Fixed point theorems for simulation functions in b-metric spaces via the \( wt \)-distance

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Communicated by S. Romaguera

Abstract

The purpose of this article is to prove some fixed point theorems for simulation functions in complete \( b \)-metric spaces with partially ordered by using \( wt \)-distance which introduced by Hussain et al. \cite{12}. Also, we give some examples to illustrate our main results.

2010 MSC: 47H09; 47H10; 54H25.

Keywords: Fixed point; simulation function; \( b \)-metric space; \( wt \)-distance; \( w \)-distance; generalized distance.

Received 02 July 2016 – Accepted 06 December 2016
1. INTRODUCTION

Since Banach’s fixed point theorem (or Banach’s contraction principle) proved by Banach [4] in 1922, many authors have extended, improved and generalized in several ways.

In 2015, Khojasteh et al. [15] introduced the notion of a simulation function to generalize Banach’s contraction principle. Recently, Roldán-López-de-Hierro et al. [18] modified the notion of a simulation function and showed the existence and uniqueness of coincidence points of two nonlinear mappings using the concept of a simulation function.

On the other hand, in 1989, Bakhtin [3] (see also Czerwik [8]) introduced the concept of a \( b \)-metric space (or a space of metric type) and proved some fixed point theorems for some contractive mappings in \( b \)-metric spaces which are generalizations of Banach’s contraction principle in metric spaces.

In 1996, Kada et al. [14] introduced some generalized metric, which is called the \( w \)-distance and gave some examples of \( w \)-distance and, using the \( w \)-distance, they also improved Caristi’s fixed point theorem, Ekeland’s variational principle and the nonconvex minimization theorem of Takahashi [20]. Later, Shioji et al. [19] studied the relationship between weakly contractive mappings and weakly Kannan mappings under the conditions, the \( w \)-distance and the symmetric \( w \)-distance. In 2012, Imdad and Rouzkard [13] proved some fixed point theorems in a complete metric space equipped with a partial ordering via the \( w \)-distance.

Recently, Hussain et al. [12] introduced the concept of the \( wt \)-distance in generalized \( b \)-metric spaces, which is a generalization of the \( w \)-distance, and also proved some fixed point theorems in a partially ordered \( b \)-metric space by using the \( wt \)-distance. Also, Abdou et al. [1] proved some common fixed point theorems in Menger probabilistic metric type spaces by using the \( wt \)-distance.

In this paper, we consider some simulation functions to show the existence of fixed points of some nonlinear mappings in complete \( b \)-metric spaces via the \( wt \)-distance. Furthermore, we also give some examples to illustrate the main results. Our result improve, extend and generalize several results given by some authors in literatures.

2. PRELIMINARIES AND GENERALIZED DISTANCES

Now, we give some definitions and their examples

**Definition 2.1.** Let \((X, \leq)\) be a partially ordered set. The elements \(x, y \in X\) are said to be comparable with respect to the order \( \leq \) if either \( x \leq y \) or \( y \leq x \).

Let us denote \( X_{\leq} \) by the subset of \( X \times X \) defined by

\[
X_{\leq} = \{(x, y) \in X \times X : x \leq y \text{ or } y \leq x \}.
\]

**Definition 2.2.** Let \((X, \leq)\) be a partially ordered set and \( f : X \to X \) be a self-mapping of \( X \). We say that

1. \( f \) is inverse increasing if, for all \( x, y \in X \), \( f(x) \leq f(y) \) implies \( x \leq y \);
2. \( f \) is nondecreasing if, for all \( x, y \in X \), \( x \leq y \) implies \( f(x) \leq f(y) \).
Definition 2.3. Let \((X, \preceq)\) be a partially ordered set and \(T : X \to X\) be a self-mapping of \(X\). Then

1. \(F(T) = \{x \in X : T(x) = x\}\), i.e., \(F(T)\) denotes the set of all fixed points of \(T\);
2. \(T\) is called a Picard operator (briefly, PO) if there exists \(x^* \in X\) such that \(F(T) = \{x^*\}\) and \(\{T^n(x)\}\) converges to \(x^*\) for all \(x \in X\);
3. \(T\) is said to be orbitally \(\mathcal{U}\)-continuous for any \(\mathcal{U} \subseteq X \times X\) if, for any \(x \in X\), \(T^n(x) \to a \in X\) as \(i \to \infty\) and \((T^n(x), a) \in \mathcal{U}\) for any \(i \in \mathbb{N}\) imply that \(T^{n+1}(x) \to Ta \in X\) as \(i \to \infty\);
4. \(T\) is said to be orbitally continuous on \(X\) if \(x \in X\) and \(T^n(x) \to a \in X\) as \(i \to \infty\) imply that \(T^{n+1}(x) \to T(a) \in X\) as \(i \to \infty\).

Definition 2.4. Let \((X, d)\) be a metric space. A function \(p : X \times X \to [0, \infty)\) is said to be the \(w\)-distance on \(X\) if the following are satisfied:

1. \(p(x, z) \leq p(x, y) + p(y, z)\) for all \(x, y, z \in X\);
2. for any \(x \in X\), \(p(x, \cdot) : X \to [0, \infty)\) is lower semi-continuous (i.e., if \(x \in X\) and \(y_n \to y \in X\), then \(p(x, y) \leq \liminf_{n \to \infty} p(x, y_n)\);
3. for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(p(z, x) \leq \delta\) and \(p(z, y) \leq \delta\) imply \(d(x, y) \leq \varepsilon\).

