A note on unibasic spaces and transitive quasi-proximities

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ABSTRACT

In this paper we prove there is a bijection between the set of all annular bases of a topological spaces \((X, \tau)\) and the set of all transitive quasi-proximities on \(X\) inducing \(\tau\). We establish some properties of those topological spaces \((X, \tau)\) which imply that \(\tau\) is the only annular basis.

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1. INTRODUCTION

W. J. Pervin showed in [9] that every topological spaces \((X, \tau)\) has a quasi-proximity \(\delta\) which induces the original topology. In this paper we give conditions for a topological space \((X, \tau)\) admits a unique compatible quasi-proximity in which the topology is the only annular basis.

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By a quasi-proximity (see [1]) on a set $X$ we will mean a relation $\delta$ between the family of subsets of $X$ satisfying the following axioms:

a) $(X, \emptyset) \notin \delta$ and $(\emptyset, X) \notin \delta$;

b) $(C, A \cup B) \in \delta$ if only if $(C, A) \in \delta$ or $(C, B) \in \delta$;

c) $(A \cup B, C) \in \delta$ if only if $(A, C) \in \delta$ or $(B, C) \in \delta$;

d) For every $x \in X$, $(\{x\}, \{x\}) \in \delta$;

e) If $(A, B) \notin \delta$, there exists a set $C \subseteq X$ such that $(A, C) \in \delta$ and $(X \setminus C, B) \notin \delta$.

A quasi-proximity $\delta$ on $X$ is a proximity on $X$ if $\delta = \delta^{-1}$, i.e., $(A, B) \in \delta$ iff $(B, A) \in \delta$.

For brevity, we write $A \delta B$ instead of $(A, B) \in \delta$ and $A \delta B$ instead of $(A, B) \notin \delta$.

Let $\delta$ be a quasi-proximity on a set $X$. For each $A \subseteq X$, define $\bar{A} = \{x \in X : \{x\} \in \delta\}$. Then the assignment $A \rightarrow \bar{A}$ is a Kuratowski-closure operator on $X$ and the corresponding topology on $X$ is denoted as $\tau_\delta$ (see [1], 1.27).


J. Ferrer in [2] trying to solve the question of whether every $T_1$ topological space with a unique compatible quasi-proximity should be hereditarily compact, he shows that it is true for product spaces as well as for locally hereditarily Lindelöf spaces.

H.-P. Künzi and S. Watson in [7] construct a $T_1$-space $X$ is not hereditarily compact, but each open subset of $X$ is the intersection of two compact open sets. The construction is carried out in ZFC, but the cardinality of the space is very large.

2. **Unibasic spaces and transitive quasi-proximities**

The main result of this section establishes a bijection between all annular bases of a topological space $(X, \tau)$ and all transitive quasi-proximities on $X$ inducing $\tau$.

A basis $B$ for a topological space $(X, \tau)$ is annular if it satisfies the following conditions:

i) $\emptyset \in B$ and $X \in B$;

ii) $B_1, B_2 \in B$ implies that $B_1 \cap B_2 \in B$ and $B_1 \cup B_2 \in B$.

**Definition 2.1.**

1. An open set $V$ in $(X, \tau)$ is everywhere basic (e.b.) if $V$ belongs to every annular basis of $X$.

2. A topological space $(X, \tau)$ is unibasic if $\tau$ is the only annular basis of $X$.

3. $(X, \tau)$ is minimally basic if $X$ has annular basis $B_0$ which is contained in every other annular basis $B$ of $X$. 

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Remark 2.2.

i) Every element of a minimum annular basis $B_0$ of $X$ is e.b. and every unibasic space is minimally basic.

ii) Every open and compact subset of a topological space $X$ is e.b.. Hence, every hereditarily compact space is unibasic.

Lemma 2.3. Let $B$ be an annular basis of a topological space $(X, \tau)$. Define $A \delta B$ iff $A \cap H \neq \emptyset$ for every $H \in C(B)$ which contains $B$. Then $\delta$ is a transitive quasi-proximity on $X$ which induces $\tau$.

