

Generalized normal product adjacency in digital topology

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ABSTRACT

We study properties of Cartesian products of digital images for which adjacencies based on the normal product adjacency are used. We show that the use of such adjacencies lets us obtain many “product properties” for which the analogous statement is either unknown or invalid if, instead, we were to use c_u -adjacencies.

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1. INTRODUCTION

We study adjacency relations based on the normal product adjacency for Cartesian products of multiple digital images. Most of the literature of digital topology focuses on images that use a c_u -adjacency; however, the results of this paper seem to indicate that for Cartesian products of digital images, the natural adjacencies to use are based on the normal product adjacency of the factor adjacencies, in the sense of preservation of many properties in Cartesian products.

2. PRELIMINARIES

We use \mathbb{N} , \mathbb{Z} , and \mathbb{R} to represent the sets of natural numbers, integers, and real numbers, respectively.

Much of the material that appears in this section is quoted or paraphrased from [15, 17], and other papers cited in this section.

We will assume familiarity with the topological theory of digital images. See, e.g., [3] for many of the standard definitions. All digital images X are assumed to carry their own adjacency relations (which may differ from one image to another). When we wish to emphasize the particular adjacency relation we write the image as (X, κ) , where κ represents the adjacency relation.

2.1. Common adjacencies. Among the commonly used adjacencies are the c_u -adjacencies. Let $x, y \in \mathbb{Z}^n$, $x \neq y$. Let u be an integer, $1 \leq u \leq n$. We say x and y are c_u -adjacent if

- there are at most u indices i for which $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

We often label a c_u -adjacency by the number of points adjacent to a given point in \mathbb{Z}^n using this adjacency. E.g.,

- In \mathbb{Z}^1 , c_1 -adjacency is 2-adjacency.
- In \mathbb{Z}^2 , c_1 -adjacency is 4-adjacency and c_2 -adjacency is 8-adjacency.
- In \mathbb{Z}^3 , c_1 -adjacency is 6-adjacency, c_2 -adjacency is 18-adjacency, and c_3 -adjacency is 26-adjacency.

Given digital images or graphs (X, κ) and (Y, λ) , the *normal product adjacency* $NP(\kappa, \lambda)$ (also called the *strong adjacency* [37] and denoted $\kappa_*(\kappa, \lambda)$ in [13]) generated by κ and λ on the Cartesian product $X \times Y$ is defined as follows.

Definition 2.1 ([1]). *Let $x, x' \in X$, $y, y' \in Y$. Then (x, y) and (x', y') are $NP(\kappa, \lambda)$ -adjacent in $X \times Y$ if and only if*

- $x = x'$ and y and y' are λ -adjacent; or
- x and x' are κ -adjacent and $y = y'$; or
- x and x' are κ -adjacent and y and y' are λ -adjacent. □

2.2. Connectedness. A subset Y of a digital image (X, κ) is κ -connected [32], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a sequence $\{y_i\}_{i=0}^m \subset Y$ such that $a = y_0$, $b = y_m$, and y_i and y_{i+1} are κ -adjacent for $0 \leq i < m$.

For two subsets $A, B \subset X$, we will say that A and B are *adjacent* when there exist points $a \in A$ and $b \in B$ such that a and b are equal or adjacent. Thus sets with nonempty intersection are automatically adjacent, while disjoint sets may or may not be adjacent. It is easy to see that a finite union of connected adjacent sets is connected.

2.3. Continuous functions. The following generalizes a definition of [32].

Definition 2.2 ([4]). *Let (X, κ) and (Y, λ) be digital images. A function $f : X \rightarrow Y$ is (κ, λ) -continuous if for every κ -connected $A \subset X$ we have that $f(A)$ is a λ -connected subset of Y .*

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be reformulated in terms of adjacency of points:

Theorem 2.3 ([32, 4]). *A function $f : X \rightarrow Y$ is continuous if and only if, for any adjacent points $x, x' \in X$, the points $f(x)$ and $f(x')$ are equal or adjacent.* \square

Note that similar notions appear in [18, 19] under the names *immersion*, *gradually varied operator*, and *gradually varied mapping*.

Theorem 2.4 ([3, 4]). *If $f : (A, \kappa) \rightarrow (B, \lambda)$ and $g : (B, \lambda) \rightarrow (C, \mu)$ are continuous, then $g \circ f : (A, \kappa) \rightarrow (C, \mu)$ is continuous.* \square

Example 2.5 ([32]). *A constant function between digital images is continuous.* \square

Example 2.6. *The identity function $1_X : (X, \kappa) \rightarrow (X, \kappa)$ is continuous.*

Proof. This is an immediate consequence of Theorem 2.3. \square

Definition 2.7. *Let (X, κ) be a digital image in \mathbb{Z}^n . Let $x, y \in X$. A κ -path of length m from x to y is a set $\{x_i\}_{i=0}^m \subset X$ such that $x = x_0$, $x_m = y$, and x_{i-1} and x_i are equal or κ -adjacent for $1 \leq i \leq m$. If $x = y$, we say $\{x\}$ is a path of length 0 from x to x .*

Notice that for a path from x to y as described above, the function $f : [0, m]_{\mathbb{Z}} \rightarrow X$ defined by $f(i) = x_i$ is (c_1, κ) -continuous. Such a function is also called a κ -path of length m from x to y .

2.4. Digital homotopy. A homotopy between continuous functions may be thought of as a continuous deformation of one of the functions into the other over a finite time period.

Definition 2.8 ([4; see also [26]). *Let X and Y be digital images. Let $f, g : X \rightarrow Y$ be (κ, κ') -continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ such that*

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \rightarrow Y$ defined by

$$F_x(t) = F(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

is $(2, \kappa')$ -continuous. That is, $F_x(t)$ is a path in Y .

- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \rightarrow Y$ defined by

$$F_t(x) = F(x, t) \text{ for all } x \in X$$

is (κ, κ') -continuous.

Then F is a digital (κ, κ') -homotopy between f and g , and f and g are digitally (κ, κ') -homotopic in Y . If for some $x \in X$ we have $F(x, t) = F(x, 0)$ for all $t \in [0, m]_{\mathbb{Z}}$, we say F holds x fixed, and F is a pointed homotopy. \square

We denote a pair of homotopic functions as described above by $f \simeq_{\kappa, \kappa'} g$. When the adjacency relations κ and κ' are understood in context, we say f and g are *digitally homotopic* (or just *homotopic*) to abbreviate “digitally (κ, κ') -homotopic in Y ,” and write $f \simeq g$.

Proposition 2.9 ([26, 4]). *Digital homotopy is an equivalence relation among digitally continuous functions $f : X \rightarrow Y$.* \square

Definition 2.10 ([5]). *Let $f : X \rightarrow Y$ be a (κ, κ') -continuous function and let $g : Y \rightarrow X$ be a (κ', κ) -continuous function such that*

$$f \circ g \simeq_{\kappa', \kappa'} 1_X \text{ and } g \circ f \simeq_{\kappa, \kappa} 1_Y.$$

Then we say X and Y have the same (κ, κ') -homotopy type and that X and Y are (κ, κ') -homotopy equivalent, denoted $X \simeq_{\kappa, \kappa'} Y$ or as $X \simeq Y$ when κ and κ' are understood. If for some $x_0 \in X$ and $y_0 \in Y$ we have $f(x_0) = y_0$, $g(y_0) = x_0$, and there exists a homotopy between $f \circ g$ and 1_X that holds x_0 fixed, and a homotopy between $g \circ f$ and 1_Y that holds y_0 fixed, we say (X, x_0, κ) and (Y, y_0, κ') are pointed homotopy equivalent and that (X, x_0) and (Y, y_0) have the same pointed homotopy type, denoted $(X, x_0) \simeq_{\kappa, \kappa'} (Y, y_0)$ or as $(X, x_0) \simeq (Y, y_0)$ when κ and κ' are understood. \square

It is easily seen, from Proposition 2.9, that having the same homotopy type (respectively, the same pointed homotopy type) is an equivalence relation among digital images (respectively, among pointed digital images).

2.5. Connectivity preserving and continuous multivalued functions.

A multivalued function $f : X \rightarrow Y$ assigns a subset of Y to each point of x . We will write $f : X \multimap Y$. For $A \subset X$ and a multivalued function $f : X \multimap Y$, let $f(A) = \bigcup_{x \in A} f(x)$.

Definition 2.11 ([30]). *A multivalued function $f : X \multimap Y$ is connectivity preserving if $f(A) \subset Y$ is connected whenever $A \subset X$ is connected.*

As is the case with Definition 2.2, we can reformulate connectivity preservation in terms of adjacencies.

Theorem 2.12 ([15]). *A multivalued function $f : X \multimap Y$ is connectivity preserving if and only if the following are satisfied:*

- For every $x \in X$, $f(x)$ is a connected subset of Y .
- For any adjacent points $x, x' \in X$, the sets $f(x)$ and $f(x')$ are adjacent. \square

Definition 2.11 is related to a definition of multivalued continuity for subsets of \mathbb{Z}^n given and explored by Escibano, Giraldo, and Sastre in [20, 21] based on subdivisions. (These papers make a small error with respect to compositions, that is corrected in [22].) Their definitions are as follows:

Definition 2.13. For any positive integer r , the r -th subdivision of \mathbb{Z}^n is

$$\mathbb{Z}_r^n = \{(z_1/r, \dots, z_n/r) \mid z_i \in \mathbb{Z}\}.$$

An adjacency relation κ on \mathbb{Z}^n naturally induces an adjacency relation (which we also call κ) on \mathbb{Z}_r^n as follows: $(z_1/r, \dots, z_n/r), (z'_1/r, \dots, z'_n/r)$ are adjacent in \mathbb{Z}_r^n if and only if (z_1, \dots, z_n) and (z'_1, \dots, z'_n) are adjacent in \mathbb{Z}^n .

Given a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, the r -th subdivision of X is

$$S(X, r) = \{(x_1, \dots, x_n) \in \mathbb{Z}_r^n \mid (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor) \in X\}.$$

Let $E_r : S(X, r) \rightarrow X$ be the natural map sending $(x_1, \dots, x_n) \in S(X, r)$ to $(\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$. □

Definition 2.14. For a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, a function $f : S(X, r) \rightarrow Y$ induces a multivalued function $F : X \multimap Y$ if $x \in X$ implies

$$F(x) = \bigcup_{x' \in E_r^{-1}(x)} \{f(x')\}. \quad \square$$

Definition 2.15. A multivalued function $F : X \multimap Y$ is called continuous when there is some r such that F is induced by some single valued continuous function $f : S(X, r) \rightarrow Y$. □

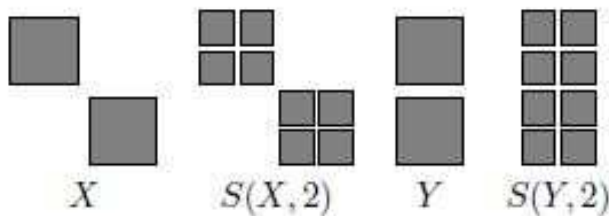


FIGURE 1. [15] Two images X and Y with their second subdivisions.