Let \(X\) be a metric space with a metric \(d\). A \(w\)-distance \(p\) on \(X\) is said to be symmetric if \(p(x, y) = p(y, x)\) for all \(x, y \in X\). Obviously, every metric is the \(w\)-distance, but not conversely.

Next, we recall some examples in [21] to show that the \(w\)-distance is a generalized metric.

Example 2.5. Let \((X, d)\) be a metric space. A function \(p : X \times X \to [0, \infty)\) defined by \(p(x, y) = c\) for all \(x, y \in X\) is a \(w\)-distance on \(X\), where \(c\) is a positive real number. But \(p\) is not a metric since \(p(x, x) = c \neq 0\) for any \(x \in X\).

Example 2.6. Let \((X, \| \cdot \|)\) be a normed linear space. A function \(p : X \times X \to [0, \infty)\) defined by \(p(x, y) = \|x\| + \|y\|\) for all \(x, y \in X\) is a \(w\)-distance on \(X\).

Example 2.7. Let \(F\) be a bounded and closed subset of a metric spaces \(X\). Assume that \(F\) contain at least two points and \(c\) is a constant with \(c \geq \delta(F)\), where \(\delta(F)\) is the diameter of \(F\). Then a function \(p : X \times X \to [0, \infty)\) defined by

\[
p(x, y) = \begin{cases} d(x, y), & \text{if } x, y \in F; \\ c, & \text{if } x \notin F \text{ or } y \notin F, \end{cases}
\]

is a \(w\)-distance on \(X\).

Definition 2.8. Let \(X\) be a nonempty set and \(s \geq 1\) be a given real number. A functional \(D : X \times X \to [0, \infty)\) is called a \(b\)-metric if, for all \(x, y, z \in X\), the following conditions are satisfied:

1. \(D(x, y) = 0\) if and only if \(x = y\);
2. \(D(x, y) = D(y, x)\);
3. \(D(x, z) \leq s[D(x, y) + D(y, z)]\).
A pair \((X, D)\) is called a \(b\)-metric space with coefficient \(s\).

In Definition 2.8, every metric space is a \(b\)-metric space with \(s = 1\) and hence the class of \(b\)-metric spaces is larger than the class of metric spaces.

Some examples of \(b\)-metric spaces are given by Berinde [5], Czerwik [9], Heinonen [11] and, further, some examples to show that every \(b\)-metric space is a real generalization of metric spaces are as follows:

**Example 2.9.** The set \(\mathbb{R}\) of real numbers together with the functional \(D : \mathbb{R} \times \mathbb{R} \to [0, \infty)\) defined by

\[
D(x, y) := |x - y|^2
\]

for all \(x, y \in \mathbb{R}\) is a \(b\)-metric space with coefficient \(s = 2\). However, we know that \(D\) is not a metric on \(X\) since the ordinary triangle inequality is not satisfied.

Indeed,

\[
D(3, 5) > D(3, 4) + D(4, 5).
\]

In 2014, Hussain et al. [12] introduced the concept of the \(wt\)-distance as follow:

**Definition 2.10.** Let \((X, D)\) be a \(b\)-metric space with constant \(K \geq 1\). A function \(P : X \times X \to [0, \infty)\) is called the \(wt\)-distance on \(X\) if the following are satisfied:

1. \(P(x, z) \leq K(P(x, y) + P(y, z))\) for all \(x, y, z \in X\);
2. for any \(x \in X\), \(P(x, \cdot) : X \to [0, \infty)\) is \(K\)-lower semi-continuous (i.e., if \(x \in X\) and \(y_n \to y \in X\), then \(P(x, y) \leq \liminf_{n \to \infty} K P(x, y_n)\));
3. for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(P(z, x) \leq \delta\) and \(P(z, y) \leq \delta\) imply \(D(x, y) \leq \varepsilon\).

**Example 2.11 ([12]).** Let \((X, D)\) be a \(b\)-metric space. Then the metric \(D\) is a \(wt\)-distance on \(X\).

**Example 2.12 ([12]).** Let \(X = \mathbb{R}\) and \(D_1 = (x - y)^2\). A function \(P : X \times X \to [0, \infty)\) defined by \(P(x, y) = \|x\|^2 + \|y\|^2\) for all \(x, y \in X\) is a \(wt\)-distance on \(X\).

**Example 2.13 ([12]).** Let \(X = \mathbb{R}\) and \(D_1 = (x - y)^2\). A function \(P : X \times X \to [0, \infty)\) defined by \(P(x, y) = \|y\|^2\) for all \(x, y \in X\) is a \(wt\)-distance on \(X\).

The following two lemmas are crucial for our results.

