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On topological groups with remainder of character k

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Abstract

In [A.V. Arhangel'skii and J. van Mill, On topological groups with a first-countable remainder, *Topology Proc. 42 (2013), 157-163] it is proved that the character of a non-locally compact topological group with a first countable remainder doesn't exceed ω_1 and a non-locally compact topological group of character ω_1 having a compactification whose remainder is first countable is given. We generalize these results in the general case of an arbitrary infinite cardinal κ.*

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1. Introduction

In [3] the authors answer in the negative to the following problem:

Problem 1.1 ([3, Problem 1.1])**.** *Suppose that G is a non-locally compact topological group with a first countable remainder. Is G metrizable?*

Also, the following necessary condition for a non-locally compact topological group to have a first countable remainder is established:

Theorem 1.2 ([3, Theorem 2.1])**.** *Suppose that G is a non-locally compact topological group with a first countable remainder. Then the character of the space G doesn't exceed ω_1.*

As a consequence of the previous result the following holds:

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Theorem 1.3 ([3, Theorem 2.4]). *If G is a non-locally compact topological group with a first countable remainder, then $|G| \leq 2^{\omega_1}$.*

In [3] it is proved that Theorem 1.2 is the best possible giving the following:

Example 1.4 ([3, Section 3]). A non locally compact topological group G of character ω_1 which has a compactification bG such that $bG \setminus G$ is first countable.

In this paper we show that the methods used by Arhangel'skii and van Mill permit to generalize Theorem 1.2 and Example 1.4 in the general case of an arbitrary infinite cardinal κ.

By a space we understand a Tychonoff topological space. By a remainder of a space we mean the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. We follow the terminology and notation in [4].

2. Generalizations of Arhangel'skii and van Mill's results in the general case of an arbitrary infinite cardinal κ

We show that Arhangel'skii and van Mill's proof of [3, Theorem 2.1], works in the general case of an arbitrary infinite cardinal κ.

Theorem 2.1. Let κ be an infinite cardinal and let G be a non-locally compact topological group. Assume that G has a compactification such that its remainder $bG \setminus G$ has character κ. Then the character of the space G doesn't exceed κ^+.

To prove Theorem 2.1, we need the following propositions 2.2 and 2.3. In particular, Proposition 2.2 is known (see for example [1], also note that the concept of free sequence was introduced in [2]). We include the proof of Proposition 2.2 for completness of the exposition.

Proposition 2.2. Suppose that Y is a space with tightness $t(Y) = \kappa$ satisfying the following condition:

(s) for any subset A of Y such that $|A| \leq \kappa^+$, the closure of A in Y is compact.

Then Y is compact.

Proof. Striving for a contradiction, assume that Y is not compact and let X be a compactification of Y. Pick an arbitrary point $x \in X \setminus Y$. Then:

Fact 1: Every nonempty G_κ-subset P of X that contains x meets Y.

Indeed, let $P = \bigcap\{V_\alpha : \alpha < \kappa\}$, where each V_α is open. For each α take an open set U_α in X such that $x \in \overline{U}_\alpha \subseteq V_\alpha$. Put $\{U_\alpha : \alpha < \kappa\} = \mathcal{U}$. We may assume without any loss of generality that $\mathcal{U}$ is closed under finite intersections. For any $U \in \mathcal{U}$ pick a point $y_U \in U \cap Y$ and let $A = \{y_U : U \in \mathcal{U}\}$. By condition (s), the set $S = \overline{A}^Y$ is compact. As the family $\mathcal{F} = \{\overline{U} \cap S : U \in \mathcal{U}\}$ has the finite intersection property, we must have $\bigcap \mathcal{F} \neq \varnothing$. Since $\bigcap \mathcal{F} \subseteq P \cap Y$, we are done.

Using Fact 1, we define for every $\xi < \kappa^+$ a point $y_\xi \in Y$ and a closed G_κ-subset P_ξ of X containing x, as follows. Let y_0 be any element of Y, and put

$P_0 = X$. Now assume that $\xi < \kappa^+$, and that the points $y_\beta \in Y$ and the closed G_κ-subsets P_β of X have been defined for every $\beta < \xi$. Denote by F_ξ the closure of the set $\{y_\beta : \beta < \xi\}$ in X. Then $F_\xi \subseteq Y$ and $x \notin F_\xi$. Since F_ξ is closed in X and X is Tychonoff, it follows that there exists a closed G_δ-subset V of x in X such that $x \in V$ and $V \cap F_\xi = \varnothing$. Put $P_\xi = V \cap \bigcap_{\beta<\xi} P_\beta$. Clearly, $x \in P_\xi$, and P_ξ is a closed G_κ-subset of X. By Fact 1, we have $P_\xi \cap Y \neq \varnothing$. This completes the transfinite construction.

Obviously, the following statements hold for any $\xi < \kappa^+$ (Fact 4 follows directly from facts 2 and 3).

Fact 2: $\overline{\{y_\beta : \beta < \xi\}} \cap P_\xi = \varnothing$.
Fact 3: $\overline{\{y_\beta : \xi \leq \beta < \kappa^+\}} \subseteq P_\xi$.
Fact 4: $\overline{\{y_\beta : \beta < \xi\}} \cap \overline{\{y_\beta : \xi \leq \beta < \kappa^+\}} = \varnothing$.

Fact 4 implies that $\eta = \{y_\xi : \xi < \kappa^+\}$ is a free sequence in X. Its closure is compact and is contained in Y. Hence this contradicts the fact that the tightness of Y is at most κ (Juhász [5, 3.12]). □

Following the argument from [3, Proposition 2.3], and using Proposition 2.2 instead of [3, Proposition 2.2] we obtain the following result.