Note [15] that the subdivision construction (and thus the notion of continuity) depends on the particular embedding of X as a subset of \mathbb{Z}^n . In particular we may have $X, Y \subset \mathbb{Z}^n$ with X isomorphic to Y but $S(X, r)$ not isomorphic to $S(Y, r)$. This in fact is the case for the two images in Figure 1, when we use 8-adjacency for all images. Then the spaces X and Y in the figure are isomorphic, each being a set of two adjacent points. But $S(X, 2)$ and $S(Y, 2)$ are not isomorphic since $S(X, 2)$ can be disconnected by removing a single point, while this is impossible in $S(Y, 2)$.

The definition of connectivity preservation makes no reference to X as being embedded inside of any particular integer lattice \mathbb{Z}^n .

Proposition 2.16 ([20, 21]). *Let $F : X \multimap Y$ be a continuous multivalued function between digital images. Then*

- *for all $x \in X$, $F(x)$ is connected; and*
- *for all connected subsets A of X , $F(A)$ is connected.* □

Theorem 2.17 ([15]). *For $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, if $F : X \multimap Y$ is a continuous multivalued function, then F is connectivity preserving.* □

The subdivision machinery often makes it difficult to prove that a given multivalued function is continuous. By contrast, many maps can easily be shown to be connectivity preserving.

Proposition 2.18 ([15]). *Let X and Y be digital images. Suppose Y is connected. Then the multivalued function $f : X \multimap Y$ defined by $f(x) = Y$ for all $x \in X$ is connectivity preserving.* □

Proposition 2.19 ([15]). *Let $F : (X, \kappa) \multimap (Y, \lambda)$ be a multivalued surjection between digital images $(X, \kappa), (Y, \lambda) \subset (\mathbb{Z}^n, \kappa)$. If X is finite and Y is infinite, then F is not continuous.* □

Corollary 2.20 ([15]). *Let $F : X \multimap Y$ be the multivalued function between digital images defined by $F(x) = Y$ for all $x \in X$. If X is finite and Y is infinite and connected, then F is connectivity preserving but not continuous.* □

Examples of connectivity preserving but not continuous multivalued functions on finite spaces are given in [15].

2.6. Other notions of multivalued continuity. Other notions of continuity have been given for multivalued functions between graphs (equivalently, between digital images). We have the following.

Definition 2.21 ([36]). *Let $F : X \multimap Y$ be a multivalued function between digital images.*

- *F has weak continuity if for each pair of adjacent $x, y \in X$, $f(x)$ and $f(y)$ are adjacent subsets of Y .*
- *F has strong continuity if for each pair of adjacent $x, y \in X$, every point of $f(x)$ is adjacent or equal to some point of $f(y)$ and every point of $f(y)$ is adjacent or equal to some point of $f(x)$.* □

Proposition 2.22 ([15]). *Let $F : X \multimap Y$ be a multivalued function between digital images. Then F is connectivity preserving if and only if F has weak continuity and for all $x \in X$, $F(x)$ is connected.* □

Example 2.23 ([15]). *If $F : [0, 1]_{\mathbb{Z}} \multimap [0, 2]_{\mathbb{Z}}$ is defined by $F(0) = \{0, 2\}$, $F(1) = \{1\}$, then F has both weak and strong continuity. Thus a multivalued function between digital images that has weak or strong continuity need not have connected point-images. By Theorem 2.12 and Proposition 2.16 it follows that neither having weak continuity nor having strong continuity implies that a multivalued function is connectivity preserving or continuous.* □

Example 2.24 ([15]). Let $F : [0, 1]_{\mathbb{Z}} \multimap [0, 2]_{\mathbb{Z}}$ be defined by $F(0) = \{0, 1\}$, $F(1) = \{2\}$. Then F is continuous and has weak continuity but does not have strong continuity. \square

Proposition 2.25 ([15]). Let $F : X \multimap Y$ be a multivalued function between digital images. If F has strong continuity and for each $x \in X$, $F(x)$ is connected, then F is connectivity preserving. \square

The following shows that not requiring the images of points to be connected yields topologically unsatisfying consequences for weak and strong continuity.

Example 2.26 ([15]). Let X and Y be nonempty digital images. Let the multivalued function $f : X \multimap Y$ be defined by $f(x) = Y$ for all $x \in X$.

- f has both weak and strong continuity.
- f is connectivity preserving if and only if Y is connected. \square

As a specific example [15] consider $X = \{0\} \subset \mathbb{Z}$ and $Y = \{0, 2\}$, all with c_1 adjacency. Then the function $F : X \multimap Y$ with $F(0) = Y$ has both weak and strong continuity, even though it maps a connected image surjectively onto a disconnected image.

2.7. Shy maps and their inverses.

Definition 2.27 ([5]). Let $f : X \rightarrow Y$ be a continuous surjection of digital images. We say f is shy if

- for each $y \in Y$, $f^{-1}(y)$ is connected, and
- for every $y_0, y_1 \in Y$ such that y_0 and y_1 are adjacent, $f^{-1}(\{y_0, y_1\})$ is connected. \square

Shy maps induce surjections on fundamental groups [5]. Some relationships between shy maps f and their inverses f^{-1} as multivalued functions were studied in [8, 15, 9]. We have the following.

Theorem 2.28 ([15, 9]). Let $f : X \rightarrow Y$ be a continuous surjection between digital images. Then the following are equivalent.

- f is a shy map.
- For every connected $Y_0 \subset Y$, $f^{-1}(Y_0)$ is a connected subset of X .
- $f^{-1} : Y \multimap X$ is a connectivity preserving multi-valued function.
- $f^{-1} : Y \multimap X$ is a multi-valued function with weak continuity such that for all $y \in Y$, $f^{-1}(y)$ is a connected subset of X . \square

2.8. Other tools. Other terminology we use includes the following. Given a digital image $(X, \kappa) \subset \mathbb{Z}^n$ and $x \in X$, the set of points adjacent to $x \in \mathbb{Z}^n$ and the neighborhood of x in \mathbb{Z}^n are, respectively,

$$N_{\kappa}(x) = \{y \in \mathbb{Z}^n \mid y \text{ is } \kappa\text{-adjacent to } x\},$$

$$N_{\kappa}^*(x) = N_{\kappa}(x) \cup \{x\}.$$

3. EXTENSIONS OF NORMAL PRODUCT ADJACENCY

In this section, we define extensions of the normal product adjacency, as follows.

Definition 3.1. Let u and v be positive integers, $1 \leq u \leq v$. Let $\{(X_i, \kappa_i)\}_{i=1}^v$ be digital images. Let $NP_u(\kappa_1, \dots, \kappa_v)$ be the adjacency defined on the Cartesian product $\prod_{i=1}^v X_i$ as follows. For $x_i, x'_i \in X_i$, $p = (x_1, \dots, x_v)$ and $q = (x'_1, \dots, x'_v)$ are $NP_u(\kappa_1, \dots, \kappa_v)$ -adjacent if and only if

- For at least 1 and at most u indices i , x_i and x'_i are κ_i -adjacent, and
- for all other indices i , $x_i = x'_i$. □

Throughout this paper, the reader should be careful to note that some of our results for $NP_u(\kappa_1, \dots, \kappa_v)$ are stated for all $u \in \{1, \dots, v\}$ and others are stated only for $u = 1$ or $u = 2$ or $u = v$.

Proposition 3.2. $NP(\kappa, \lambda) = NP_2(\kappa, \lambda)$. I.e., given $x, x' \in (X, \kappa)$ and $y, y' \in (Y, \lambda)$, $p = (x, y)$ and $p' = (x', y')$ are $NP(\kappa, \lambda)$ -adjacent in $X \times Y$ if and only if p and p' are $NP_2(\kappa, \lambda)$ -adjacent.

Proof. This follows immediately from Definitions 2.1 and 3.1. □

Theorem 3.3 ([13]). For $X \in \mathbb{Z}^m$, $Y \in \mathbb{Z}^n$, $NP_2(c_m, c_n) = c_{m+n}$, i.e., the normal product adjacency for $(X, c_m) \times (Y, c_n)$ coincides with the c_{m+n} -adjacency for $X \times Y$. □

Examples are also given in [13] that show that if $X \in \mathbb{Z}^m$, $Y \in \mathbb{Z}^n$, and $a < m$ or $b < n$, then $NP_2(c_a, c_b) \neq c_{a+b}$.

The following shows that NP_v obeys a recursive property.

Proposition 3.4. Let $v > 2$. Then

$$NP_v(\kappa_1, \dots, \kappa_v) = NP_2(NP_{v-1}(\kappa_1, \dots, \kappa_{v-1}), \kappa_v).$$

Proof. Let $x_i, x'_i \in X_i$ for $1 \leq i \leq v$. Then $p = (x_1, \dots, x_v)$ and $p' = (x'_1, \dots, x'_v)$ are $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent if and only if for at least 1 and at most v indices i , x_i and x'_i are κ_i -adjacent and for all other indices i , $x_i = x'_i$. Hence $p = (x_1, \dots, x_v)$ and $p' = (x'_1, \dots, x'_v)$ are $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent if and only if either

- x_i and x'_i are κ_i -adjacent for from 1 to $v-1$ indices among $\{1, \dots, v-1\}$, $x_i = x'_i$ for all other indices among $\{1, \dots, v-1\}$, and $x_v = x'_v$; or
- x_i and x'_i are κ_i -adjacent for from 1 to $v-1$ indices among $\{1, \dots, v-1\}$, $x_i = x'_i$ for all other indices among $\{1, \dots, v-1\}$, and x_v and x'_v are κ_v -adjacent.

Thus, $p = (x_1, \dots, x_v)$ and $p' = (x'_1, \dots, x'_v)$ are $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent if and only if p and p' are $NP_2(NP_{v-1}(\kappa_1, \dots, \kappa_{v-1}), \kappa_v)$ -adjacent. □

Notice Proposition 3.4 may fail to extend to $NP_u(\kappa_1, \dots, \kappa_v)$ if $u < v$, as shown in the following (suggested by an example in [13]).

Example 3.5. Let $x_i, x'_i \in (X_i, \kappa_i)$, $i \in \{1, 2, 3\}$. Suppose x_1 and x'_1 are κ_1 -adjacent, x_2 and x'_2 are κ_2 -adjacent, and $x_3 = x'_3$. Then (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) are $NP_2(\kappa_1, \kappa_2, \kappa_3)$ -adjacent in $X_1 \times X_2 \times X_3$, but (x_1, x_2) and (x'_1, x'_2) are not $NP_1(\kappa_1, \kappa_2)$ -adjacent in $X_1 \times X_2$. Thus,

$$NP_2(\kappa_1, \kappa_2, \kappa_3) \neq NP_2(NP_1(\kappa_1, \kappa_2), \kappa_3). \quad \square$$

Theorem 3.6. Let $f, g : (X, \kappa) \rightarrow (Y, \lambda)$ be functions. Let $H : X \times [0, m]_{\mathbb{Z}} \rightarrow Y$ be a function such that $H(x, 0) = f(x)$ and $H(x, m) = g(x)$ for all $x \in X$. Then H is a homotopy if and only if H is $(NP_1(\kappa, c_1), \lambda)$ -continuous.