**Lemma 2.14 ([12]).** Let \((X, D)\) be a \(b\)-metric space with constant \(K \geq 1\) and \(P\) be a \(wt\)-distance on \(X\). Let \(\{x_n\}, \{y_n\}\) be two sequences in \(X\) and \(\{\alpha_n\}, \{\beta_n\}\) two sequences in \(0, \infty\) converging to zero. Then the following conditions hold: for all \(x, y, z \in X\),

1. if \(P(x_n, y) \leq \alpha_n\) and \(P(x_n, z) \leq \beta_n\) for all \(n \in \mathbb{N}\), then \(y = z\). In particular, if \(P(x, y) = 0\) and \(P(x, z) = 0\), then \(y = z\);
2. if \(P(x_n, y_n) \leq \alpha_n\) and \(P(x_n, z) \leq \beta_n\) for all \(n \in \mathbb{N}\), then \(\{y_n\}\) converges to \(z\);
Definition 3.1. A simulation function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

1. $\zeta(0, 0) = 0$;
2. $\zeta(t, s) < s - t$ for all $s, t > 0$;
3. if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0$, then
   \[
   \limsup_{n \to \infty} \zeta(t_n, s_n) < 0.
   \]

Now, we recall some examples of the simulation function given by Khojasteh et al. [15].

Example 3.2. Let $\zeta_i : [0, \infty) \times [0, \infty) \to \mathbb{R}$ for $i = 1, 2, 3$ be defined by

1. $\zeta_1(t, s) = \psi(s) - \phi(t)$ for all $t, s \in [0, \infty)$, where $\phi, \psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$;
2. $\zeta_2(t, s) = s - \frac{f(t, s)}{g(t, s)}$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty) \times [0, \infty) \to (0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$;
3. $\zeta_3(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, \infty)$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

Then $\zeta_i$ for $i = 1, 2, 3$ are a simulation function.

Recently, Roldán-López-de-Hierro et al. [18] modified the notion of a simulation function as follow:

Definition 3.3 ([18]). A simulation function $\tilde{\zeta} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

1. $\tilde{\zeta}(0, 0) = 0$;
2. $\tilde{\zeta}(t, s) < s - t$ for all $s, t > 0$;
3. if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then
   \[
   \limsup_{n \to \infty} \tilde{\zeta}(t_n, s_n) < 0.
   \]

Note that the classes of all simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ denote by $Z$ and every simulation function in the original sense of Khojasteh et al. [15] is also a simulation function in the sense of Roldán-López-de-Hierro et al. [18], but the converse is not true as in the following example.
Example 3.4 ([18]). Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta \in \mathbb{Z}$ be the function defined by

$$\zeta(t, s) = \begin{cases} 2s - 2t, & \text{if } s < t, \\ ks - t, & \text{otherwise.} \end{cases}$$

Then $\zeta$ is a simulation function in the sense of Definition 3.3, but $\zeta$ does not satisfy the condition $(\zeta_3)$ of Definition 3.1.

**Definition 3.5.** Let $(X, d)$ be a complete metric space. A mapping $T : X \rightarrow X$ is called $\mathbb{Z}$-contraction if there exists $\zeta \in \mathbb{Z}$ such that

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0$$

for all $x, y \in X$.

**Remark 3.6.** If we take $\zeta(t, s) = \lambda s - t$ for all $s, t \geq 0$, where $\lambda \in [0, 1)$ in Definition 3.5, then the $\mathbb{Z}$-contraction become to the Banach contraction.

4. Fixed point theorems for simulation functions

In this section, we consider the concept of a simulation function and show the existence of a fixed point for such mapping in complete $b$-metric spaces via the $wt$-distance. First, we improve the notion of a simulation function for our considerations as follow:

**Definition 4.1.** Let $K$ be a given real number such that $K \geq 1$. A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

$(\zeta_1)$ $\zeta(0, 0) = 0$;

$(\zeta_2)$ $\zeta(Kt, s) < s - Kt$ for all $s, t > 0$;

$(\zeta_3)$ if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0, \infty)$ such that

$$\limsup_{n \rightarrow \infty} K t_n = \limsup_{n \rightarrow \infty} s_n > 0 \text{ and } t_n < s_n \text{ for all } n \in \mathbb{N},$$

then

$$\limsup_{n \rightarrow \infty} \zeta(Kt_n, s_n) < 0.$$

**Example 4.2.** Let $\lambda, K \in \mathbb{R}$ be such that $\lambda < 1$ and $K \geq 1$. Define the mapping $\hat{\zeta} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ by

$$\zeta(Kt, s) = \begin{cases} s - Kt, & \text{if } s < t, \\ \lambda s - Kt \frac{Ks + 1}{Ks + 1}, & \text{otherwise.} \end{cases}$$

Clearly, $\zeta$ verifies $(\zeta_1)$, and $\zeta$ satisfies $(\zeta_2)$. Indeed,

$$s, t > 0, \begin{cases} 0 < s < t \Rightarrow \zeta(Kt, s) = s - Kt, \\ 0 < t < s \Rightarrow \zeta(Kt, s) = \frac{\lambda s - Kt}{Ks + 1} < \frac{s - Kt}{Ks + 1} < s - Kt. \end{cases}$$

