A decomposition of normality via a generalization of $\kappa$-normality

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ABSTRACT

A simultaneous generalization of $\kappa$-normality and weak $\theta$-normality is introduced. Interrelation of this generalization of normality with existing variants of normality is studied. In the process of investigation a new decomposition of normality is obtained.

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KEYWORDS: regularly open set; regularly closed set; $\theta$-open set; $\theta$-closed set; $\kappa$-normal (mildly) normal space; almost normal space; (weakly) (functionally) $\theta$-normal space; weakly $\kappa$-normal space; $\Delta$-normal space; strongly seminormal space.

1. INTRODUCTION AND PRELIMINARIES

Several generalized notions of normality such as almost normal, $\kappa$-normal, $\Delta$-normal, $\theta$-normal, semi-normal, Quasi-normal, $\pi$-normal, densely normal etc. exist in the literature. Recently, Interrelation among some of these variants of normality was studied in [4] and factorizations of normality are obtained in [4, 5, 12, 14]. In this paper, we tried to exhibit the interrelations that exist among these generalized notions of normality and introduced a simultaneous generalization of $\kappa$-normality and weak $\theta$-normality called weak $\kappa$-normality. Interestingly, the class of weakly $\kappa$-normal spaces contains the class of almost compact spaces whereas the class of $\kappa$-normal spaces does not contain the class of almost compact spaces. This newly introduced notion of weak normality
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is utilized to obtain a factorization of normality. Moreover, it is verified that
some covering properties which need not imply \( \kappa \)-normality implies weak \( \kappa \)-normality.

Let \( X \) be a topological space and let \( A \subset X \). Throughout the present paper,
the closure and interior of a set \( A \) will be denoted by \( \overline{A} \) (or \( clA \)) and \( intA \) (or \( A^o \)) respectively. A set \( U \subset X \) is said to be regularly open [15] if \( U = \text{in} \overline{U} \). The complement of a regularly open set is called regularly closed. It is observed that
an intersection of two regularly closed sets need not be regularly closed. A finite
union of regularly open sets is called \( \pi \)-open set and a finite intersection of regular
closed sets is called \( \pi \)-closed set. It is obvious that
an intersection of two \( \pi \)-closed sets need not be \( \pi \)-closed. A finite
union (intersection) of \( \pi \)-closed sets is \( \pi \)-closed, but the infinite union (intersection)
of \( \pi \)-closed sets need not be \( \pi \)-closed (See [11]). A point \( x \in X \) is called a
\( \theta \)-limit point (respectively \( \delta \)-limit point) [21] of \( A \) if every closed (respectively
regularly open) neighbourhood of \( x \) intersects \( A \). Let \( cl_\theta A \) (respectively \( cl_\delta A \))
denotes the set of all \( \theta \)-limit point (respectively \( \delta \)-limit point) of \( A \). The set \( A \) is called \( \theta \)-closed (respectively \( \delta \)-closed) if \( A = cl_\theta A \) (respectively \( A = cl_\delta A \)). The complement of a \( \theta \)-closed (respectively \( \delta \)-closed) set will be referred to as a \( \theta \)-open (respectively \( \delta \)-open) set. The family of \( \theta \)-open sets as well as the family of \( \delta \)-open sets form topologies on \( X \). The topology formed by the set of \( \delta \)-open sets is the semiregularization topology whose basis is the family of
regularly open sets.

Let \( Y \) be a subspace of \( X \). A subset \( A \) of \( X \) is concentrated on \( Y \) [2] if \( A \)
is contained in the closure of \( A \cap Y \) in \( X \). A subset \( A \) of \( Y \) is said to be
strongly concentrated on \( Y \) [6] if \( A \subset \overline{(A \cap Y)^c} \). It is obvious that
every strongly concentrated set is concentrated. We say that \( X \) is normal on \( Y \) if
every two disjoint closed subsets of \( X \) concentrated on \( Y \) can be separated by
disjoint open neighbourhoods in \( X \) [2]. Similarly, \( X \) is said to be weakly normal on \( Y \) [6] if for every disjoint closed subsets \( A \) and \( B \) of \( X \) strongly concentrated
on \( Y \), there exist disjoint open sets in \( X \) separating \( A \) and \( B \) respectively.