Proof. In the following, we consider arbitrary $(NP_1(\kappa, c_1), \lambda)$ -adjacent (x, t) and (x', t') in $X \times [0, m]_{\mathbb{Z}}$ with $x, x' \in X$ and $t, t' \in [0, m]_{\mathbb{Z}}$. Such points offer the following cases.

- (1) x and x' are κ -adjacent and $t = t'$; or
- (2) $x = x'$ and t and t' are c_1 -adjacent, i.e., $|t - t'| = 1$.

Let H be a homotopy. Then f and g are continuous, and given $(NP_1(\kappa, c_1), \lambda)$ -adjacent (x, t) and (x', t') in $X \times [0, m]_{\mathbb{Z}}$, we consider the cases listed above.

- In case 1, since H is a homotopy, $H(x, t)$ and $H(x', t) = H(x', t')$ are equal or λ -adjacent.
- In case 2, since H is a homotopy, $H(x, t)$ and $H(x', t') = H(x, t')$ are equal or λ -adjacent.

Therefore, H is $(NP_1(\kappa, c_1), \lambda)$ -continuous.

Suppose H is $(NP_1(\kappa, c_1), \lambda)$ -continuous. Then for κ -adjacent x, x' , $f(x) = H(x, 0)$ and $f(x') = H(x', 0)$ are equal or λ -adjacent, so f is continuous. Similarly, $g(x) = H(x, m)$ and $g(x') = H(x', m)$ are equal or λ -adjacent, so g is continuous. Also, the continuity of H implies that $H(x, t)$ and $H(x', t)$ must be equal or λ -adjacent, so the induced function H_t is (κ, λ) -continuous. For c_1 -adjacent t, t' , the continuity of H implies that $H(x, t)$ and $H(x, t')$ are equal or λ -adjacent, so the induced function H_x is continuous. By Definition 2.8, H is a homotopy. \square

4. NP_u AND MAPS ON PRODUCTS

Given functions $f_i : (X_i, \kappa_i) \rightarrow (Y_i, \lambda_i)$, $1 \leq i \leq v$, the function

$$\prod_{i=1}^v f_i : (\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v)) \rightarrow (\prod_{i=1}^v Y_i, NP_u(\lambda_1, \dots, \lambda_v))$$

is defined by

$$\prod_{i=1}^v f_i(x_1, \dots, x_v) = (f_1(x_1), \dots, f_v(x_v)), \text{ where } x_i \in X_i.$$

The following generalizes a result in [9, 13].

Theorem 4.1. Let $f_i : (X_i, \kappa_i) \rightarrow (Y_i, \lambda_i)$, $1 \leq i \leq v$. Then the product map

$$f = \prod_{i=1}^v f_i : (\prod_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_v)) \rightarrow (\prod_{i=1}^v Y_i, NP_v(\lambda_1, \dots, \lambda_v))$$

is continuous if and only if each f_i is continuous.

Proof. In the following, we let $p = (x_1, \dots, x_v)$ and $p' = (x'_1, \dots, x'_v)$, where $x_i, x'_i \in X_i$.

Suppose each f_i is continuous and p and p' are $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent. Then for all indices i , x_i and x'_i are equal or κ_i -adjacent, so $f_i(x_i)$ and $f_i(x'_i)$ are equal or λ_i -adjacent. Therefore, $f(p)$ and $f(p')$ are equal or $NP_v(\lambda_1, \dots, \lambda_v)$ -adjacent. Thus, f is continuous.

Suppose f is continuous and for all indices i , x_i and x'_i are κ_i -adjacent. Then $f(p)$ and $f(p')$ are equal or $NP_v(\lambda_1, \dots, \lambda_i)$ -adjacent. Therefore, for each index i , $f_i(x_i)$ and $f_i(x'_i)$ are equal or λ_i -adjacent. Thus, each f_i is continuous. \square

The statement analogous to Theorem 4.1 is not generally true if c_u -adjacencies are used instead of normal product adjacencies, as shown in the following.

Example 4.2. Let $X = \{(0, 0), (1, 0)\} \subset \mathbb{Z}^2$. Let $Y = \{(0, 0), (1, 1)\} \subset \mathbb{Z}^2$. Clearly, there is an isomorphism $f : (X, c_2) \rightarrow (Y, c_2)$. Consider $X' = X \times \{0\} \subset \mathbb{Z}^3$ and $Y' = Y \times \{0\} \subset \mathbb{Z}^3$. Note that the product map $f \times 1_{\{0\}}$ is not (c_1, c_1) -continuous, since X' is c_1 -connected and $Y' = (f \times 1_{\{0\}})(X')$ is not c_1 -connected. \square

The following is a generalization of a result of [25].

Theorem 4.3. The projection maps $p_i : (\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v)) \rightarrow (X_i, \kappa_i)$ defined by $p_i(x_1, \dots, x_v) = x_i$ for $x_i \in (X_i, \kappa_i)$, are all continuous, for $1 \leq u \leq v$.

Proof. Let $p = (x_1, \dots, x_v)$ and $p' = (x'_1, \dots, x'_v)$ be $NP_u(\kappa_1, \dots, \kappa_v)$ -adjacent in $(\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v))$, where $x_i, x'_i \in X_i$. Then for all indices i , $x_i = p_i(p)$ and $x'_i = p_i(p')$ are equal or κ_i -adjacent. Thus, p_i is continuous. \square

The statement analogous to Theorem 4.3 is not generally true if a c_u -adjacency is used instead of a normal product adjacency, as shown in the following.

Example 4.4 ([13]). Let $X = [0, 1]_{\mathbb{Z}} \subset \mathbb{Z}$. Let $Y = \{(0, 0), (1, 1)\} \subset \mathbb{Z}^2$. Then the projection map $p_2 : (X \times Y, c_3) \rightarrow (Y, c_1)$ is not continuous, since $X \times Y$ is c_3 -connected and Y is not c_1 -connected. \square

We see in the next result that isomorphism is preserved by taking Cartesian products with a normal product adjacency.

Theorem 4.5. Let $X = \prod_{i=1}^v X_i$. Let $f_i : (X_i, \kappa_i) \rightarrow (Y_i, \lambda_i)$, $1 \leq i \leq v$.

- For $1 \leq u \leq v$, if the product map $f = \prod_{i=1}^v f_i : (X, NP_u(\kappa_1, \dots, \kappa_v)) \rightarrow (\prod_{i=1}^v Y_i, NP_u(\lambda_1, \dots, \lambda_v))$ is an isomorphism, then for $1 \leq i \leq v$, f_i is an isomorphism.
- If f_i is an isomorphism for all i , then the product map $f = \prod_{i=1}^v f_i : (X, NP_v(\kappa_1, \dots, \kappa_v)) \rightarrow (\prod_{i=1}^v Y_i, NP_v(\lambda_1, \dots, \lambda_v))$ is an isomorphism.

Proof. Let f be an isomorphism. Then each f_i must be one-to-one and onto.

Let $x_i \in X_i$. Let $I_i : X_i \rightarrow X$ be defined by

$$I_i(x) = (x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_v).$$

Define $I'_i : Y_i \rightarrow Y$ similarly. Clearly, I_i is $(\kappa_i, NP_u(\kappa_1, \dots, \kappa_v))$ -continuous and I'_i is $(\lambda_i, NP_u(\lambda_1, \dots, \lambda_v))$ -continuous. Let $p_i : X \rightarrow X_i$ and $p'_i : Y \rightarrow Y_i$ be the projections to the i -th coordinate. By Theorems 2.4 and 4.3, $f_i = p'_i \circ f \circ I_i$ and $f_i^{-1} = p_i \circ f^{-1} \circ I'_i$ are continuous. Hence, f_i is an isomorphism.

Let $f_i : X_i \rightarrow Y_i$ be an isomorphism. One sees easily that f is one-to-one and onto, and by Theorem 4.1, f is continuous. The inverse function f^{-1} is the product function of the f_i^{-1} , hence is continuous by Theorem 4.1. Thus, f is an isomorphism. \square

The statement analogous to Theorem 4.5 is not generally true for all c_u -adjacencies, as shown by the following.

Example 4.6. Let $X = \{(0, 0), (1, 1)\} \subset \mathbb{Z}^2$. Let $Y = \{(0, 0), (1, 0)\} \subset \mathbb{Z}^2$. Clearly, (X, c_2) and (Y, c_2) are isomorphic. Consider $X' = X \times \{0\} \subset \mathbb{Z}^3$ and $Y' = Y \times \{0\} \subset \mathbb{Z}^3$. Note (X', c_1) and (Y', c_1) are not isomorphic, since the former is c_1 -disconnected and the latter is c_1 -connected. \square

5. NP_v AND CONNECTEDNESS

Theorem 5.1. Let (X_i, κ_i) be digital images, $i \in \{1, 2, \dots, v\}$. Then (X_i, κ_i) is connected for all i if and only $(\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v))$ is connected.

Proof. Suppose (X_i, κ_i) is connected for all i . Let $x_i, x'_i \in X_i$. Then there are paths P_i in X_i from x_i to x'_i . Let $p = (x_1, \dots, x_v), p' = (x'_1, \dots, x'_v) \in \prod_{i=1}^v X_i$. Then $\bigcup_{i=1}^v P'_i$, where

$$\begin{aligned} P'_1 &= P_1 \times \{(x_2, \dots, x_v)\}, \\ P'_i &= \{(x'_1, \dots, x'_{i-1})\} \times P_i \times \{(x_{i+1}, \dots, x_v)\} \text{ for } 2 \leq i < v, \\ P'_v &= \{(x'_1, \dots, x'_{v-1})\} \times P_v, \end{aligned}$$

is a path in $\prod_{i=1}^v X_i$ from p to p' . Since p and p' are arbitrarily selected points in $\prod_{i=1}^v X_i$, it follows that $(\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v))$ is connected.

If $(\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v))$ is connected, then $(X_i, \kappa_i) = p_i(\prod_{i=1}^v X_i)$ is connected, by Definition 2.2 and Theorem 4.3. \square

The statement analogous to Theorem 5.1 is not generally true if a c_u -adjacency is used instead of $NP_u(\kappa_1, \dots, \kappa_v)$ for $X \times Y$, as shown by the following.

Example 5.2 ([13]). Let $X = [0, 1]_{\mathbb{Z}}$, $Y = \{(0, 0), (1, 1)\} \subset \mathbb{Z}^2$. Then $X \times Y$ is c_2 -connected, but Y is not c_1 -connected. Also, X is c_1 -connected and Y is c_2 -connected, but $X \times Y$ is not c_1 -connected. \square

6. NP_v AND HOMOTOPY RELATIONS

In this section, we show that normal products preserve a variety of digital homotopy relations. These include homotopy type and several generalizations introduced in [17]. These generalizations - homotopic similarity, long homotopy type, and real homotopy type - all coincide with homotopy type on pairs of finite digital images; however, for each of these relationships, an example is given

in [17] of a pair of digital images, at least one member of which is infinite, such that the two images have the given relation but are not homotopy equivalent.