Proof. Clearly $X \delta \emptyset$ and $\emptyset \delta X$. If $(A \cup B) \delta C$, we must have $A \delta C$ or $B \delta C$. Indeed, $A \delta C$ and $B \delta C$ imply the existence of $H_1, H_2 \in C(B)$ such that $H_1 \cap H_2 \supseteq C$, $A \cap H_1 = \emptyset = B \cap H_2$. Therefore $(A \cup B) \cap H_1 \cap H_2 = \emptyset$ and $H_1 \cap H_2$ is an element of $C(B)$ containing $C$, that, $(A \cup B) \delta C$, a contradiction. In a similar way one may prove that $C \delta (A \cup B)$ implies that $C \delta A$ or $C \delta B$. It is obvious that $\{x\} \delta \{x\}$ for each $x \in X$. Finally, suppose that $A \delta B$. Therefore, there exists an element $H \in C(B)$ such that $H \supseteq B$ and $A \cap H = \emptyset$. Therefore, $A \delta H$ and $(X \setminus H) \delta B$.

Observe now that $(X \setminus H) \delta H$ for every $H \in C(B)$ and $T(X \setminus H, H) = X \times X \setminus [(X \setminus H) \times H] = (H \times X) \cup [X \times (X \setminus H)]$. Hence, if $A \delta B$ and $H \in C(B)$ satisfies $B \subseteq H \subseteq X \setminus A$, we have $T(X \setminus H, H) \subseteq [(X \setminus A) \times X] \cup [X \times (X \setminus B)] = T(A, B)$. This proves that the quasi-uniformity $U_\delta$ is transitive.

Finally, we must prove that $\tau_\delta = \tau$. For this, take any set $C \subseteq X$ and consider the set $C_1 = \{x \in X: \{x\} \delta C\}$. It is enough to prove that $C_1 = C$. If $x \in X \setminus C$, there exists a set $B \in B$ such that $x \in B \subseteq X \setminus C$. Therefore, $X \setminus B \in C(B)$ and $X \setminus B \supseteq C$, that is, $\{x\} \delta C$ and $X \setminus C \subseteq X \setminus C_1$. On the other hand, if $x \in X \setminus C_1$, i.e., if $\{x\} \delta C$, there exists a set $H \in C(B)$ such that $H \supseteq C \cap x \notin H$. Therefore, $x \in X \setminus C$ and the proof is complete. □

A quasi-proximity $\delta$ on a set $X$ is:

1) Point-symmetric if $A \delta \{x\}$ implies $\{x\} \delta A$. Equivalently, $\delta$ is point-symmetric if $\tau_\delta \subseteq \tau_{\delta-1}$.

2) Locally-symmetric if $A \delta G$ for every $\tau$-neighborhood $G$ of $x$ implies that $\{x\} \delta A$.

Notation 2.4. If $\mathcal{G}$ is a family of subsets of $X$, we define: $C(\mathcal{G}) = \{H: X \setminus H \in \mathcal{G}\}$.

Let $B$ be an annular basis of a topological space $(X, \tau)$ is:

i) Disjunctive (or a Wallman basis) if whenever $x \in B \in B$, there exists an element $H_x \in C(B)$ such that $x \in H_x \subseteq B$.

ii) Regular if whenever $x \in B \in B$, there exists an element $D \in B$ and an element $H \in C(B)$ such that $x \in D \subseteq H \subseteq B$.

iii) Normal is for every pair $H, K$ of disjoint elements of $C(B)$, there exists a pair $B, D$ of disjoint elements of $B$ such that $H \subseteq B$ and $K \subseteq D$. 

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Theorem 2.5. Let $\mathcal{B}$ be an annular basis of a topological space $(X, \tau)$ and let $\delta$ be the quasi-proximity on $X$ associated to $\mathcal{B}$. Then:

i) $\mathcal{B}$ is disjunctive iff $\delta$ is point-symmetric.

ii) $\mathcal{B}$ is regular iff $\delta$ is locally symmetric.

iii) $\mathcal{B}$ is normal iff $\delta$ is of Wallman type$^1$.