A space \( X \) is called densely normal if there exists a dense subspace \( Y \) of \( X \)
such that \( X \) is normal on \( Y \) [2]. A topological space \( X \) is said to be weakly densely normal [6] if there exist a proper dense subspace \( Y \) of \( X \) such that \( X \) is weakly normal on \( Y \). It is easy to see that every densely normal space is
weakly densely normal and every weakly densely normal space is \( \kappa \)-normal. On
the other hand, the converses are not true, as were shown in [10] and [6].

Lemma 1.1. A subset \( A \) of a topological space \( X \) is \( \theta \)-open if and only if for
each \( x \in A \), there is an open set \( U \) such that \( x \in U \subset \overline{U} \subset A \).

Definition 1.2. A topological space \( X \) is said to be

(i) quasi-normal [23] if any two disjoint \( \pi \)-closed subsets \( A \) and \( B \) of \( X \)
there exist two open disjoint subsets \( U \) and \( V \) of \( X \) such that \( A \subset U \)
and \( B \subset V \).
(ii) \( \pi \)-normal [11] if for any two disjoint closed subsets \( A \) and \( B \) of \( X \) one of which is \( \pi \)-closed, there exist two open disjoint subsets \( U \) and \( V \) of \( X \) such that \( A \subset U \) and \( B \subset V \).

(iii) \( \Delta \)-normal [9] if every pair of disjoint closed sets one of which is \( \delta \)-closed are contained in disjoint open sets.

(iv) weakly \( \Delta \)-normal [9] if every pair of disjoint \( \delta \)-closed sets are contained in disjoint open sets.

(v) \( \theta \)-normal [12] if every pair of disjoint \( \theta \)-closed sets one of which is \( \theta \)-closed are contained in disjoint open sets;  

(vi) weakly \( \theta \)-normal [12] if every pair of disjoint \( \theta \)-closed sets are contained in disjoint open sets;  

(vii) functionally \( \theta \)-normal [12] if for every pair of disjoint closed sets \( A \) and \( B \) one of which is \( \theta \)-closed there exists a continuous function \( f : X \to [0,1] \) such that \( f(A) = 0 \) and \( f(B) = 1 \).

(xvi) \( \Delta \)-regular [9] if for every closed set \( F \) and each open set \( U \) containing \( F \), there exist regular open sets \( V \) such that \( F \subset V \subset U \).

2. Weakly \( \kappa \)-normal spaces

Definition 2.1. A \( \theta \)-closed set \( A \) is said to be a regularly \( \theta \)-closed set if \( \text{int} A = A \). The complement of a regularly \( \theta \)-closed set will be regularly \( \theta \)-open.

Clearly every regularly \( \theta \)-closed set is regularly closed as well as \( \theta \)-closed but the converse need not be true.

Example 2.2. Let \( X \) be the set of positive integers. Define a topology on \( X \) by taking every odd integer to be open and a set \( U \subset X \) is open if for every
even integer \( p \in U \), the predecessor and the successor of \( p \) are also in \( U \). Here the set \( \{2k, 2k + 1, 2k + 2 : k \in \mathbb{Z}^+\} \) is a regularly closed set which is not \( \theta \)-closed.

**Example 2.3.** Let \( X \) denote the interior of the unit square \( S \) in the plane together with the points \((0, 0)\) and \((1, 0)\), i.e. \( X = S^0 \cup \{(0, 0), (1, 0)\} \). Every point in \( S^0 \) has the usual Euclidean neighbourhoods. The points \((0, 0)\) and \((1, 0)\) have neighbourhoods of the form \( U_n \) and \( V_n \) respectively, where, \( U_n = \{(0, 0)\} \cup \{(x, y) : 0 < x < 1/2, 0 < y < 1/n\} \) and \( V_n = \{(1, 0)\} \cup \{(x, y) : 1/2 < x < 1, 0 < y < 1/n\} \). Clearly, the sets \( \{(0, 0)\} \) and \( \{(1, 0)\} \) are \( \theta \)-closed but not regularly \( \theta \)-closed.