By contrast with Euclidean topology, in which a bounded space such as a single point and an unbounded space such as \mathbb{R}^n with Euclidean topology can have the same homotopy type, a finite digital image and an image with infinite diameter - e.g., a single point and (\mathbb{Z}^n, c_1) - cannot share the same homotopy type. However, examples in [17] show that a finite digital image and an image with infinite diameter can share homotopic similarity, long homotopy type, or real homotopy type.

6.1. Homotopic maps and homotopy type.

Theorem 6.1. *Let (X_i, κ_i) and (Y_i, λ_i) be digital images, $1 \leq i \leq v$. Let $X = (\prod_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_i))$. Let $Y = (\prod_{i=1}^v Y_i, NP_v(\lambda_1, \dots, \lambda_i))$. Let $f_i, g_i : X_i \rightarrow Y_i$ be continuous and let $H_i : X_i \times [0, m_i]_{\mathbb{Z}} \rightarrow Y_i$ be a homotopy from f_i to g_i . Then there is a homotopy H between the product maps $F = \prod_{i=1}^v f_i : X \rightarrow Y$ and $G = \prod_{i=1}^v g_i : X \rightarrow Y$. If the homotopies H_i are pointed, then H is pointed.*

Proof. Let $M = \max\{m_i\}_{i=1}^v$. Let $H'_i : X_i \times [0, M]_{\mathbb{Z}} \rightarrow Y_i$ be defined by

$$H'_i(x, t) = \begin{cases} H_i(x, t) & \text{for } 0 \leq t \leq m_i; \\ H_i(x, m_i) & \text{for } m_i \leq t \leq M. \end{cases}$$

Clearly, H'_i is a homotopy from f_i to g_i .

Let $H : X \times M \rightarrow Y$ be defined by

$$H((x_1, \dots, x_v), t) = (H'_1(x_1, t), \dots, H'_v(x_v, t)).$$

It is easily seen that H is a homotopy from F to G , and that if each H_i is pointed, then H is pointed. \square

The following theorem generalizes results of [17].

Theorem 6.2. *Suppose*

$$(6.1) \quad X_i \simeq_{\kappa_i, \lambda_i} Y_i \text{ for } 1 \leq i \leq v.$$

Then

$$(6.2) \quad X = \prod_{i=1}^v X_i \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v)} Y = \prod_{i=1}^v Y_i.$$

Further, if the homotopy equivalences (6.1) are all pointed with respect to $x_i \in X_i$ and $y_i \in Y_i$, then the homotopy equivalence (6.2) is pointed with respect to $(x_1, \dots, x_v) \in X$ and $(y_1, \dots, y_v) \in Y$.

Proof. We give a proof for the unpointed assertion. With minor modifications, the pointed assertion is proven similarly.

By hypothesis, there exist continuous functions $f_i : X_i \rightarrow Y_i$ and $g_i : Y_i \rightarrow X_i$ and homotopies $H_i : X_i \times [0, m_i]_{\mathbb{Z}} \rightarrow X_i$ from $g_i \circ f_i$ to 1_{X_i} and $K_i : Y_i \times [0, n_i]_{\mathbb{Z}} \rightarrow Y_i$ from $f_i \circ g_i$ to 1_{Y_i} .

Let $M = \max\{m_i\}_{i=1}^v$. Then $H'_i : X_i \times [0, M]_{\mathbb{Z}} \rightarrow X_i$, defined by

$$H'_i(x, t) = \begin{cases} H_i(x, t) & \text{for } 0 \leq t \leq m_i; \\ H_i(x, m_i) & \text{for } m_i \leq t \leq M, \end{cases}$$

is clearly a homotopy from $g_i \circ f_i$ to 1_{X_i} .

Let $F = \prod_{i=1}^v f_i : X \rightarrow Y$. Let $G = \prod_{i=1}^v g_i : Y \rightarrow X$. By Theorem 4.1, F and G are continuous. Let $H : X \times [0, M]_{\mathbb{Z}} \rightarrow X$ be defined by

$$H(x_1, \dots, x_v, t) = (H'_1(x_1, t), \dots, H'_v(x_v, t)).$$

Then H is easily seen to be a homotopy from $G \circ F$ to $\prod_{i=1}^v 1_{X_i} = 1_X$.

We can similarly show that $F \circ G \simeq 1_Y$. Therefore, $X \simeq Y$. □

The statements analogous to Theorems 6.1 and 6.2 are not generally true if a c_u -adjacency is used instead of a normal product adjacency for the Cartesian product. Consider, e.g., X and Y as in Example 4.6. Let $f : Y \rightarrow X$ be a (c_1, c_2) -isomorphism. Then f is (c_1, c_2) -homotopic to the constant map $\overline{(0, 0)}$ of Y to $(0, 0)$. However, $f \times 1_{\{0\}}$ is not even (c_1, c_1) -continuous, hence is not (c_1, c_1) -homotopic to $\overline{(0, 0)} \times 1_{\{0\}}$. Although X and Y are (c_2, c_1) -homotopy equivalent, $X' = X \times \{0\}$ and $Y' = Y \times \{0\}$ are not (c_1, c_1) -homotopy equivalent, since X' is not c_1 -connected and Y' is c_1 -connected.

6.2. Homotopic similarity.

Definition 6.3 ([17]). *Let X and Y be digital images. We say (X, κ) and (Y, λ) are homotopically similar, denoted $X \simeq_{\kappa, \lambda}^s Y$, if there exist subsets $\{X_j\}_{j=1}^{\infty}$ of X and $\{Y_j\}_{j=1}^{\infty}$ of Y such that:*

- $X = \bigcup_{j=1}^{\infty} X_j, Y = \bigcup_{j=1}^{\infty} Y_j$, and, for all $j, X_j \subset X_{j+1}, Y_j \subset Y_{j+1}$.
- There are continuous functions $f_j : X_j \rightarrow Y_j, g_j : Y_j \rightarrow X_j$ such that $g_j \circ f_j \simeq_{\kappa, \kappa} 1_{X_j}$ and $f_j \circ g_j \simeq_{\lambda, \lambda} 1_{Y_j}$.
- For $m \leq n, f_n|_{X_m} \simeq_{\kappa, \lambda} f_m$ in Y_m and $g_n|_{Y_m} \simeq_{\lambda, \kappa} g_m$ in X_m .

If all of these homotopies are pointed with respect to some $x_1 \in X_1$ and $y_1 \in Y_1$, we say (X, x_1) and (Y, y_1) are pointed homotopically similar, denoted $(X, x_1) \simeq_{\kappa, \lambda}^s (Y, y_1)$ or $(X, x_1) \simeq^s (Y, y_1)$ when κ and λ are understood. □

Theorem 6.4. *Let $X_i \simeq_{\kappa_i, \lambda_i}^s Y_i, 1 \leq i \leq v$. Let $X = \prod_{i=1}^v X_i, Y = \prod_{i=1}^v Y_i$. Then $X \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v)}^s Y$. If the similarities $X_i \simeq_{\kappa_i, \lambda_i}^s Y_i$ are pointed at $x_i \in X_i, y_i \in Y_i$, then the similarity $X \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v)}^s Y$ is pointed at $x_0 = (x_1, \dots, x_v) \in X, y_0 = (y_1, \dots, y_v) \in Y$.*

Proof. We give a proof for the unpointed assertion. A virtually identical argument can be given for the pointed assertion.

By hypothesis, for $j \in \mathbb{N}$ there exist digital images $X_{i,j} \subset X_i, Y_{i,j} \subset Y_i$ such that $X_{i,j} \subset X_{i,j+1}, X_i = \bigcup_{j=1}^{\infty} X_{i,j}, Y_{i,j} \subset Y_{i,j+1}, Y_i = \bigcup_{j=1}^{\infty} Y_{i,j}$, and continuous functions $f_{i,j} : X_{i,j} \rightarrow Y_{i,j}, g_{i,j} : Y_{i,j} \rightarrow X_{i,j}$, such that $g_{i,j} \circ f_{i,j} \simeq_{\kappa_i, \kappa_i} 1_{X_{i,j}}, f_{i,j} \circ g_{i,j} \simeq_{\lambda_i, \lambda_i} 1_{Y_{i,j}}$, and $m \leq n$ implies $f_{i,n}|_{X_{i,m}} \simeq_{\kappa_i, \lambda_i} f_{i,m}$ in $Y_{i,m}$ and $g_{i,n}|_{Y_{i,m}} \simeq_{\lambda_i, \kappa_i} g_{i,m}$ in $X_{i,m}$.

Let $X_j = \prod_{i=1}^v X_{i,j}, Y_j = \prod_{i=1}^v Y_{i,j}$. Clearly we have $X = \bigcup_{j=1}^{\infty} X_j, Y = \bigcup_{j=1}^{\infty} Y_j, X_j \subset X_{j+1}, Y_j \subset Y_{j+1}$. Let $f_j = \prod_{i=1}^v f_{i,j} : X_j \rightarrow Y_j, g_j =$

$\prod_{i=1}^v g_{i,j} : Y_j \rightarrow X_j$. By Theorem 4.1, f_j is $(NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v))$ -continuous and g_j is $(NP_v(\lambda_1, \dots, \lambda_v), NP_v(\kappa_1, \dots, \kappa_v))$ -continuous. By Theorem 6.1,

$$g_j \circ f_j \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\kappa_1, \dots, \kappa_v)} 1_{X_j} \text{ and } f_j \circ g_j \simeq_{NP_v(\lambda_1, \dots, \lambda_v), NP_v(\lambda_1, \dots, \lambda_v)} 1_{Y_j}.$$

Also by Theorem 6.1, $m \leq n$ implies $f_n|X_m \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v)} f_m$ in Y_m and $g_n|Y_m \simeq_{NP_v(\lambda_1, \dots, \lambda_v), NP_v(\kappa_1, \dots, \kappa_v)} g_m$ in X_m . This completes the proof. \square

6.3. Long homotopy type.

Definition 6.5 ([17]). Let (X, κ) and (Y, λ) be digital images. Let $f, g : X \rightarrow Y$ be continuous. Let $F : X \times \mathbb{Z} \rightarrow Y$ be a function such that

- for all $x \in X$, there exists $N_{F,x} \in \mathbb{N}$ such that $t \leq -N_{F,x}$ implies $F(x, t) = f(x)$ and $t \geq N_{F,x}$ implies $F(x, t) = g(x)$.
- For all $x \in X$, the induced function $F_x : \mathbb{Z} \rightarrow Y$ defined by

$$F_x(t) = F(x, t) \text{ for all } t \in \mathbb{Z}$$

is (c_1, λ) -continuous.

- For all $t \in \mathbb{Z}$, the induced function $F_t : X \rightarrow Y$ defined by

$$F_t(x) = F(x, t) \text{ for all } x \in X$$

is (κ, λ) -continuous.