Next, we will show that $\zeta$ satisfies $(\zeta_3)$. If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that

$$\limsup_{n \rightarrow \infty} K t_n = \limsup_{n \rightarrow \infty} s_n > 0 \text{ and } t_n < s_n \text{ for all } n \in \mathbb{N},$$
then
\[
\limsup_{n \to \infty} \zeta(Kt_n, s_n) = \limsup_{n \to \infty} \left( \frac{\lambda s_n - Kt_n}{Ks_n + 1} \right) < \limsup_{n \to \infty} \left( \frac{s_n - Kt_n}{Kt_n + 1} \right) < \limsup_{n \to \infty} \left( \frac{s_n - Kt_n}{Kt_n} \right) < \limsup_{n \to \infty} \left( \frac{s_n}{Kt_n} - \frac{Kt_n}{Kt_n} \right) \leq \limsup_{n \to \infty} \left( \frac{s_n}{Kt_n} \right) - \liminf_{n \to \infty} (1) \leq 1 - 1 = 0.
\]

Then \( \zeta \) is a simulation function in the sense of Definition 4.1, but \( \zeta \) does not satisfy the condition (\( \zeta_s \)) of Definition 3.1. Indeed, if we take \( K = 1, \ t_n = 2\sqrt{2} \) and \( s_n = 2\sqrt{2} - \frac{1}{n} \), for all \( n \in \mathbb{N} \). Then, \( s_n < t_n \)

\[
\limsup_{n \to \infty} \zeta(t_n, s_n) = \limsup_{n \to \infty} \left( 2\sqrt{2} - \frac{1}{n} - 2\sqrt{2} \right) = \limsup_{n \to \infty} \left( -\frac{1}{n} \right) = 0.
\]

**Theorem 4.3.** Let \((X, \leq)\) be a partially ordered set, \((X, D)\) be a complete \( b \)-metric space with constant \( K \geq 1 \) and \( P \) be a \( wt \)-distance on \( X \). Suppose that \( T : X \to X \) is a nondecreasing mapping satisfying the following conditions:

(i) there exists \( \zeta \in \mathcal{Z} \) such that

\[
\zeta(KP(Tx, T^2x), P(x, Tx)) \geq 0
\]

for all \((x, Tx) \in X_\leq;\)

(ii) for all \( x \in X \) with \((x, Tx) \in X_\leq, \)

\[
\inf\{P(x, y) + P(x, Tx)\} > 0
\]

for all \( y \in X \) with \( y \neq Ty;\)

(iii) there exists \( x_0 \in X \) such that \((x_0, Tx_0) \in X_\leq.\)

Then \( T \) has a fixed point in \( X \). Moreover, if \( Tx = x \), then \( P(x, x) = 0 \).

**Proof.** If \( Tx_0 = x_0 \), then we are done. Suppose that the conclusion is not true. Then there exists \( x_0 \in X \) such that \((x_0, Tx_0) \in X_\leq.\) Since \( T \) is nondecreasing, we have \((Tx_0, T^2x_0) \in X_\leq.\) Continuing this process, we obtain \((T^n x_0, T^m x_0) \in X_\leq \) for all \( n, m \in \mathbb{N} \). Now, we claim that

\[
\lim_{n \to \infty} P(T^n x_0, T^{n+1} x_0) = 0.
\]

By the assumption (i) and the property of \( \zeta \), we observe that

\[
0 \leq \zeta(KP(T^n x_0, T^{n+1} x_0), P(T^{n-1} x_0, T^n x_0)) \leq P(T^{n-1} x_0, T^n x_0) - KP(T^n x_0, T^{n+1} x_0)
\]

for all \( n \in \mathbb{N} \). Since \( K \geq 1 \) and using (4.3), we get

\[
P(T^n x_0, T^{n+1} x_0) \leq KP(T^n x_0, T^{n+1} x_0) \leq P(T^{n-1} x_0, T^n x_0).
\]
This means that the sequence \( \{P(T^n x_0, T^{n+1} x_0)\} \) is a decreasing sequence of nonnegative real numbers and so it is convergent to some \( r \geq 0 \). Suppose that \( r > 0 \).

**Case I.** If \( K > 1 \), letting \( n \rightarrow \infty \) in (4.4), we get \( r \leq Kr \leq r \) which is a contradiction.

**Case II.** If \( K = 1 \), putting \( t_n = P(T^{n+1} x_0, T^{n+2} x_0) \) and \( s_n = P(T^n x_0, T^{n+1} x_0) \), the sequences \( \{Kt_n\} \) and \( \{sn\} \) have the same positive limit. Also, the sequences \( \{Kt_n\} \) and \( \{sn\} \) have the same positive limit superior and verify that \( t_n < s_n \) for all \( n \in \mathbb{N} \). By the condition (\( \zeta_3 \)) of definition 4.1 we have

\[
\limsup_{n \to \infty} (K P(T^{n+1} x_0, T^{n+2} x_0), P(T^n x_0, T^{n+1} x_0)) = \limsup_{n \to \infty} (Kt_n, s_n) < 0,
\]

which is a contradiction. Therefore \( r = 0 \), that is, the claim (4.3) holds. Next, we show that

\[
\limsup_{m,n \to \infty} P(T^n x_0, T^m x_0) = 0.
\]