**Definition 2.4.** A topological space \( X \) is said to be weakly \( \kappa \)-normal if for every pair of disjoint regularly \( \theta \)-closed sets \( A \) and \( B \) there exist disjoint open sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

From the definitions it is obvious that every \( \kappa \)-normal space is weakly \( \kappa \)-normal and every weakly \( \theta \)-normal space is weakly \( \kappa \)-normal. The following diagram illustrates the interrelations that exist between weakly \( \kappa \)-normal spaces and variants of normality that exist in literature. But none of the implications below is reversible (See [7], [9], [11], [12], [14], [18] and examples below).

![Diagram](attachment:image.png)

**Example 2.5.** The space defined in Example 2.2 is weakly \( \kappa \)-normal but not \( \kappa \)-normal.

**Example 2.6.** The example of a Tychonoff \( \kappa \)-normal space which is not densely normal was given by Just and Tartir [10]. Since every regular space is \( \theta \)-regular [12], this space is \( \theta \)-regular but not normal. Thus the space is not weakly \( \theta \)-normal as every \( \theta \)-regular, weakly \( \theta \)-normal space is normal [12].
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**Theorem 2.7.** A topological space $X$ is weakly $\kappa$-normal if and only if for every regularly $\theta$-closed set $A$ and a regularly $\theta$-open set $U$ containing $A$ there is an open set $V$ such that $A \subset V \subset \overline{V} \subset U$.

**Proof.** Let $X$ be a weakly $\kappa$-normal space and $U$ be a regularly $\theta$-open set containing a regularly $\theta$-closed set $A$. Then $A$ and $X - U$ are disjoint regularly $\theta$-closed sets in $X$. Since $X$ is weakly $\kappa$-normal, there are disjoint open sets $V$ and $W$ containing $A$ and $X - U$, respectively. Since $X - W$ is closed, $A \subset V \subset X - W \subset U$. Conversely, let $A$ and $B$ be two disjoint regularly $\theta$-closed sets in $X$. Then $U = X - B$ is a regularly $\theta$-open set containing the regularly $\theta$-closed set $A$. Thus by the hypothesis there exists an open set $V$ such that $A \subset V \subset \overline{V} \subset U$. Then $V$ and $X - V$ are disjoint open sets containing $A$ and $B$, respectively. Hence $X$ is weakly $\kappa$-normal. $\Box$

**Theorem 2.8.** Let $X$ be a finite topological space. For a subset $A$ of $X$, the following statements are equivalent.

(a) $A$ is clopen.
(b) $A$ is $\theta$-closed.
(c) $A$ is $\theta$-open.

**Proof.** The implication (a) $\implies$ (b) is obvious. To prove (b) $\implies$ (a), let $A$ be a closed subset of $X$. Then $(X - A)$ is $\theta$-open in $X$. By Lemma 1.3.4, for each $x \in X - A$ there exists an open set $U_x$ containing $x$ such that $x \in U_x \subset \overline{U_x} \subset X - A$. Since $X$ is finite, $\bigcup_{x \in X - A} U_x = X - A$, is the union of finitely many closed sets and hence closed. Thus $A$ is open. By hypothesis $A$ is $\theta$-closed and hence closed. Consequently, $A$ is clopen. The proofs of (a) $\implies$ (c) and (c) $\implies$ (a) are similar and hence omitted. $\Box$

From the above result the following observation is obvious.

**Remark 2.9.** Every finite topological space is weakly $\kappa$-normal whereas finite topological spaces need not be $\kappa$-normal.