Then F is a long homotopy from f to g . If for some $x_0 \in X$ and $y_0 \in Y$ we have $F(x_0, t) = y_0$ for all $t \in \mathbb{N}^*$, we say F is a pointed long homotopy. We write $f \simeq_{\kappa, \lambda}^L g$, or $f \simeq^L g$ when the adjacencies κ and λ are understood, to indicate that f and g are long homotopic functions. \square

We have the following.

Theorem 6.6. Let $f_i, g_i : (X_i, \kappa_i) \rightarrow (Y_i, \lambda_i)$ be continuous functions that are long homotopic, $1 \leq i \leq v$. Then $f = \prod_{i=1}^v f_i$ and $g = \prod_{i=1}^v g_i$ are long homotopic maps from $(\prod_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_v))$ to $(\prod_{i=1}^v Y_i, NP_v(\lambda_1, \dots, \lambda_v))$. If the long homotopies $f_i \simeq^L g_i$ are pointed with respect to $x_i \in X_i$ and $y_i \in Y_i$, then the long homotopy $f \simeq^L g$ is pointed with respect to $(x_1, \dots, x_v) \in \prod_{i=1}^v X_i$ and $(y_1, \dots, y_v) \in \prod_{i=1}^v Y_i$.

Proof. We give a proof for the unpointed assertion. Minor modifications yield a proof for the pointed assertion.

Let $h_i : X_i \times \mathbb{Z} \rightarrow Y_i$ be a long homotopy from f_i to g_i . For all $x_i \in X_i$, there exists $N_{F_i, x_i} \in \mathbb{N}$ such that $t \leq -N_{F_i, x_i}$ implies $h_i(x_i, t) = f_i(x_i)$ and $t \geq N_{F_i, x_i}$ implies $h_i(x_i, t) = g_i(x_i)$. For all $x = (x_1, \dots, x_v) \in \prod_{i=1}^v X_i$, let $N_x = \max\{N_{F_i, x_i} \mid 1 \leq i \leq v\}$. Let $h = \prod_{i=1}^v h_i : \prod_{i=1}^v X_i \times \mathbb{Z} \rightarrow \prod_{i=1}^v Y_i$. Then $t \leq -N_x$ implies $h(x, t) = f(x)$ and $t \geq N_x$ implies $h(x, t) = g(x)$.

For all $x \in \prod_{i=1}^v X_i$, the induced function $h_x(t) = (h_1(x_1, t), \dots, h_v(x_v, t))$ is $(c_1, NP_v(\lambda_1, \dots, \lambda_v))$ -continuous, by an argument similar to that given in the proof of Theorem 6.1.

For all $t \in \mathbb{Z}$, the induced function $h_t(x) = (h_i(x_1, t), \dots, h_v(x_v, t))$ is $(NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v))$ -continuous, by an argument similar to that given in the proof of Theorem 6.1. The assertion follows. \square

Definition 6.7 ([17]). Let $f : (X, \kappa) \rightarrow (Y, \lambda)$ and $g : (Y, \lambda) \rightarrow (X, \kappa)$ be continuous functions. Suppose $g \circ f \simeq^L 1_X$ and $f \circ g \simeq^L 1_Y$. Then we say (X, κ) and (Y, λ) have the same long homotopy type, denoted $X \simeq_{\kappa, \lambda}^L Y$ or simply $X \simeq^L Y$. If there exist $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$, $g(y_0) = x_0$, the long homotopy $g \circ f \simeq^L 1_X$ holds x_0 fixed, and the long homotopy $f \circ g \simeq^L 1_Y$ holds y_0 fixed, then (X, x_0, κ) and (Y, y_0, λ) have the same pointed long homotopy type, denoted $(X, x_0) \simeq_{\kappa, \lambda}^L (Y, y_0)$ or $(X, x_0) \simeq^L (Y, y_0)$. \square

Theorem 6.8. Let $X_i \simeq_{\kappa_i, \lambda_i}^L Y_i$, $1 \leq i \leq v$. Let $X = \prod_{i=1}^v X_i$, $Y = \prod_{i=1}^v Y_i$. Then $X \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v)}^L Y$. If for each i the long homotopy equivalence $X_i \simeq_{\kappa_i, \lambda_i}^L Y_i$ is pointed with respect to $x_i \in X_i$ and $y_i \in Y_i$, then the long homotopy equivalence $X \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v)}^L Y$ is pointed with respect to $x_0 = (x_1, \dots, x_v) \in X$ and $y_0 = (y_1, \dots, y_v) \in Y$.

Proof. This follows easily from Definition 6.7 and Theorem 6.6. \square

6.4. Real homotopy type.

Definition 6.9 ([17]). Let (X, κ) be a digital image, and $[0, 1] \subset \mathbb{R}$ be the unit interval. A function $f : [0, 1] \rightarrow X$ is a real [digital] $[\kappa]$ -path in X if:

- there exists $\epsilon_0 > 0$ such that f is constant on $(0, \epsilon_0)$ with constant value equal or κ -adjacent to $f(0)$, and
- for each $t \in (0, 1)$ there exists $\epsilon_t > 0$ such that f is constant on each of the intervals $(t - \epsilon_t, t)$ and $(t, t + \epsilon_t)$, and these two constant values are equal or κ -adjacent, with at least one of them equal to $f(t)$, and
- there exists $\epsilon_1 > 0$ such that f is constant on $(1 - \epsilon_1, 1)$ with constant value equal or κ -adjacent to $f(1)$.

If $t = 0$ and $f(0) \neq f((0, \epsilon_0))$, or $0 < t < 1$ and the two constant values $f((t - \epsilon_t, t))$ and $f((t, t + \epsilon_t))$ are not equal, or $t = 1$ and $f(1) \neq f((1 - \epsilon_1, 1))$, we say t is a jump of f .

Proposition 6.10 ([17]). Let $p, q \in (X, \kappa)$. Let $f : [a, b] \rightarrow X$ be a real κ -path from p to q . Then the number of jumps of f is finite.

Definition 6.11 ([17]). Let (X, κ) and (Y, κ') be digital images, and let $f, g : X \rightarrow Y$ be (κ, κ') continuous. Then a real [digital] homotopy of f and g is a function $F : X \times [0, 1] \rightarrow Y$ such that:

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$
- for all $x \in X$, the induced function $F_x : [0, 1] \rightarrow Y$ defined by

$$F_x(t) = F(x, t) \text{ for all } t \in [0, 1]$$

is a real κ -path in X .

- for all $t \in [0, 1]$, the induced function $F_t : X \rightarrow Y$ defined by

$$F_t(x) = F(x, t) \text{ for all } x \in X$$

is (κ, κ') -continuous.

If such a function exists we say f and g are real homotopic and write $f \simeq^{\mathbb{R}} g$. If there are points $x_0 \in X$ and $y_0 \in Y$ such that $F(x_0, t) = y_0$ for all $t \in [0, 1]$, we say f and g are pointed real homotopic.

Definition 6.12 ([17]). We say digital images (X, κ) and (Y, κ') have the same real homotopy type, denoted $X \simeq_{\kappa, \kappa'}^{\mathbb{R}} Y$ or $X \simeq^{\mathbb{R}} Y$ when κ and κ' are understood, if there are continuous functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f \simeq^{\mathbb{R}} 1_X$ and $f \circ g \simeq^{\mathbb{R}} 1_Y$. If there exist $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$, $g(y_0) = x_0$, and the real homotopies above are pointed with respect to x_0 and y_0 , we say X and Y have the same pointed real homotopy type, denoted $(X, x_0) \simeq_{\kappa, \kappa'}^{\mathbb{R}} (Y, y_0)$ or $(X, x_0) \simeq^{\mathbb{R}} (Y, y_0)$.

Theorem 6.13. Suppose

$$(6.3) \quad X_i \simeq_{\kappa_i, \lambda_i}^{\mathbb{R}} Y_i \text{ for } 1 \leq i \leq v.$$

Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. Then

$$(6.4) \quad X \simeq_{NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v)}^{\mathbb{R}} Y.$$

If the equivalences (6.3) are all pointed with respect to $x_i \in X_i$ and $y_i \in Y_i$, then the equivalence (6.4) is pointed with respect to $x_0 = (x_1, \dots, x_v) \in X$ and $y_0 = (y_1, \dots, y_v) \in Y$.

Proof. We give a proof for the unpointed assertion. With minor modifications, the same argument yields the pointed assertion.

By hypothesis, there exist continuous functions $f_i : X_i \rightarrow Y_i$, $g_i : Y_i \rightarrow X_i$ and real homotopies $h_i : X_i \times [0, 1] \rightarrow X_i$ from $g_i \circ f_i$ to 1_{X_i} , $k_i : Y_i \times [0, 1] \rightarrow Y_i$ from $f_i \circ g_i$ to 1_{Y_i} .

Let $f = \Pi_{i=1}^v f_i : X \rightarrow Y$. Let $g = \Pi_{i=1}^v g_i : Y \rightarrow X$. For $x = (x_1, \dots, x_v) \in X$ with $x_i \in X_i$, let $H : X \times [0, 1] \rightarrow X$ be defined by

$$H(x, t) = (h_1(x_1, t), \dots, h_v(x_v, t)).$$

Then $H(x, 0) = G \circ F(x)$ and $H(x, 1) = x$.

For $x \in X$, the induced function H_x has jumps only at the finitely many (by Proposition 6.10) jumps of the functions h_i . It follows that H_x is a real $NP_v(\lambda_1, \dots, \lambda_v)$ -path in Y .

Let $x' = (x'_1, \dots, x'_v)$ be $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent to x in X . Then, for any $t \in [0, 1]$, $H_t(x') = (H_1(x'_1, t), \dots, H_v(x'_v, t))$ is $NP_v(\lambda_1, \dots, \lambda_v)$ -adjacent to $H_t(x) = (H_1(x_1, t), \dots, H_v(x_v, t))$, since each H_i is a real homotopy. Hence H_t is continuous.

Thus, H is a real homotopy from $G \circ F$ to 1_X . A similar argument lets us conclude that $F \circ G \simeq^{\mathbb{R}} 1_Y$. Therefore, $X \simeq^{\mathbb{R}} Y$. \square

7. NP_v AND RETRACTIONS

Definition 7.1 ([2, 3]). Let $Y \subset (X, \kappa)$. A (κ, κ) -continuous function $r : X \rightarrow Y$ is a retraction, and A is a retract of X , if $r(y) = y$ for all $y \in Y$. \square

Theorem 7.2. Let $A_i \subset (X_i, \kappa_i)$, $i \in \{1, \dots, v\}$. Then A_i is a retract of X_i for all i if and only if $\prod_{i=1}^v A_i$ is a retract of $(\prod_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_v))$.

Proof. Suppose, for all i , A_i is a retract of X_i . Let $r_i : X_i \rightarrow A_i$ be a retraction. Then, by Theorem 4.1, $\prod_{i=1}^v r_i : \prod_{i=1}^v X_i \rightarrow \prod_{i=1}^v A_i$ is continuous, and therefore is easily seen to be a retraction.

