\prod_{I\in\Gamma} X_I$. Shortly, if we use the notations $\mathcal{T} =$

$(\bigotimes_{I\in\Gamma} \mathcal{S}_I)|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}}$, $\mathcal{V} = (\bigotimes_{I\in\Gamma} \tau_I)|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}}$ and $\mathcal{Z} = (\bigotimes_{I\in\Gamma} \kappa_I)|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}}$ for the induced texturing, topology and cotopology, respectively, then now we will prove that ψ is bicontinuous with respect to the ditopologies $(\bigotimes_{s\in\Lambda} \tau_s, \bigotimes_{s\in\Lambda} \kappa_s)$ and $(\mathcal{V}, \mathcal{Z})$:

Let $G \in \bigotimes_{s\in\Lambda} \tau_s = \tau_\Lambda$ and $\psi^{-1}[G] \not\subseteq Q_{\{x_I\}_{I\in\Gamma}}$. In this case, $G \not\subseteq Q_{\psi(\{x_I\}_{I\in\Gamma})}$ and so $G \not\subseteq Q_{\{x_{I_s}\}_{s\in\Lambda}}$. Thus, there exists $B \in \mathcal{B}_{\tau_\Lambda}$ which is the base for the product topology τ_Λ, such that $B \subseteq G$ and $B \not\subseteq Q_{\psi(\{x_I\}_{I\in\Gamma})}$. Note here that $B = \bigcap_{j\in J_0\subseteq\Lambda} \pi_j^{-1}[G_j]$, where $G_j \in \tau_j$ and $j \in J_0$ for the finite set $J_0 \subseteq \Lambda$. Thus, we have $\psi^{-1}(\bigcap_{j\in J_0\subseteq\Lambda} \pi_j^{-1}[G_j]) \subseteq \psi^{-1}[G]$ and so $\bigcap_{j\in J_0} (\pi_j \circ \psi)^{-1}[G_j] \subseteq \psi^{-1}[G]$.

On the other hand, the equality $\pi_j \circ \psi = \pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}}$ is obvious by the definition of projection map $\pi_{I_j} : \prod_{I\in\Gamma} X_I \to X_{I_j} = X_j$ and by the facts $j \in \Lambda$ and $I_j = \{j\} \subseteq \Lambda$ which means that $I_j \in \Gamma$ for $j \in J_0$.

Additionally, if take φ as the inverse of ψ, then we have $\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}} \circ \varphi = \pi_j$. Here, the restriction $\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}}$ is bicontinuous since I_j. projection map π_{I_j} is bicontinuous.

Hence, if $A = \bigcap_{j\in J_0} (\pi_j \circ \psi)^{-1}[G_j] = \bigcap_{j\in J_0} (\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}})^{-1}[G_j]$ then $A \in \mathcal{B}_{\mathcal{V}}$. Here, $\mathcal{B}_{\mathcal{V}}$ denotes the base for topology $\mathcal{V}$. In this case, the fact $A \subseteq \psi^{-1}[G]$ is clear.

Now let us prove $A \not\subseteq Q_{\{x_I\}_{I\in\Gamma}}$: Firstly, recall $B \not\subseteq Q_{\{x_{I_s}\}_{s\in\Lambda}}$ and so $\pi_j^{-1}[G_j] \not\subseteq Q_{\{x_{I_s}\}_{s\in\Lambda}}$ for all $j \in J_0$. That is, $\pi_j(\{x_{I_s}\}) \in G_j$ and $(\pi_j \circ \psi)(\{x_I\}_{I\in\Gamma}) \in G_j$ for all $j \in J_0$. Therefore, $\{x_I\}_{I\in\Gamma} \in (\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}})^{-1}[G_j]$ is clear for all $j \in J_0$. Finally, $\{x_I\}_{I\in\Gamma} \in \bigcap_{j\in J_0} (\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}})^{-1}[G_j] = A$, and so $A \not\subseteq Q_{\{x_I\}_{I\in\Gamma}}$ since the related texturings are plain. Hence $\psi^{-1}[G] \in \mathcal{V}$ and ψ is continuous.