**Theorem 2.10 (**[13]**). A space $X$ is almost regular if and only if for every open set $U$ in $X$, $\text{int}\overline{U}$ is $\theta$-open.

**Theorem 2.11.** In an almost regular space, the following statements are equivalent.

(a) $X$ is $\kappa$-normal.
(b) $X$ is weakly $\kappa$-normal.

**Proof.** The proof of (a) $\implies$ (b) directly follows from definitions. To prove (b) $\implies$ (a), let $X$ be an almost regular, weakly $\kappa$-normal space. Let $A$ and $B$ be two disjoint regularly closed sets in $X$. By Theorem 2.10, $A$ and $B$ are disjoint regularly $\theta$-closed sets in $X$. Thus by weak $\kappa$-normality of $X$, there exist disjoint open sets separating $A$ and $B$. Hence $X$ is $\kappa$-normal. $\Box$

**Theorem 2.12.** In an almost regular space, every $\pi$-closed set is $\theta$-closed.
Proof. Let $X$ be an almost regular space and let $A \subset X$ be $\pi$-closed in $X$. Thus $A$ is finite intersection of $\pi$-closed sets in $X$. Since in an almost regular space every regularly closed set is $\theta$-closed [16] and finite intersection of $\theta$-closed sets is $\theta$-closed [21], $A$ is $\theta$-closed. □

Theorem 2.13. Every almost regular, weakly $\theta$-normal space is quasi normal.

Proof. Let $X$ be an almost regular, weakly $\theta$-normal space. Let $A$ and $B$ be two disjoint $\pi$-closed sets in $X$. By Theorem 2.12, $A$ and $B$ are disjoint $\theta$-closed sets which can be separated by disjoint open set as $X$ is weakly $\theta$-normal. □

Theorem 2.14. Every almost regular, $\theta$-normal space is $\pi$-normal.

It is well known that every compact Hausdorff space is normal. However, in the absence of Hausdorffness or regularity a compact space may fail to be normal. Thus it is useful to know which topological property weaker than Hausdorffness with compactness implies normality. The property of being a $T_1$ space fails to do the job since the cofinite topology on an infinite set is a compact $T_1$ space which is not normal. However, it is well known that Every compact $R_0$-space is normal (See [17]). In [12], it is shown that every compact space in particular every paracompact space in absence of any separation axioms is $\theta$-normal. It is also known that every Lindelöf spaces need not be $\kappa$-normal. However, by the following theorem of [12] it follows that every Lindelöf space is weakly $\kappa$-normal. Similarly, almost compactness need not implies $\kappa$-normality, but by Theorem of [12], every almost compact space is weakly $\kappa$-normal.

Theorem 2.15 ([12]). Every Lindelöf space is weakly $\theta$-normal.

Corollary 2.16. Every Lindelöf space is weakly $\kappa$-normal.

Corollary 2.17. Every almost regular, Lindelöf space is $\kappa$-normal.

Proof. The prove immediately follows from Theorem 2.11, since in an almost regular space every weakly $\kappa$-normal space is $\kappa$-normal. □

Theorem 2.18 ([12]). Every almost compact space is weakly $\theta$-normal.

Corollary 2.19. Every almost compact space is weakly $\kappa$-normal.

Corollary 2.20. Every almost regular, almost compact space is $\kappa$-normal.

Proof. The prove immediately follows from Theorem 2.11, since in an almost regular space every weakly $\kappa$-normal space is $\kappa$-normal. □

Remark 2.21. Corollary 2.17 and Corollary 2.20 were independently proved in [18]. In contrary to the above results the following example establishes that Lindelöf spaces need not be $\kappa$-normal and almost compactness need not imply $\kappa$-normality.

Example 2.22. Let $X$ be the set of positive integers with the topology as defined in Example 2.2 and $Y = \{1, 2, 3, ..., 11\}$. Then the subspace topology on $Y$ is compact but not $\kappa$-normal as disjoint regularly closed sets $\{2, 3, 4\}$ and $\{6, 7, 8\}$ can not be separated by disjoint open sets.
Definition 2.23 ([13]). A space $X$ is said to be $\theta$-compact if every open covering of $X$ by $\theta$-open sets has a finite subcollection that covers $X$.