Proof. We prove only iii). Suppose $\delta$ is of Wallman type and let $H, K \in C(\mathcal{B})$ be disjoint. Since $H$ and $K$ are $\delta$-remote, there exists a neighborhood $G$ of $H$ such that $H\delta(X \setminus G)$ and $K\delta G$. This last condition implies the existence of an elements of $H_1 \in C(\mathcal{B})$ such that $K \subseteq X \setminus H_1 \subseteq X \setminus G$. The first condition implies the existence of an element $K_1 \in C(\mathcal{B})$ such that $X \setminus G \subseteq K_1 \subseteq X \setminus H$. Hence, $X \setminus K_1$ and $X \setminus H_1$ are disjoint elements of $\mathcal{B}$ and $\mathcal{B}$ is normal.

Assume now that $\mathcal{B}$ is normal. Let $A, B$ be $\delta$-remote. Let $H, K \in C(\mathcal{B})$ be disjoint sets such that $A \subseteq H$ and $B \subseteq K$. Since $\mathcal{B}$ is normal, there exist disjoint elements $C, D \in \mathcal{B}$ such that $H \subseteq C$ and $K \subseteq D$. Defining $G = C$, we have $H\delta(X \setminus G)$ and $K\delta G$, i.e., $\delta$ is of Wallman type. □

Corollary 2.6. Every transitive point-symmetric quasi-proximity of Wallman type is locally symmetric and its induced topology is completely regular.

Lemma 2.7. Let $\delta$ be a transitive quasi-proximity on a topological space $(X, \tau)$ and suppose that $\tau_3 = \tau$. Then $\mathcal{B} = \{V \in \tau: V\delta(X \setminus V)\}$ is an annular basis of $(X, \tau)$.

Proof. Clearly $\emptyset \in \mathcal{B}$ and $X \in \mathcal{B}$. Suppose now that $B_1, B_2$ both belong to $\mathcal{B}$. If $B_1 \cup B_2 \notin \mathcal{B}$, we would have $(B_1 \cup B_2)\delta(X \setminus B_1) \cap (X \setminus B_2)$. Therefore $B_1\delta(X \setminus B_1) \cap (X \setminus B_2)$ or $B_2\delta(X \setminus B_1) \cap (X \setminus B_2)$. This would imply that $B_1\delta(X \setminus B_1)$ or $B_2\delta(X \setminus B_2)$, a contradiction. Hence, $B_1 \cup B_2 \in \mathcal{B}$. In a similar fashion we prove that $B_1 \cap B_2 \in \mathcal{B}$. It remains to prove that $\mathcal{B}$ is a basis of $(X, \tau)$. Suppose then that $x \in V \in \tau$. Therefore $\{x\}\delta(X \setminus V)$ (recall $\tau_3 = \tau$). Let $R \in U_\delta$ be a transitive entourage contained in $T(\{x\}, X \setminus V)$. Let us prove that $R(x) \subseteq V$. If $y \in R(x)$, we have $(x, y) \in R \subseteq T(\{x\}, X \setminus V) = [(X \setminus \{x\}) \times X] \cup [\{x\} \times V]$. Therefore, $(x, y) \in \{x\} \times V$, that is, $y \in V$. Besides, $R(x)\delta(X \setminus R(x))$ because $R(x)\delta(X \setminus R(x))$ would imply that $[R(x) \times (X \setminus R(x))] \cap S \neq \emptyset$ for every $S \in U_\delta$, and, in particular, $[R(x) \times (X \setminus R(x))] \cap R \neq \emptyset$. But since $R$ is transitive, this last statement is clearly false. Hence, we must have that $R(x)\delta(X \setminus R(x))$. Since this implies that $R(x) \cap (X \setminus R(x)) = \emptyset$, we deduce that $R(x)$ is open. Therefore, $R(x) \in \mathcal{B}$ and $\mathcal{B}$ is an annular basis of $(X, \tau)$. □

Let $(X, \tau)$ be a topological space with topology $\tau$. For $G \in \tau$ let $S_G = (G \times G) \cup ((X \setminus G) \times X)$. The filter generated by $\{S_G: G \in \tau\}$ is a quasi-uniformity $\mathcal{P}$ for $X$ called Pervin quasi-uniformity (see [8]).