Dually, it is easy to verify that ψ is cocontinuous by dealing with the closed sets. Then ψ is bicontinuous. As the final step, that the map φ as the inverse of ψ is bicontinuous can be shown in a like manner. □

The above theorem could be also summarized for the subcategory $\mathbf{ifPDicomp_2}$ consisting of dicompact [11] and bi-T_2 (bi-Hausdorff) [4] objects of the category **ifPDitop**. Hence, with the above arguments, note that:

Corollary 5.2. *The infinite ditopological products of the objects which belong to* $\mathbf{ifPDicomp_2}$ *can be expressed via inverse limits, in terms of the finite ditopological products in* $\mathbf{ifPDicomp_2}$ *of those objects.*

Proof. For all the details about category of dicompact spaces see [11], and from [4], note that $(S, \mathcal{S}, \tau, \kappa)$ is bi-T_2 if and only if for $s, t \in S$, $Q_s \not\subseteq Q_t \implies \exists H \in \tau, K \in \kappa$ with $H \subseteq K$, $P_s \not\subseteq K$ and $H \not\subseteq Q_t$. Thus, the required characaterization is seen as a result of Theorem 5.1. Indeed, by the facts that the jointly closed subtexture spaces and the product spaces of dicompact, bi-T_2 ditopological spaces are dicompact and bi-T_2 from [18, Theorem 4.16] and Tychonoff property, respectively, and from [18, Theorem 4.17 a)], the proof is completed. $\square$

Definition 5.3. A property P is called *ditopological property* if it is a property defined for ditopological texture spaces, as a natural counterpart of the classical notion, named topological property.

According to this, we have the following as a final result, as well.

Corollary 5.4. *Let P be a ditopological property which is hereditary with respect to the jointly closed subsets of a ditopological space and finitely multiplicative (that is, P is preserved under the finite multiplications of ditopological spaces). In this case, $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{ifPDitop}$ is $\mathbf{ifPDitop}$-isomorphic to the inverse limit of an inverse system constructed over a directed set Λ, via bi-T_2 spaces $(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha) \in \mathrm{Ob}\,\mathbf{ifPDitop}$, $\alpha \in \Lambda$, which have the property P if and only if $(S, \mathcal{S}, \tau, \kappa)$ is $\mathbf{ifPDitop}$-isomorphic to a jointly closed subspace of the product space $(\prod_{\alpha\in\Lambda} S_\alpha, \bigotimes_{\alpha\in\Lambda} \mathcal{S}_\alpha, \bigotimes_{\alpha\in\Lambda} \tau_\alpha, \bigotimes_{\alpha\in\Lambda} \kappa_\alpha)$.*

Proof. Necessity. Suppose that $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{ifPDitop}$ is isomorphic to the inverse limit space $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty) \in \mathrm{Ob}\,\mathbf{ifPDitop}$ of the inverse system $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv}_{\mathbf{ifPDitop}}$ over a directed set Λ, where $S_\infty = \lim\limits_{\leftarrow} \{S_\alpha\}$. Also, if recall that the inverse limit space S_∞ is jointly closed in the product $\prod_{\alpha\in\Lambda} S_\alpha$ by [18, Theorem 4.17 a)], then the required assertion is proved.