The following result is useful to show that every almost regular, $\theta$-compact space is $\kappa$-normal as well as weakly $\theta$-normal.

Theorem 2.24 ([16]). Let $A \subset X$ be $\theta$-closed and let $x \notin A$. Then there exist regular open sets which separate $x$ and $A$.

Theorem 2.25. In an almost regular space, every $\theta$-compact space is weakly $\theta$-normal.

Proof. Let $X$ be an almost regular $\theta$-compact space. Let $A$ and $B$ be any two disjoint $\theta$-closed subsets of $X$. By Theorem 2.24, for every $a \in A$, there exist disjoint regularly open sets $U_a$ and $V_a$ containing $a$ and $B$ respectively. Since $X$ is almost regular, $U_a$ and $V_a$ are disjoint $\theta$-open sets containing $a$ and $B$. Now the collection $\{U_a : a \in A\}$ is a $\theta$-open cover of $A$. Then $A \subset \bigcup_{a \in A} U_a = O$.

Since arbitrary union of $\theta$-open sets is $\theta$-open, $X - O = D$ is $\theta$-closed. Since $A$ is a $\theta$-closed set disjoint from $D$, by Theorem 2.24, for every $d \in D$, there exist disjoint regularly open sets $S_d$ and $T_d$ containing $A$ and $d$ respectively. Again by almost regularity of $X$, $T_d$ is a $\theta$-open set which is disjoint from $A$. Now the collection $U = \{U_a : a \in A\} \cup \{T_d : d \in D\}$ is a $\theta$-open covering of $X$. By $\theta$-compactness of $X$, $\text{mathcal}U$ has a finite subcollection $V$ which covers $X$. Let the members of $V$ which intersects $A$ be $W$. Each member of $W$ is of the form $U_a$ for some $a \in A$ as for each $d \in D$, $T_d \cap A = \phi$. Suppose $W = \{U_{a_i} : i = 1, 2, 3, ..., n\}$. Then $\bigcap_{i=1}^{n} U_{a_i} = U$ and $\bigcap_{i=1}^{n} V_{a_i} = V$ are disjoint open sets containing $A$ and $B$ respectively. Hence $X$ is weakly $\theta$-normal. □

Corollary 2.26. In an almost regular space, every $\theta$-compact space is weakly $\kappa$-normal.

Proof. The proof immediately follows from the fact that every $\theta$ normal space is weakly $\kappa$-normal. □

Corollary 2.27. In an almost regular space, every $\theta$-compact space is $\kappa$-normal.

Proof. The proof immediately follows from Theorem 2.11. □

Corollary 2.28. In an almost regular space, every almost compact space is weakly $\kappa$-normal.

Proof. The proof is immediate as every almost compact space is $\theta$-compact [13]. □

3. Decompositions of normality

Theorem 3.1. An $T_1$-space is almost normal if and only if it is almost $\beta$-normal and weakly $\kappa$-normal.
Proof. Necessary part is obvious. Conversely, let $X$ be a $T_1$-almost $\beta$ normal, weakly $\kappa$-normal space. Since $X$ is $T_1$-almost $\beta$-normal, by Theorem 2.9 of [3], $X$ is almost regular. So by Theorem 2.11, $X$ is $\kappa$-normal. Hence $X$ is almost normal as every almost $\beta$-normal, $\kappa$-normal space is almost normal [3]. □

Corollary 3.2. An $T_1$ space is normal if and only if it is almost $\beta$-normal, weakly $\kappa$-normal and semi-normal.

Proof. The prove follows from the result that every almost normal, semi normal space is normal [18]. □

Definition 3.3. A space $X$ is said to be strongly seminormal if for every closed set $A$ contained in an open set $U$ there exists a regularly $\theta$-open set $V$ such that $A \subset V \subset U$.