Suppose there is a retraction $r : \prod_{i=1}^v X_i \rightarrow \prod_{i=1}^v A_i$. We construct retractions $r_j : X_j \rightarrow A_j$ as follows. Let $a_i \in A_i$. Define $I_j : X_j \rightarrow \prod_{i=1}^v X_i$ by

$$I_j(x) = (a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_v).$$

Clearly, I_j is continuous. Then $r_j = p_j \circ r \circ I_j$ is continuous, by Theorem 2.4 and Corollary 4.3, and is easily seen to be a retraction. \square

Let $A \subset (X, \kappa)$. We say A is a *deformation retract* of X if there is a κ -homotopy $H : X \times [0, m]_{\mathbb{Z}} \rightarrow X$ from 1_X to a retraction of X to A . If $H(a, t) = a$ for all $(a, t) \in Y \times [0, m]_{\mathbb{Z}}$, we say H is a *strong deformation* and A is a *strong deformation retract* of X . We have the following.

Theorem 7.3. Let $A_i \subset (X_i, \kappa_i)$, $i \in \{1, \dots, v\}$. Then A_i is a (strong) deformation retract of X_i for all i if and only if $A = \prod_{i=1}^v A_i$ is a (strong) deformation retract of $X = (\prod_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_v))$.

Proof. Suppose A_i is a deformation retract of X_i , $1 \leq i \leq v$. It follows from Theorems 6.1 and 7.2 and that A is a deformation retract of X . If each A_i is a strong deformation retract of X_i , then by using the argument in the proof of Theorem 6.1 we can construct a homotopy from 1_X to a retraction of X to A that holds every point of a fixed, so A is a strong deformation retract of X .

Suppose A is a (strong) deformation retract of X . This means there is a homotopy $H : X \times [0, m]_{\mathbb{Z}} \rightarrow X$ from 1_X to a retraction r of X onto A (such that $H(a, t) = a$ for all $(a, t) \in A \times [0, m]_{\mathbb{Z}}$). Let $I_i : X_i \rightarrow X$ be as in Theorem 7.2. Let $H_i : X_i \times [0, m]_{\mathbb{Z}} \rightarrow X_i$ be defined by $H_i(x, t) = p_i(H(I_i(x), t))$. Then H_i is a homotopy between $p_i \circ I_i = 1_{X_i}$ and $p_i \circ r \circ I_i$ (such that $H_i(a_i, t) = a_i$ for all $a_i \in A_i$). Since

$$p_i \circ r \circ I_i(X_j) \subset p_i \circ r(X) = p_i(A) = A_i$$

and $a \in A_i$ implies $p_i \circ r \circ I_i(a) = a$, $p_i \circ r \circ I_i$ is a retraction. Thus, A_i is a (strong) deformation retract of X_i . \square

8. NP_v AND THE DIGITAL BORSUK-ULAM THEOREM

The Borsuk-Ulam Theorem of Euclidean topology states that if $f : S^n \rightarrow \mathbb{R}^n$ is a continuous function, where \mathbb{R}^n is n -dimensional Euclidean space and S^n is

the unit sphere in \mathbb{R}^{n+1} , i.e.,

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\},$$

then there exists $x \in S^n$ such that $f(-x) = f(x)$. A “layman’s example” of this theorem for $n = 2$ is that there are opposite points $x, -x$ on the earth’s surface with the same temperature and the same barometric pressure.

We say a set $X \subset \mathbb{Z}^n$ is *symmetric with respect to the origin* if for every $x \in X$, $-x \in X$.

The assertion analogous to the Borsuk-Ulam theorem is not generally true in digital topology. An example is given in [7] of a continuous function $f : S \rightarrow \mathbb{Z}$ from a simple closed curve $S \subset (\mathbb{Z}^2, c_2)$, a digital analog of S^1 , into the digital line \mathbb{Z} , with S symmetric with respect to the origin, such that $f(x) \neq f(-x)$ for all $x \in S$. However, the papers [7, 34] give conditions under which a continuous function f from a digital version S_n of S^n to \mathbb{Z}^n must have a point $x \in S_n$ for which $f(x)$ and $f(-x)$ are equal or adjacent.

In particular, [34] uses the “boundary” of a digital box as a digital model of a Euclidean sphere. Let $B_n = \Pi_{i=1}^n [-a_i, a_i]_{\mathbb{Z}}$, for $a_i \in \mathbb{N}$. Let

$$\delta B_n = \bigcup_{i=1}^n \{(x_1, \dots, x_n) \in B_n \mid x_i \in \{-a_i, a_i\}\}.$$

Theorem 8.1. *We have the following.*

- [7] Let (S, κ) be a digital simple closed curve in \mathbb{Z}^n such that S is symmetric with respect to the origin. Let $f : S \rightarrow \mathbb{Z}$ be a (κ, c_1) -continuous function. Then for some $x \in S$, $f(x)$ and $f(-x)$ are equal or c_1 -adjacent, i.e., $|f(x) - f(-x)| \leq 1$.
- [34] Let $u \in \{1, n-1\}$ and let $f : \delta B_n \rightarrow \mathbb{Z}^{n-1}$ be a (c_n, c_u) -continuous function. Then for some $x \in \delta B_n$, $f(x)$ and $f(-x)$ are equal or c_u -adjacent. \square

Notice that for $X_i \subset \mathbb{Z}^{n_i}$, $\Pi_{i=1}^v X_i$ is symmetric with respect to the origin of $\mathbb{Z}^{\sum_{i=1}^v n_i}$ if and only if X_i is symmetric with respect to the origin of \mathbb{Z}^{n_i} for all indices i .

Suppose $m, n \in \mathbb{N}$, $1 \leq m \leq n$. Let’s say a digital image $S \subset \mathbb{Z}^{n+1}$ that is symmetric with respect to the origin *has the (m, κ, λ) -Borsuk-Ulam property* if for every (κ, λ) -continuous function $f : S \rightarrow \mathbb{Z}^m$ there exists $x \in X$ such that $f(x)$ and $f(-x)$ are equal or λ -adjacent in \mathbb{Z}^m .

We have the following.

Theorem 8.2. *Suppose*

- $v > 1$,
- $S_i \subset \mathbb{Z}^{n_i+1}$ is symmetric with respect to the origin of \mathbb{Z}^{n_i+1} for $1 \leq i \leq v$, and

- $\prod_{i=1}^v S_i$ has the $(m, NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v))$ -Borsuk-Ulam property for some adjacencies κ_i for \mathbb{Z}^{n_i+1} and λ_i for \mathbb{Z}^{n_i} , where $m = \sum_{i=1}^v n_i$.

Then, for all i , S_i has the $(n_i, \kappa_i, \lambda_i)$ -Borsuk-Ulam property.

Proof. Notice $\prod_{i=1}^v S_i \subset \mathbb{Z}^{m+v}$. Let $f : \prod_{i=1}^v S_i \rightarrow \mathbb{Z}^m$ be any function that is $(NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v))$ -continuous. By hypothesis, there exists $x \in \prod_{i=1}^v S_i$ such that $f(x)$ and $f(-x)$ are equal or $NP_v(\lambda_1, \dots, \lambda_v)$ -adjacent.

In particular, we can let f be the product of arbitrary continuous functions $f_i : S_i \rightarrow \mathbb{Z}^{n_i}$, since if $f_i : S_i \rightarrow \mathbb{Z}^{n_i}$ is (κ_i, λ_i) -continuous, then by Theorem 4.1, $f = \prod_{i=1}^v f_i$ is $(NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v))$ -continuous. Therefore, there exists $x = (x_1, \dots, x_v) \in X$ where $x_i \in S_i$ such that $f(x) = (f_1(x_1), \dots, f_v(x_v))$ and $f(-x) = (f_1(-x_1), \dots, f_v(-x_v))$ are equal or are $NP_v(\lambda_1, \dots, \lambda_v)$ -adjacent. Hence, for all indices i , $f_i(x_i)$ and $f_i(-x_i)$ are equal or λ_i -adjacent.

Since the f_i were arbitrarily chosen, the assertion follows. \square

9. NP_u AND THE APPROXIMATE FIXED POINT PROPERTY

In both topology and digital topology,

- a *fixed point* of a continuous function $f : X \rightarrow X$ is a point $x \in X$ satisfying $f(x) = x$;
- if every continuous $f : X \rightarrow X$ has a fixed point, then X has the *fixed point property* (FPP).

However, a digital image X has the FPP if and only if X has a single point [10]. Therefore, it turns out that the *approximate fixed point property* is more interesting for digital images.

Definition 9.1 ([10]). *A digital image (X, κ) has the approximate fixed point property (AFPP) if every continuous $f : X \rightarrow X$ has an approximate fixed point, i.e., a point $x \in X$ such that $f(x)$ is equal or κ -adjacent to x .* \square

A number of results concerning the AFPP were presented in [10], including the following.

Theorem 9.2 ([10]). *Suppose (X, κ) has the AFPP. Let $h : X \rightarrow Y$ be a (κ, λ) -isomorphism. Then (Y, λ) has the AFPP.* \square

Theorem 9.3 ([10]). *Suppose Y is a retract of (X, κ) . If (X, κ) has the AFPP, then (Y, κ) has the AFPP.* \square

The following is a generalization of Theorem 5.10 of [10].

Theorem 9.4. *Let (X_i, κ_i) be digital images, $1 \leq i \leq v$. Then for any $u \in \mathbb{Z}$ such that $1 \leq u \leq v$, if $(\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v))$ has the AFPP then (X_i, κ_i) has the AFPP for all i .*

Proof. Let $X = (\prod_{i=1}^v X_i, NP_u(\kappa_1, \dots, \kappa_v))$.

Suppose X has the AFPP. Let $x_i \in X_i$. Let

$$X'_1 = X_1 \times \{(x_2, \dots, x_v)\},$$

$$X'_i = \{(x_1, \dots, x_{i-1})\} \times X_i \times \{(x_{i+1}, \dots, x_v)\} \text{ for } 2 \leq i < v,$$

$$X'_v = \{(x_1, \dots, x_{v-1})\} \times X_v.$$

Clearly, each X'_i is a retract of X and is isomorphic to X_i . By Theorems 9.2 and 9.3, X_i has the AFPP. \square

10. NP_v AND FUNDAMENTAL GROUPS

Several versions of the fundamental group for digital images exist in the literature, including those of [35, 27, 4, 16]. In this paper, we use the version of [4], which was shown in [16] to be equivalent to the version developed in the latter paper. Other papers cited in this section use the version of the digital fundamental group presented in [4].

The author of [25] attempted to study the fundamental group of a Cartesian product of digital simple closed curves. Errors of [25] were corrected in [13].

The notion of a covering map [25] is often useful in computing the fundamental group. The following is a somewhat simpler characterization of a covering map than that given in [25].