Suppose that this is not true. Then we can find \( \varepsilon_0 > 0 \) with the sequences \( \{m_k\}, \{n_k\} \) such that, for any \( m_k > n_k \) such that

\[
P(T^{m_k} x_0, T^{m_k} x_0) > \varepsilon_0
\]

for all \( k \in \{1, 2, 3, \ldots \} \). We can assume that \( m_k \) is a minimum index such that (4.6) holds. Then we also have

\[
P(T^{m_k} x_0, T^{m_k-1} x_0) \leq \varepsilon_0.
\]

Hence we have

\[
\varepsilon_0 < P(T^{m_k} x_0, T^{m_k} x_0) \leq K [P(T^{m_k} x_0, T^{m_k-1} x_0) + P(T^{m_k-1} x_0, T^{m_k} x_0)] < K \varepsilon_0 + K P(T^{m_k-1} x_0, T^{m_k} x_0).
\]

Taking limit superior as \( k \to \infty \) in the above inequality and using (4.2), we have

\[
\varepsilon_0 < \limsup_{k \to \infty} P(T^{m_k} x_0, T^{m_k} x_0) \leq K \varepsilon_0.
\]

Now, we claim that \( \limsup_{n \to \infty} P(T^{m_k+1} x_0, T^{m_k+1} x_0) < \varepsilon_0 \). If

\[
\limsup_{k \to \infty} P(T^{m_k+1} x_0, T^{m_k+1} x_0) \geq \varepsilon_0,
\]

then there exists \( \{k_r\} \) and \( \delta > 0 \) such that

\[
\limsup_{r \to \infty} P(T^{m_{k_r}+1} x_0, T^{m_{k_r}+1} x_0) = \delta \geq \varepsilon_0.
\]

By the assumption (i) and the property of \( \zeta \), we have

\[
0 \leq \zeta (K P(T^{m_{k_r}+1} x_0, T^{m_{k_r}+1} x_0), P(T^{m_k} x_0, T^{m_k} x_0)) \leq P(T^{m_k} x_0, T^{m_k} x_0) - K P(T^{m_{k_r}+1} x_0, T^{m{k_r}+1} x_0).
\]

Hence,

\[
KP(T^{m_{k_r}+1} x_0, T^{m_{k_r}+1} x_0) \leq P(T^{m_k} x_0, T^{m_k} x_0),
\]
it follows from (4.8), (4.9) and (4.11), we get that
\[ K\delta = \limsup_{r \to \infty} KP(T^{nk_r+1}x_0, T^{nk_r+1}x_0) \leq \limsup_{r \to \infty} P(T^{nk_r}x_0, T^{nk_r}x_0) \leq K\varepsilon_0 \leq K\delta. \]

Therefore the sequence \( \{ Kt_{k_r} := KP(T^{nk_r+1}x_0, T^{nk_r+1}x_0) \} \) and \( \{ s_{k_r} := P(T^{nk_r}x_0, T^{nk_r}x_0) \} \) have the same positive limit superior and verify that \( t_{k_r} < s_{k_r} \) for all \( r \in \mathbb{N} \). By the property \((\zeta_3)\), we conclude that
\[
0 \leq \limsup_{r \to \infty} \zeta(KP(T^{nk_r+1}x_0, T^{nk_r+1}x_0), P(T^{nk_r}x_0, T^{nk_r}x_0)) = \limsup_{r \to \infty} \zeta(Kt_{k_r}, s_{k_r}) < 0,
\]
which is a contradiction and hence (4.5) holds. It follows from Lemma 2.14 (iii) that \( \{ T^n x_0 \} \) is a Cauchy sequence. Since \( X \) is a complete \( b \)-metric space, the sequence \( \{ T^n x_0 \} \) converges to some element \( z \in X \). From the fact that \( \lim_{m,n \to \infty} P(T^{m}x_0, T^{n}x_0) = 0 \), for each \( \varepsilon > 0 \), there exists \( N_\varepsilon \in \mathbb{N} \) such that \( n > N_\varepsilon \) implies
\[ P(T^{n}x_0, T^{n}x_0) < \varepsilon. \]

Since \( P(x, \cdot) \) is \( K \)-lower semi-continuous and the sequence \( \{ T^n x_0 \} \) converges to \( z \), we have
\[
P(T^{n}x_0, z) \leq \liminf_{n \to \infty} KP(T^{n}x_0, T^{n}x_0) \leq K\varepsilon.
\]
Setting \( \varepsilon = \frac{1}{K} \) and \( N_\varepsilon = n_k \), by (4.12), we have
\[
\lim_{k \to \infty} P(T^{n_k}x_0, z) = 0.
\]

Now, we prove that \( z \) is a fixed point of \( T \). Suppose that \( Tz \neq z \). Since
\[
(T^{n_k}x_0, T^{n_k+1}x_0) \in X \leq
\]
for each \( n \in \mathbb{N} \), using the assumption (ii), (4.2) and (4.13), we have
\[
0 < \inf\{ P(T^{n_k}x_0, z) + P(T^{n_k}x_0, T^{n_k+1}x_0) \} \to 0
\]
as \( n \to \infty \), which is a contradiction. Therefore, \( Tz = z \).