Theorem 3.4. Every normal space is strongly seminormal.

Proof. Let $A$ be a closed set and $U$ be an open set containing $A$. Let $B = X - U$. Then $A$ and $B$ are disjoint closed sets in $X$. By Urysohn’s lemma there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Let $V = f^{-1}(0,1/2)$ and $W = f^{-1}(1/2,1]$. Then $A \subset V \subset X - W \subset U$. Thus $A \subset V \subset V' \subset X - W \subset U$. We claim that $V'$ is a regularly $\theta$-open set. $V'$ is regularly open, only we have to show that $V'$ is $\theta$-open. Let $x \in V'$. Then $f(x) \in (0,1/2)$. So there is a closed neighbourhood $N$ of $f(x)$ contained in $[0,1/2)$. Let $U_x = (f^{-1}(N))^\circ$. Then $x \in U_x \subset f^{-1}(N) \subset V'$. By Lemma 1.1, $V'$ is $\theta$-open. Hence $X$ is strongly seminormal. □

Theorem 3.5. Every strongly seminormal space is seminormal.

Theorem 3.6. Every strongly seminormal space is $\theta$-regular.

The following implications are obvious but none of these is reversible.

![Implications Diagram]

Example 3.7. Let $X$ be the set of positive integers with the topology as defined in Example 2.2, then $X$ is seminormal but not strongly seminormal.

Example 3.8. The space given in [10] by Just and Tartir is an example of a Tychonoff $\kappa$-normal space which is not densely normal. Since every seminormal $\kappa$-normal space is normal [18], thus becoming densely normal, this space is not seminormal but is $\theta$-regular as every regular space is $\theta$-regular.

Theorem 3.9. A space $X$ is normal if and only if it is strongly seminormal and weakly $\kappa$-normal.
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**Proof.** The necessary part i.e., a normal space is strongly seminormal as well as weakly \( \kappa \)-normal directly follows from the definition. Conversely, let \( X \) be a strongly seminormal and weakly \( \kappa \)-normal space. let \( A \) and \( B \) be two disjoint closed sets in \( X \). Thus \( A \) is a closed set contained in an open set \( U = X - B \). Since \( X \) is strongly seminormal, there exists an regularly \( \theta \)-open set \( V \) such that \( A \subset V \subset U \). Now \( X - V \) is a regularly \( \theta \)-closed set contained in an open set \( X - A \). Again by strong seminormality of \( X \), there exists a regularly \( \theta \)-open set \( W \) such that \( X - V \subset W \subset X - A \). Thus \( X - V \) and \( X - W \) are two disjoint regularly \( \theta \)-closed sets in \( X \) containing \( B \) and \( A \) respectively. By weak \( \kappa \)-normality of \( X \), there exist two disjoint open sets \( O \) and \( P \) separating \( X - W \) and \( X - V \). Hence \( X \) is normal. \( \square \)

**Corollary 3.10.** In the class of strongly seminormal spaces, the following statements are equivalent.

(a) \( X \) is normal.
(b) \( X \) is \( \Delta \)-normal.
(c) \( X \) is \( \text{wf} \Delta \)-normal.
(d) \( X \) is weakly \( \Delta \)-normal.
(e) \( X \) is functionally \( \theta \)-normal.
(f) \( X \) is \( \theta \)-normal.
(g) \( X \) is weakly functionally \( \theta \)-normal.
(h) \( X \) is weakly \( \text{ quasi} \)-normal.
(i) \( X \) is \( \pi \)-normal.
(j) \( X \) is quasi \( \text{ normal} \).
(k) \( X \) is almost normal.
(l) \( X \) is \( \kappa \)-normal.
(m) \( X \) is weakly \( \kappa \)-normal.

**Remark 3.11.** In [9], it is shown that in the class of \( \Delta \)-regular spaces statements (a)-(d) of Corollary 3.10 are equivalent and in the class of \( \theta \)-regular spaces statements (a)-(h) are equivalent.

**References**