Proposition 2.3. Suppose that X is a nowhere locally compact space with remainder Y such that $\chi(Y) = \kappa$, where κ is an infinite cardinal. Then the π-character of the space X doesn't exceed κ^+ at some point of X.

Proof of Theorem 2.1. It follows from Proposition 2.3 that there exists a π-base $\mathcal{P}$ of G at the neutral element e of G such that $|\mathcal{P}| \leq \kappa^+$. Then, clearly, the family $\mu = \{UU^{-1} : U \in \mathcal{P}\}$ is a base of G at e such that $|\mu| \leq \kappa^+$. □

Theorem 2.4. If G is a non-locally compact topological group with remainder Y such that $\chi(Y) = \kappa$, then $|G| \leq 2^{\kappa^+}$.

Proof. Let bG be a compactification of the space G such that the remainder $Y = bG \setminus G$ has character κ. By Theorem 2.1, the character of the space G doesn't exceed κ^+. Since $\chi(Y) = \kappa$ and Y and G are both dense in bG, we conclude that $\chi(bG) \leq \kappa^+$. Since bG is compact, it follows that $|bG| \leq 2^{\kappa^+}$. Hence, $|G| \leq 2^{\kappa^+}$. □

Following the method in [3, Section 3] we construct the following example.

Example 2.5. A non-locally compact topological group G of character κ^+ which has a compactification bG such that $bG \setminus G$ has character κ.

Let X be a space with a dense subset D and consider the subspace

$$X(D) = (X \times \{0\}) \cup (D \times \{1\})$$

of the Alexandroff duplicate of X.
Observe that $X(D)$ is compact if X is compact.

The idea used by authors in [3, Section 3] is to replace every isolated point of the form $(d,1)$ in $X(D)$ by a copy of a fixed non-empty space Y. Also they note that if both X and Y are compact, then so is $X(D,Y)$ and that the function $\pi : X(D,Y) \to X \times \{0\}$ that collapses each set of the form $\{d\} \times Y \times \{1\}$ to $(d,0)$ is a retraction.

Let $\kappa \geq \omega$ and let $K = 2^\kappa(2^\kappa)$, i.e., the Alexandroff duplicate of the Cantor cube 2^κ. Following the idea used in [3, Section 3] and using this building block repeatedly, we will construct an inverse sequence of compact spaces $X_\alpha, \alpha < \kappa^+$.

In particular following step by step [3, Section 3] and defining $X_0 = 2^\kappa$ instead of 2^ω, we construct all X_α, where $\alpha < \omega_1$ and $X_{\omega_1} = \varprojlim\{X_\alpha, \pi^\alpha_\beta\}$. Let $\pi^{\omega_1}_\alpha : X_{\omega_1} \to X_\alpha$ denotes the projection for all $\alpha < \omega_1$.

Also the points $p \in X_{\omega_1}$, for which $\pi^{\omega_1}_\alpha(p)$ is isolated for every successor ordinal number $\alpha < \omega_1$, form a dense subspace H in X_{ω_1}.

Now put $X_{\omega_1+1} = X_{\omega_1}(H)$, and let $\pi^{\omega_1+1}_{\omega_1}$ be the standard retraction. We continue as before, replacing each isolated point by a copy of K, etc. Let $X_{\omega_1+\omega_1}$ be the inverse limit of spaces $X_{\omega_1+\beta}, \beta < \omega_1$. Continuing in this way for all $\alpha < \omega_2$, we get an inverse sequence $\{X_\alpha, \pi^\alpha_\beta\}$ of compact spaces having character equal to κ. Let $X_{\omega_2} = \varprojlim\{X_\alpha, \pi^\alpha_\beta\}$ with retractions $\pi^{\omega_2}_\alpha : X_{\omega_2} \to X_\alpha$ for all $\alpha < \omega_2$.

Continuing in this way for all $\alpha < \kappa^+$, we get an inverse sequence $\{X_\alpha, \pi^\alpha_\beta\}$ of compact spaces having character equal to κ. Let $X_{\kappa^+} = \varprojlim\{X_\alpha, \pi^\alpha_\beta\}$ with retractions $\pi^{\kappa^+}_\alpha : X_{\kappa^+} \to X_\alpha$ for all $\alpha < \kappa^+$.

The following fact holds.

If $p \in X_{\kappa^+}$ and there exists a successor ordinal number $\alpha < \kappa^+$ such that $\pi^{\kappa^+}_\alpha(p)$ is not isolated, then

$$(\pi^{\kappa^+}_\alpha)^{-1}(\{\pi^{\kappa^+}_\alpha(p)\}) = \{p\}.$$

Hence, X_{κ^+} has character equal to κ at p.

The points $p \in X_{\kappa^+}$, for which $\pi^{\kappa^+}_\alpha(p)$ is isolated for every successor ordinal number $\alpha < \kappa^+$, form a dense subspace G in X_{κ^+}. The space G is easily seen to be homeomorphic to the space $(2^\kappa)^{\kappa^+}$ with the G_κ-topology (where the topology on 2^κ is the standard product topology). The reason that we get the G_κ-topology is clear: because if $p \in G$, then, for every $\alpha < \kappa^+$, we have that $\pi^{\kappa^+}_{\alpha+1}(p)$ is isolated. Hence, G is a topological group, and so we are done.

3. Suggestions for further research

It seems natural to pose the following question:

Question 3.1. Is it possible to generalize Theorem 2.1 and Example 2.5 in some "generalized" class of topological groups? In particular, is this possible in the class of paratopological groups?

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