Sufficiency. Let $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha)\}_{\alpha\in\Lambda}$ be a family consisting of the objects in $\mathbf{ifPDitop}$, which have the properties bi-T_2 and P. Assume that $(S, \mathcal{S}, \tau, \kappa)$ is $\mathbf{ifPDitop}$-isomorphic to a jointly closed subspace $(U, (\bigotimes \mathcal{S}_\alpha)|_U, (\bigotimes \tau_\alpha)|_U, (\bigotimes \kappa_\alpha)|_U)$ of the product space $(\prod S_\alpha, \bigotimes \mathcal{S}_\alpha, \bigotimes \tau_\alpha, \bigotimes \kappa_\alpha)$. By Theorem 5.1, it is known that the product $\prod S_\alpha$ can be expressed as the inverse limit of an inverse system consisting of finite cartesian product spaces $\prod_{i=1}^{n} S_i$ for $n \in \mathbb{N}$. Hence, with the same notations used in Corollary 4.6, $(U, (\bigotimes \mathcal{S}_\alpha)|_U, (\bigotimes \tau_\alpha)|_U, (\bigotimes \kappa_\alpha)|_U))$ becomes the inverse limit of inverse system $\mathcal{A} = \{(\overline{U}_n, (\bigotimes_{i=1}^{n} \mathcal{S}_i)|_{\overline{U}_n}, (\bigotimes_{i=1}^{n} \tau_i)|_{\overline{U}_n}, (\bigotimes_{i=1}^{n} \kappa_i)|_{\overline{U}_n}), \varphi_{nm}\}_{n\geq m}$ constructed by the bonding maps $\varphi_{nm} : \overline{U}_n \to \overline{U}_m$ for $n \geq m$, as well as the jointly closed subspaces $(\overline{U}_n, (\bigotimes_{i=1}^{n} \mathcal{S}_i)|_{\overline{U}_n}, (\bigotimes_{i=1}^{n} \tau_i)|_{\overline{U}_n}, (\bigotimes_{i=1}^{n} \kappa_i)|_{\overline{U}_n})$ of finite cartesian product spaces $(\prod_{i=1}^{n} S_i, \bigotimes_{i=1}^{n} \mathcal{S}_i, \bigotimes_{i=1}^{n} \tau_i, \bigotimes_{i=1}^{n} \kappa_i)$ for every $n \in \mathbb{N}$. Here, $\overline{U}_n$ denotes the closure of U_n for each n, with respect to the joint topology appropriate for the finite product space of the spaces $(S_i, \mathcal{S}_i, \tau_i, \kappa_i)$, $i = 1, 2, ..., n$.

On the other hand, since each space S_α, $\alpha \in \Lambda$ has the property bi-T_2 from [4], the product space $\prod S_\alpha$ has the property bi-T_2 and so the ditopologies on subsets $\overline{U}_n$ have the property bi-T_2 as well. Furthermore, each finite product space $\prod_{i=1}^{n} S_i$ has the property P since each space S_α, $\alpha \in \Lambda$ has the property P by hypothesis. Thus, the jointly closed subspaces $\overline{U}_n$, $n \in \mathbb{N}$ have the common property P as P is hereditary with respect to the jointly closed subspaces. Consequently, $\mathcal{A}$ is the required inverse system in **ifPDitop** and by the fact $\lim\limits_{\leftarrow} \mathcal{A} = U$, the proof is concluded. □

6. Conclusion

This paper studied some further categorical aspects of the inverse systems (projective spectrums) and inverse limits constructed in the subcategory **ifPDitop** of ditopological plain spaces.

As one of the investigations here, an identity natural transformation which is peculiar to the theory of inverse systems and inverse limits, as well as consisting of the adjoint and isomorphism functors introduced between the suitable related main subcategories of **Bitop** and **ifPDitop**, consisting of the spaces which satisfy a special separation axiom, is established. As another one, we proved a representation theorem which shows any infinite textural product of the objects in category **ifPDitop** can be expressed as the inverse limit of the inverse system in $\mathbf{Inv_{ifPDitop}}$, constructed by the finite products of those objects in **ifPDitop**. Besides that, the textural products of dicompact bi-T_2 ditopological spaces are characterized in terms of finite products, via inverse limits.

There are considerable difficulties involved in constructing a suitable theory of inverse systems for general ditopological spaces. Hence, we confined our attention to the inverse systems - limits constructed in the special category **ifPDitop** and we leaved as an open problem the task of extending the further results obtained here to more general categories established in the theory of ditopological spaces.

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