Theorem 10.1 ([6]). *Let (E, κ) and (B, λ) be digital images. Let $g : E \rightarrow B$ be a (κ, λ) -continuous surjection. Then g is a (κ, λ) -covering map if and only if for each $b \in B$, there is an index set M such that*

- $g^{-1}(N_\lambda^*(b, 1, B)) = \bigcup_{i \in M} N_\kappa^*(e_i, 1, E)$ where $e_i \in g^{-1}(b)$;
- if $i, j \in M$ and $i \neq j$ then $N_\kappa^*(e_i, 1, E) \cap N_\kappa^*(e_j, 1, E) = \emptyset$; and
- the restriction map $g|_{N_\kappa^*(e_i, 1, E)} : N_\kappa^*(e_i, 1, E) \rightarrow N_\lambda^*(b, 1, B)$ is a (κ, λ) -isomorphism for all $i \in M$. \square

Example 10.2 ([25]). *Let $C \subset \mathbb{Z}^n$ be a simple closed κ -curve, as realized by a (c_1, κ) -continuous surjection $f : [0, m-1]_{\mathbb{Z}} \rightarrow C$ such that $f(0)$ and $f(m-1)$ are κ -adjacent. Define $g : \mathbb{Z} \rightarrow C$ by $g(z) = f(z \bmod m)$. Then g is a covering map. \square*

Proposition 10.3 ([13]). *Suppose for $i \in \{1, 2\}$, $g_i : E_i \rightarrow B_i$ is a (κ_i, λ_i) -covering map. Then $g_1 \times g_2 : E_1 \times E_2 \rightarrow B_1 \times B_2$ is a $(NP_2(\kappa_1, \kappa_2), NP_2(\lambda_1, \lambda_2))$ -covering map. \square*

Corollary 10.4. *Suppose for $i \in \{1, \dots, v\}$, $g_i : E_i \rightarrow B_i$ is a (κ_i, λ_i) -covering map. Then $\prod_{i=1}^v g_i : \prod_{i=1}^v E_i \rightarrow \prod_{i=1}^v B_i$ is a $(NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v))$ -covering map. \square*

Proof. This follows from Propositions 10.3 and 3.4, and Theorem 10.1. \square

A digital image with the homotopy type of a single point is called *contractible*. For the following theorem, it is useful to know that a digital simple closed curve S is not contractible if and only if $|S| > 4$ [4, 7].

Theorem 10.5 ([29, 4, 25]). *Let $S \subset (\mathbb{Z}^n, \kappa)$ be a digital simple closed κ -curve that is not contractible. Let $s_0 \in S$. Then the fundamental group of S is $\Pi_1^\kappa(S, s_0) \approx \mathbb{Z}$. \square*

The following theorem was discussed in [25], but the argument given for it in [25] had errors. A correct proof was given in [13].

Theorem 10.6 ([13]). *Let $S_i \subset (\mathbb{Z}^{n_i}, c_{n_i})$, for $i \in \{1, 2\}$, be a noncontractible digital simple closed curve. Let $s_i \in S_i$. Then the fundamental group*

$$\Pi_1^{c_{n_1+n_2}}(S_1 \times S_2, (s_1, s_2)) \approx \mathbb{Z}^2. \quad \square$$

□

The significance of the adjacency $c_{n_1+n_2}$ in the proof of Theorem 10.6 is that, per Theorem 3.3, $NP(c_{n_1}, c_{n_2}) = c_{n_1+n_2}$. Thus, trivial modifications of the proof given in [13] for Theorem 10.6 yield the following generalization.

Theorem 10.7. *For $i \in \{1, \dots, v\}$, let $S_i \subset (\mathbb{Z}^{n_i}, \kappa_i)$ be a noncontractible digital simple closed curve. Let $s_i \in S_i$. Then the fundamental group*

$$\Pi_1^{NP_v(\kappa_1, \dots, \kappa_v)}(\Pi_{i=1}^v S_i, (s_1, \dots, s_v)) \approx \mathbb{Z}^v. \quad \square$$

Many results concerning digital covering maps depend on the radius 2 local isomorphism property (e.g., [24, 6, 11, 12, 13, 7, 14]). We have the following.

Definition 10.8 ([24]). *Let $n \in \mathbb{N}$. A (κ, λ) -covering (E, p, B) is a radius n local isomorphism if, for all $i \in M$, the restriction map $p|_{N_{\kappa}^*(e_i, n)} : N_{\kappa}^*(e_i, n) \rightarrow N_{\lambda}^*(b_i, n)$ is an isomorphism, where e_i, b_i, M are as in Theorem 10.1.*

Lemma 10.9. *Let $x_i \in (X_i, \kappa_i)$. Then*

$$N_{NP_v(\kappa_1, \dots, \kappa_v)}^*((x_1, \dots, x_n), n) = \Pi_{i=1}^v N_{\kappa_i}^*(x_i, n).$$

Proof. Let $x = (x_1, \dots, x_v)$. Let $y \in N_{NP_v(\kappa_1, \dots, \kappa_v)}^*(x, n)$. For some $m \leq n$, there is a path $\{y_i\}_{i=0}^m$ from x to y . Let $y_i = (y_{i,1}, \dots, y_{i,v})$ where $y_{i,j} \in X_i$. Since y_i and y_{i+1} are $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent, $y_{i,j}$ and $y_{i,j+1}$ are equal or κ_i -adjacent. Therefore, $\{y_{i,j}\}_{j=1}^m$ is a κ_i path in X_i from $y_{i,0}$ to $y_{i,m}$. Hence $N_{NP_v(\kappa_1, \dots, \kappa_v)}^*((x_1, \dots, x_n), n) \subset \Pi_{i=1}^v N_{\kappa_i}^*(x_i, n)$.

Let $y = (y_1, \dots, y_v) \in \Pi_{i=1}^v N_{\kappa_i}^*(x_i, n)$. For each i and for some $m_i \leq n$, there is a κ_i -path $P_i = \{y_{i,j}\}_{j=1}^{m_i}$ from x_i to y_i . There is no loss of generality in assuming $m_i = n$, since we can take $P_i = \{y_{i,j}\}_{j=1}^n$ where $y_{i,j} = y_{i,m_i}$ for $m_i \leq j \leq n$. Then for each $i < n$, $y'_i = (y_{i,1}, \dots, y_{i,v})$ and $y'_{i+1} = (y_{i+1,1}, \dots, y_{i+1,v})$ are equal or $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent. Then $\{y'_i\}_{i=1}^n$ is an $NP_v(\kappa_1, \dots, \kappa_v)$ -path from x to y . Thus, $\Pi_{i=1}^v N_{\kappa_i}^*(x_i, n) \subset N_{NP_v(\kappa_1, \dots, \kappa_v)}^*(x, n)$. The assertion follows. □

Theorem 10.10. *For $1 \leq i \leq v$, let $p_i : (E_i, \kappa_i) \rightarrow (B_i, \lambda_i)$ be continuous and let $n \in \mathbb{N}$. If (E_i, p_i, B_i) is a covering and a radius n local isomorphism for all i , then the product function*

$$\Pi_{i=1}^v p_i : \Pi_{i=1}^v E_i \rightarrow \Pi_{i=1}^v B_i$$

is a $(NP_v(\kappa_1, \dots, \kappa_v), NP_v(\lambda_1, \dots, \lambda_v))$ covering map that is a radius n local isomorphism.

Proof. This follows from Corollary 10.4 and Lemma 10.9. □

11. NP_v AND MULTIVALUED FUNCTIONS

We study properties of multivalued functions that are preserved by NP_v .

11.1. Weak and strong continuity.

Theorem 11.1. *Let $F_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued functions for $1 \leq i \leq v$. Let $X = \prod_{i=1}^v X_i$, $Y = \prod_{i=1}^v Y_i$, and $F = \prod_{i=1}^v F_i : (X, NP_v(\kappa_1, \dots, \kappa_v)) \multimap (Y, NP_v(\lambda_1, \dots, \lambda_v))$. Then F has weak continuity if and only if each F_i has weak continuity.*

Proof. Let x_i and x'_i be κ_i -adjacent or equal in X_i . Then $x = (x_1, \dots, x_v)$ and $x' = (x'_1, \dots, x'_v)$ are $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent or equal in X .

The multivalued function F has weak continuity \Leftrightarrow for x, x' as above, $F(x)$ and $F(x')$ are $NP_v(\lambda_1, \dots, \lambda_v)$ -adjacent subsets of $Y \Leftrightarrow$ for each i and for all x_i, x'_i as above, $F_i(x_i)$ and $F_i(x'_i)$ are λ_i -adjacent subsets of $Y_i \Leftrightarrow$ for each i , F_i has weak continuity. \square

Theorem 11.2. *Let $F_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued functions for $1 \leq i \leq v$. Let $X = \prod_{i=1}^v X_i$, $Y = \prod_{i=1}^v Y_i$, and $F = \prod_{i=1}^v F_i : (X, NP_v(\kappa_1, \dots, \kappa_v)) \multimap (Y, NP_v(\lambda_1, \dots, \lambda_v))$. Then F has strong continuity if and only if each F_i has strong continuity.*

Proof. Let x_i and x'_i be κ_i -adjacent or equal in X_i . Then $x = (x_1, \dots, x_v)$ and $x' = (x'_1, \dots, x'_v)$ are $NP_v(\kappa_1, \dots, \kappa_v)$ -adjacent or equal in X .

The multivalued function F has strong continuity \Leftrightarrow for x, x' as above, every point of $F(x)$ is $NP_v(\lambda_1, \dots, \lambda_v)$ -adjacent or equal to a point of $F(x')$ and every point of $F(x')$ is $NP_v(\lambda_1, \dots, \lambda_v)$ -adjacent or equal to a point of $F(x) \Leftrightarrow$ for each i and for all x_i, x'_i as above, every point of $F_i(x_i)$ is λ_i -adjacent or equal to a point of $F_i(x'_i)$ and every point of $F_i(x'_i)$ is λ_i -adjacent or equal to a point of $F_i(x_i) \Leftrightarrow$ for each i , F_i has strong continuity. \square

11.2. Continuous multifunctions.

Lemma 11.3. *Let $X \subset \mathbb{Z}^m$, $Y \subset \mathbb{Z}^n$. Let $F : (X, c_a) \multimap (Y, c_b)$ be a continuous multivalued function. Let $f : (S(X, r), c_a) \rightarrow (Y, c_b)$ be a continuous function that induces F . Let $s \in \mathbb{N}$. Then there is a continuous function $f_s : (S(X, rs), c_a) \rightarrow (Y, c_b)$ that induces F .*

Proof. Given a point $x = (x_1, \dots, x_m) \in S(X, rs)$, there is a unique point $I(x) = x' = (x'_1, \dots, x'_m) \in S(X, r)$ such that x' "contains" x in the sense that the fractional part of each component of x , $x_i - [x_i]$, "truncates" to the fractional part of the corresponding component of x' , $x'_i - [x'_i]$, i.e.,

$$x'_i - [x'_i] \leq x_i - [x_i] < x'_i - [x'_i] + 1/r.$$

(See Figure 2.) Define $f_s(x) = f(I(x))$.