If \( Tx = x \), we distinguish two cases.

**Case I** If \( K = 1 \), then
\[
0 \leq \zeta(P(Tx, T^2x), P(x, Tx)) = \zeta(P(x, x), P(x, x)) \leq P(x, x) - P(x, x) = 0.
\]

Hence \( \zeta(P(Tx, T^2x), P(x, Tx)) = 0 \) and so, by \((\zeta_1)\), we obtain \( P(x, x) = 0 \).

**Case II** If \( K > 1 \), then
\[
0 \leq \zeta(KP(Tx, T^2x), P(x, Tx)) = \zeta(KP(x, x), P(x, x)) \leq P(x, x) - KP(x, x) = (1 - K)P(x, x),
\]
it follow that \( P(x, x) \leq 0 \) and thus we must have \( P(x, x) = 0 \). This completes the proof. \( \square \)

Now, we give an example to illustrate Theorem 4.3.
Example 4.4. Let $X = [0, 1]$ and $D(x, y) = (x - y)^2$ with the $wt$-distance $P$ on $X$ defined by $P(x, y) = |y|^2$. We consider the following set:

$$X_\ll = \{(x, y) \in X \times X : x = y \text{ or } x, y \in \{0\} \cup \{\frac{1}{2^n} : n \geq 1\}\}
$$

with the usual ordering. Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } x = \frac{1}{2^n}, \ n \geq 1, \\ 0, & \text{otherwise.} \end{cases}
$$

for all $x \in X$. Obviously, $T$ is nondecreasing. Also, $T$ satisfies the condition (ii). Indeed, for any $n \in \mathbb{N}$, we have $\frac{1}{2^n} \neq T(\frac{1}{2^n})$. Moreover, for each $n \in \mathbb{N}$, we have

$$\inf \left\{ P\left(\frac{1}{2^m}, \frac{1}{2^n}\right) + P\left(\frac{1}{2^m}, \frac{1}{2^n} - \frac{1}{2^{m+1}}\right) \colon m \in \mathbb{N} \right\} = \frac{1}{2^{2n}} > 0.
$$

Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ define by

$$\zeta(t, s) = \frac{s - Kt}{1 + Ks} \text{ for all } s, t \in [0, \infty).
$$

Similarly, in Example 4.2, the function define as above is simulation function in the sense of Definition 4.1. Now, we show that $T$ satisfies the condition (i). Let given $x = \frac{1}{2^n}$ with $(\frac{1}{2^n}, T(\frac{1}{2^n})) \in X_\ll$. Then we have

$$\zeta(2P(Tx, T^2x), P(x, Tx)) = \zeta(2P(\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}), P(\frac{1}{2^n}, \frac{1}{2^{n+1}}))$$

$$= \frac{1}{2^{2n+4}} - \frac{1}{2^{2n+2}}
$$

$$= \frac{1}{2^{2n+2} + 2^{2n+2}}
$$

$$= \frac{2^{2n+2} - 2^{2n+2}}{(2^{2n+2})(2^{2n+3})} \cdot \frac{2^{2n+1}}{2^{2n+1} + 1}
$$

$$= \frac{2^{2n+2}(2 - 1)}{(2^{2n+4})(2^{2n+1} + 1)}
$$

$$= \frac{2^{2n+2}}{(2^{2n+4})(2^{2n+1} + 1)}
$$

$$> 0.
$$

Therefore, all the hypothesis of Theorem 4.3 are satisfied and, further, $x = 0$ is a fixed point of $T$. 
Corollary 4.5. Let \((X, \leq)\) be a partially ordered set and \((X, D)\) be a complete metric type space with constant \(K \geq 1\) and \(P\) be a \(w\)-distance on \(X\). Suppose that \(T : X \to X\) is a nondecreasing mapping satisfying the following conditions:

(i) there exists \(\alpha \in [0, \frac{1}{K})\) such that

\[
P(Tx, T^2x) = \alpha P(x, Tx)
\]

for all \(x \leq Tx\);

(ii) for all \(x \in X\) with \(x \leq Tx\),

\[
inf \{P(x, y) + P(x, Tx)\} > 0
\]

for all \(y \in X\) with \(y \neq Ty\);

(iii) there exists \(x_0 \in X\) such that \(x_0 \leq Tx_0\).

Then \(T\) has a fixed point in \(X\).

Theorem 4.6. Let \((X, \leq)\) be a partially ordered set and \((X, D)\) be a complete \(b\)-metric space with constant \(K \geq 1\) and \(P\) be a \(w\)-distance on \(X\). Suppose that \(T : X \to X\) is a nondecreasing mapping and there exists \(\zeta \in \mathbb{Z}\) such that

\[
\zeta(KP(Tx, T^2x), P(x, Tx)) \geq 0
\]

for all \((x, Tx) \in X\). Assume that one of the following conditions holds:

(i) for all \(x \in X\) with \((x, Tx) \in X\),

\[
inf \{P(x, y) + P(x, Tx)\} > 0
\]

for all \(y \in X\) with \(y \neq Ty\);

(ii) if both \(\{x_n\}\) and \(\{Tx_n\}\) converge to \(z\), then \(z = Tz\);

(iii) \(T\) is continuous on \(X\).

If there exists \(x_0 \in X\) such that \((x_0, Tx_0) \in X\), then \(T\) has a fixed point in \(X\). Moreover, if \(Tx = x\), then \(P(x, x) = 0\).

Proof. In the case of \(T\) satisfying the condition (i), the conclusion was proved in Theorem 4.3. Let us prove that (ii) \(\Rightarrow\) (i). Suppose that the condition (ii) holds. Let \(y \in X\) with \(y \neq Ty\) such that

\[
inf \{P(x, y) + P(x, Tx) : (x, Tx) \in X\} = 0.
\]

Then we can find a sequence \(\{z_n\}\) such that \((z_n, Tz_n) \in X\) and

\[
inf \{P(z_n, y) + P(z_n, Tz_n)\} = 0.
\]

So we have

\[
\lim_{n \to \infty} P(z_n, y) = \lim_{n \to \infty} P(z_n, Tz_n) = 0.
\]

Again, by Lemma 2.14, we have \(\lim_{n \to \infty} Tz_n = y\). Moreover, \(\lim_{n \to \infty} T^2z_n = y\). In fact, since

\[
0 \leq \zeta(KP(Tz_n, T^2z_n), P(z_n, Tz_n)) \leq P(z_n, Tz_n) - KP(Tz_n, T^2z_n),
\]

it follow from (4.14) and \(K \geq 1\), we get that

\[
\lim_{n \to \infty} P(Tz_n, T^2z_n) \leq \lim_{n \to \infty} KP(Tz_n, T^2z_n) \leq \lim_{n \to \infty} P(z_n, Tz_n) = 0.
\]
Letting $x_n = Tz_n$, the sequences $\{x_n\}$ and $\{Tx_n\}$ converge to $y$. Hence, by the assumption (ii), $y = Ty$ and so (ii) $\implies$ (i). Obviously, (iii) $\implies$ (ii). This completes the proof. □

Now, we prove new theorems by replacing some conditions in Theorem 4.3 with other conditions.

**Theorem 4.7.** Let $(X, \leq)$ be a partially ordered set and $(X, D)$ be a complete $b$-metric space with constant $K \geq 1$ and $P$ be a wt-distance on $X$. Suppose that $T : X \to X$ is a nondecreasing satisfying the following conditions:

(i) there exists $\zeta \in \mathcal{Z}$ such that

$$\zeta(KP(Tx, T^2x), P(x, Tx)) \geq 0$$

for all $(x, Tx) \in X \leq$;

(ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X \leq$,

(iii) either $T$ is orbitally continuous at $x_0$ or

(iv) $T$ is orbitally $X_{\leq}$-continuous and there exists a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$ converges to some element $x_*$ in $X$ such that $(T^{n_k}x_0, x_*) \in X \leq$

for any $k \in \mathbb{N}$.

Then $T$ has a fixed point in $X$. Moreover if $Tx = x$, then $P(x, x) = 0$.

**Proof.** If $Tx_0 = x_0$, then we are done. Suppose that the conclusion is not true. Then there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X \leq$. Since $T$ is monotone, we have $(Tx_0, T^2x_0) \in X \leq$. Continuing this process, we have a sequence $\{T^n x_0\}$ such that

$$(T^n x_0, T^m x_0) \in X \leq$$

for any $n, m \in \mathbb{N}$. As in the same argument in Theorem 4.3, we can see that

$$\lim_{n \to \infty} P(T^n x_0, T^{n+1} x_0) = 0. \quad (4.15)$$

Moreover,

$$\lim_{m, n \to \infty} P(T^n x_0, T^m x_0) = 0. \quad (4.16)$$

and $\{T^n x_0\}$ is a Cauchy sequence converges to some element $z \in X$. Next, we prove that $z$ is a fixed point of $T$. If the condition (iii) holds, then $T^{n+1}x_0 \to Tz$. By $P(x, \cdot)$ is $K$-lower semi-continuous and (4.16), we have

$$P(T^n x_0, z) \leq \liminf_{m \to \infty} KP(T^n x_0, T^m x_0) \leq \alpha_n \ (\text{say}) \quad (4.17)$$

and

$$P(T^n x_0, Tz) \leq \liminf_{m \to \infty} KP(T^n x_0, T^{m+1} x_0) \leq \beta_n, \ (\text{say}) \quad (4.18)$$

where the sequences $\{\alpha_n := \frac{\alpha_n}{K}\}$ and $\{\beta_n := \frac{\beta_n}{K}\}$ which converges to 0. By Lemma 2.14 (i), we conclude that $z = Tz$.

Suppose that the condition (iv) hold. From the fact that $\{T^{n_k}x_0\} \to z$ as $k \to \infty$, $(T^{n_k}x_0, z) \in X \leq$ and $T$ is orbitally $X_{\leq}$-continuous, it follows that $\{T^{n_k+1}x_0\} \to Tz$ as $k \to \infty$. Similarly, since $P(x, \cdot)$ is $K$-lower semi-continuous
as above, we conclude that \( z = Tz \) and the remaining part of the proof follow from the proof of Theorem 4.3.