We must show f_s is a continuous multivalued function that induces F . If x, x' are c_a -adjacent in $S(X, rs)$, then one sees easily that $I(x)$ and $I(x')$ are

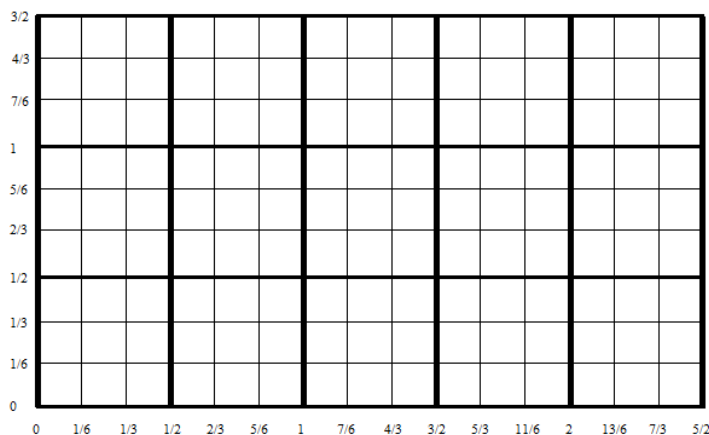


FIGURE 2. The digital image $X = [0, 2]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}$ with its partitions $S(X, 2)$ with member coordinates on heavy lines, and $S(X, 6)$ with member coordinates on both heavy and light lines. In the notation used in the proof of Lemma 11.3, we have, e.g., $I(7/6, 2/3) = (1, 1/2)$.

equal or c_a -adjacent in $S(X, r)$. Hence $f_s(x) = f(I(x))$ and $f_s(x') = f(I(x'))$ are equal or c_b -adjacent in Y . Thus, f_s is continuous. For $w \in X$ we have

$$F(w) = \bigcup_{y \in E_r^{-1}(w)} f(y) = \bigcup_{u \in E_{rs}^{-1}(w)} f_s(u).$$

Therefore, f induces F . □

For multivalued functions $F_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$, $1 \leq i \leq v$, define the product multivalued function

$$\prod_{i=1}^v F_i : (\prod_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_v)) \multimap (\prod_{i=1}^v Y_i, NP_v(\lambda_1, \dots, \lambda_v))$$

by

$$(\prod_{i=1}^v F_i)(x_1, \dots, x_v) = \prod_{i=1}^v F_i(x_i).$$

Theorem 11.4. *Given multivalued functions $F_i : (X_i, c_{a_i}) \multimap (Y_i, c_{b_i})$, $1 \leq i \leq v$, if each F_i is continuous then the product multivalued function*

$$\prod_{i=1}^v F_i : (\prod_{i=1}^v X_i, NP_v(c_{a_1}, \dots, c_{a_v})) \multimap (\prod_{i=1}^v Y_i, NP_v(c_{b_1}, \dots, c_{b_v}))$$

is continuous.

Proof. If each F_i is continuous, there exists a continuous $f_i : (S(X_i, r_i), c_{a_i}) \rightarrow (Y_i, c_{b_i})$ that generates F_i . By Lemma 11.3, we may assume that all the r_i are

equal. Thus, for some positive integer r , we have $f_i : (S(X_i, r), c_{a_i}) \rightarrow (Y_i, c_{b_i})$ generating F_i .

By Theorem 4.1, the product multivalued function

$$\Pi_{i=1}^v f_i : (\Pi_{i=1}^v S(X_i, r), NP_v(c_{a_1}, \dots, c_{a_v})) \rightarrow (\Pi_{i=1}^v (Y_i, NP_v(c_{b_1}, \dots, c_{b_v})))$$

is continuous. It is clear that this function generates the multivalued function $\Pi_{i=1}^v F_i$. \square

The paper [20] has several results concerning the following notions.

Definition 11.5 ([20]). *Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital image and $Y \subset X$. We say that Y is a κ -retract of X if there exists a κ -continuous multivalued function $F : X \multimap Y$ (a multivalued κ -retraction) such that $F(y) = \{y\}$ if $y \in Y$. If moreover $F(x) \subset N_{c_n}^*(x)$ for every $x \in X \setminus Y$, we say that F is a multivalued (N, κ) -retraction, and Y is a multivalued (N, κ) -retract of X .*

We generalize Theorem 7.2 as follows.

Theorem 11.6. *For $1 \leq i \leq v$, let $A_i \subset (X_i, \kappa_i) \subset \mathbb{Z}^{n_i}$. Suppose $F_i : X_i \multimap A_i$ is a continuous multivalued function for all i . Then F_i is a multivalued retraction for all i if and only if $F = \Pi_{i=1}^v F_i : \Pi_{i=1}^v X_i \multimap \Pi_{i=1}^v A_i$ is a multivalued $NP_v(\kappa_1, \dots, \kappa_v)$ -retraction. Further, F_i is an (N, κ_i) -retraction for all i if and only if F is a multivalued $(N, NP_v(\kappa_1, \dots, \kappa_v))$ -retraction.*

Proof. Let $X = \Pi_{i=1}^v X_i$, $A = \Pi_{i=1}^v A_i$.

Suppose each F_i is a multivalued retraction. By Theorem 11.4, the product multivalued function F is continuous.

Clearly, $F(X) \subset A$. Also, given $a = (a_1, \dots, a_v) \in A$, we have

$$F(a) = \Pi_{i=1}^v F_i(a_i) = \Pi_{i=1}^v \{a_i\} = \{a\}.$$

Therefore, $F(X) = A$, and F is a multivalued retraction.

Conversely, suppose F is a multivalued retraction. By Theorem 11.4, each F_i is continuous. Also, since $F(X) = A$, we must have, for each i , $F_i(X_i) = A_i$, and since F is a retraction, $F_i(a) = \{a\}$ for $a \in A_i$. Therefore, F_i is a multivalued retraction.

Further, from Lemma 10.9, for $x = (x_1, \dots, x_v) \in X$, $N_{NP_v(c_{n_1}, \dots, c_{n_v})}^*(x) = \Pi_{i=1}^v N_{c_{n_i}}(x_i)$. It follows that F_i is an (N, κ_i) -retraction for all i if and only if F is a multivalued $(N, NP_v(\kappa_1, \dots, \kappa_v))$ -retraction. \square

11.3. Connectivity preserving multifunctions.

Theorem 11.7. *Let $f_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be a multivalued function between digital images, $1 \leq i \leq v$. Then the product map*

$$\Pi_{i=1}^v f_i : (\Pi_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_v)) \multimap (\Pi_{i=1}^v Y_i, NP_v(\lambda_1, \dots, \lambda_v))$$

is a connectivity preserving multifunction if and only if each f_i is a connectivity preserving multifunction.

Proof. Let $X = \prod_{i=1}^v X_i$, $Y = \prod_{i=1}^v Y_i$, $F = \prod_{i=1}^v f_i : X \multimap Y$. Assume

$$x = (x_1, \dots, x_v), \quad x' = (x'_1, \dots, x'_v)$$

with $x_i, x'_i \in X_i$. Using Theorem 2.12, we argue as follows.

F is connectivity preserving

\Leftrightarrow

- For every $x \in X$, $F(x) = \prod_{i=1}^v F_i(x_i)$ is a connected subset of Y , and
- For adjacent $x, x' \in X$, $F(x) = \prod_{i=1}^v F_i(x_i)$ and $F(x') = \prod_{i=1}^v F_i(x'_i)$ are adjacent subsets of Y .

\Leftrightarrow

- For every $x_i \in X_i$, $F_i(x_i)$ is a connected subset of Y_i , and
- For adjacent $x_i, x'_i \in X_i$, $F_i(x_i)$ and $F_i(x'_i)$ are adjacent subsets of Y_i .

\Leftrightarrow

each F_i is connectivity preserving. □

12. NP_v AND SHY MAPS

The following generalizes a result of [9].

Theorem 12.1. *Let $f_i : (X_i, \kappa_i) \rightarrow (Y_i, \lambda_i)$ be a continuous surjection between digital images, $1 \leq i \leq v$. Then the product map*

$$f = \prod_{i=1}^v f_i : (\prod_{i=1}^v X_i, NP_v(\kappa_1, \dots, \kappa_v)) \rightarrow (\prod_{i=1}^v Y_i, NP_v(\lambda_1, \dots, \lambda_v))$$

is shy if and only if each f_i is a shy map.

Proof. Suppose the product map is shy. Since $f_i = p_i \circ f \circ I_i$, where I_i is the continuous injection of the proof of Theorem 7.2, it follows from Theorems 2.4 and 4.3 that f_i is continuous. Also, since f is surjective, f_i must be surjective.

Let Y'_i be a λ_i -connected subset of Y_i . By Theorem 5.1, $\prod_{i=1}^v Y'_i$ is connected in $\prod_{i=1}^v Y_i$. Since the product map is shy, we have from Theorem 2.28 that

$$X' = f^{-1}(\prod_{i=1}^v Y'_i) = \prod_{i=1}^v f_i^{-1}(Y'_i)$$

is $NP_v(\kappa_1, \dots, \kappa_v)$ -connected. Then $f_i^{-1}(Y'_i) = p_i(X')$ is κ_i -connected. From Theorem 2.28, it follows that f_i is shy.

Conversely, suppose each f_i is shy. By Theorem 4.1, the product map $\prod_{i=1}^v f_i$ is continuous, and it is easily seen to be surjective.

Let $y_i \in Y_i$. Then $(\prod_{i=1}^v f_i)^{-1}(y_1, \dots, y_v) = \prod_{i=1}^v f_i^{-1}(y_i)$ is connected, by Definition 2.27 and Theorem 5.1.

Let y_i, y'_i be λ_i -adjacent in Y_i , and let $y = (y_1, \dots, y_v)$, $y' = (y'_1, \dots, y'_v)$. Then y and y' are adjacent in Y , and $(\prod_{i=1}^v f_i)^{-1}(\{y, y'\}) = \prod_{i=1}^v f_i^{-1}(\{y_i, y'_i\})$ is connected, by Definition 2.27 and Theorem 5.1.

Thus, by Definition 2.27, $\prod_{i=1}^v f_i$ is shy. □

The statement analogous to Theorem 12.1 is not generally true if c_u -adjacencies are used instead of normal product adjacencies, as shown in the following.

Example 12.2. Recall Example 4.2, in which $X = \{(0, 0), (1, 0)\} \subset \mathbb{Z}^2$, $Y = \{(0, 0), (1, 1)\} \subset \mathbb{Z}^2$. There is a (c_1, c_2) -isomorphism $f : X \rightarrow Y$. Consider $X' = X \times \{0\} \subset \mathbb{Z}^3$, $Y' = Y \times \{0\} \subset \mathbb{Z}^3$. Although the maps f and $1_{\{0\}}$ are, respectively, (c_1, c_2) - and (c_1, c_1) -isomorphisms and therefore are, respectively, (c_1, c_2) - and (c_1, c_1) -shy, the product map $f \times 1_{\{0\}} : X' \rightarrow Y'$ is not (c_1, c_1) -shy, by Theorem 2.28, since, as observed in Example 4.2, X' is c_1 -connected and Y' is not c_1 -connected. \square

13. FURTHER REMARKS

We have studied adjacencies that are extensions of the normal product adjacency for finite Cartesian products of digital images. We have shown that such adjacencies preserve many properties for finite Cartesian products of digital images that, in some cases, are not preserved by the use of the c_u -adjacencies most commonly used in the literature of digital topology.

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