□

**Corollary 4.8.** Let \( (X, \leq) \) be a partially ordered set and \( (X, D) \) be a complete metric space and \( p \) be a \( \omega \)-distance on \( X \). Suppose that \( T : X \to X \) is a nondecreasing satisfying the following conditions:

(i) there exists \( \zeta \in \mathbb{Z} \) such that

\[
\zeta(p(Tx, T^2x), p(x, Tx)) \geq 0
\]

for all \((x, Tx) \in X_\leq\);

(ii) there exists \( x_0 \in X \) such that \((x_0, Tx_0) \in X_\leq\),

(iii) either \( T \) is orbitally continuous at \( x_0 \) or

(iv) \( T \) is orbitally \( X_\leq \)-continuous and there exists a subsequence \( \{T^{n_k}x_0\} \) of \( \{T^n x_0\} \) converges to some element \( x_* \in X \) such that \((T^{n_k}x_0, x_*) \in X_\leq \) for any \( k \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \). Moreover if \( Tx = x \), then \( p(x, x) = 0 \).

**Corollary 4.9.** Let \( (X, \leq) \) be a partially ordered set and \( (X, D) \) be a complete \( b \)-metric space with constant \( K \geq 1 \) and \( P \) be a \( \omega \)-distance on \( X \). Suppose that \( T : X \to X \) is a nondecreasing satisfying the following conditions:

(i) there exists \( \lambda \in (0, \frac{1}{K}) \) such that

\[
P(Tx, T^2x) \leq \lambda P(x, Tx)
\]

for all \((x, Tx) \in X_\leq\);

(ii) there exists \( x_0 \in X \) such that \((x_0, Tx_0) \in X_\leq\),

(iii) either \( T \) is orbitally continuous at \( x_0 \) or

(iv) \( T \) is orbitally \( X_\leq \)-continuous and there exists a subsequence \( \{T^{n_k}x_0\} \) of \( \{T^n x_0\} \) converges to some element \( x_* \in X \) such that \((T^{n_k}x_0, x_*) \in X_\leq \) for any \( k \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \). Moreover, if \( Tx = x \), then \( P(x, x) = 0 \).

**Example 4.10.** Let \( X = [0, 1] \) and \( D(x, y) = (x - y)^2 \) with the \( \omega \)-distance \( P \) on \( X \) defined by \( P(x, y) = |y|^2 \). We consider the following set:

\[
X_\leq = \{(x, y) \in X \times X : x = y \text{ or } x, y \in \{0\} \cup \left\{ \frac{1}{n} : n \geq 1 \right\} \},
\]

where \( \leq \) is the usual ordering. Let \( T : X \to X \) be a mapping define by

\[
T(x) = \begin{cases} 
  x^2, & \text{if } x = \frac{1}{n}, n \geq 2, \\
  x, & \text{otherwise.}
\end{cases}
\]

Then \( T \) is a nondecreasing mapping. Also, \( x = 0 \) is an element in \( X \) such that \( 0 \leq T(0) = 0 \) and so \((0, T(0)) \in X_\leq \). Hence \( T \) satisfies the condition (ii).

Next, we show that \( T \) satisfies the condition (i) of Theorem 4.7 with the simulation function in given in Example 4.4. If \( x \neq \frac{1}{n} \) for all \( n \geq 2 \), then
\((x, T(x)) \in X \leq \) and it is easy to see that \(T\) satisfies the condition (i). If \(x = \frac{1}{n}\) for all \(n \geq 2\), then \((\frac{1}{n}, T\frac{1}{n}) \in X \leq\). Further, we have

\[
\zeta(2P(Tx, T^2x), P(x, Tx)) = \zeta\left(2P\left(\frac{1}{n^2}, \frac{1}{n}\right), P\left(\frac{1}{n}, \frac{1}{n^2}\right)\right)
\]

\[
= \zeta\left(2\left(\frac{1}{n^2}\right)^2, \left(\frac{1}{n}\right)^2\right)
\]

\[
= \left(\frac{1}{n^2}\right)^2 - 2\left(\frac{1}{n}\right)^2
\]

\[
= \frac{1 + 2 \cdot \left(\frac{1}{n}\right)^2}{n^8 - 2n^4 + n^4 + 2}
\]

\[
= \frac{n^8 - 2n^4}{n^4 + 2}
\]

\[
= \frac{n^8(n^4 + 2)}{n^4 - 2}
\]

\[
> 0.
\]

Hence \(T\) satisfies the condition (i). Furthermore, for each \(x \in X\), \(T^{n+1}(x) \to 0 \in X\) as \(i \to \infty\), and also \(T^{n+1}(x) \to T(0) \in X\) as \(i \to \infty\). Hence all the conditions of Theorem 4.7 are satisfied. Furthermore, \(x = 0\) is fixed points of \(T\).

**Acknowledgements.** This project was supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Research Cluster (CLASSIC), Faculty of Science, KMUTT. The first author was supported by Thailand Research Fund (Grant No. TRG5880221) and Kasetsart University Research and Development Institute (KURDI). Also, Yeol Je Cho was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and future Planning (2014R1A2A2A01002100).

The authors are also grateful to the referee by several useful suggestions that have improved the first version of the paper.

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