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$^1$Two sets $A, B \subseteq X$ are said to be $\delta$-remote if there exist disjoint sets $H, K \subseteq X$ such that $A \subseteq H$, $B \subseteq K$, $(X \setminus H)\delta H$ and $(X \setminus K)\delta K$. A quasi-proximity $\delta$ on a set $X$ is of Wallman type if for every pair of $\delta$-remote sets $A, B$, there exists a neighborhood $G$ of $A$ such that $A\delta(X \setminus G)$ and $B\delta G$. 

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Theorem 2.8. Let \((X, \tau)\) be a topological space. Then there exists a bijective correspondence between the collection of annular bases of \((X, \tau)\) and the collection of totally bounded transitive quasi-proximities on \(X\) which induce \(\tau\). Hence, \((X, \tau)\) is minimally basic iff the family of totally bounded transitive quasi-uniformities on \(X\) inducing \(\tau\) has a minimum element and \((X, \tau)\) is unibasic iff \(P = U_{\emptyset^0}\) is the only totally bounded transitive quasi-uniformity inducing \(\tau\).

Theorem 2.9. Let \(B\) be an everywhere-basic set on a topological space \((X, \tau)\) and suppose that \(B \neq X\). If \(K \subseteq X\) is closed and \(K \subseteq B\), then \(K\) is compact.

Proof. Suppose that \(K\) is not compact. Then there exists a family \(G = \{B_i: i \in J\} \subseteq \tau\) such that \(K \subseteq \cup\{B_i: i \in J\} \subseteq B\), but for each finite subset \(J_0 \subseteq J\), we have \(K \setminus \cup\{B_i: i \in J_0\} \neq \emptyset\). If \(B' = \{L \in \tau: L \subseteq \cup\{B_i: i \in J_0\}\}\) for some \(J_0 \subseteq J\), finite and let \(B'' = \{L \in \tau: L \cap K = \emptyset\} \cup \{X\}\), it is easy to check that \(B = \{L_1 \cup L_2: L_1 \in B'\) and \(L_2 \in B''\}\) is an annular basis of \((X, \tau)\). But \(B \not\subseteq B\), contradicting the fact that \(B\) is everywhere basic. Hence, \(K\) must be compact. \(\square\)

Definition 2.10. A topological space \((X, \tau)\) is \(R_0\) if whenever \(x \in V \in \tau\) there exists a closed set \(H_x\) such that \(x \in H_x \subseteq V\) and \((X, \tau)\) is \(R_1\) if whenever \(x, y \in X\) and \(\{x\} \neq \{y\}\), there exist disjoint open sets \(V, W\) such that \(x \in V\) and \(y \in W\).

A topological space \((X, \tau)\) is \(R_0\) if only if \(\tau\) is a Wallman basis of \((X, \tau)\). Also \((X, \tau)\) is regular if only if \(\tau\) admits a regular Wallman basis. It is also clear that every \(R_1\) space is \(R_0\) and every regular or Hausdorff space is \(R_1\).

Theorem 2.11. Let \(B\) be an everywhere basic subset of an \(R_1\) topological space \((X, \tau)\) such that \(B \neq X\). Then \(B\) is compact.

Proof. According to Theorem (2.9), it is enough to prove that \(Fr(B) = \emptyset\). Assume, on the contrary, there exists a point \(p \in Fr(B)\). Define \(B_1 = \{V \in \tau: p \not\in V\}\) and \(B_2 = \{W \in \tau: p \in W\}\). If \(B = \{V \cup W: V \in B_1\) and \(W \in B_2\}\), it is clear that \(B\) is an annular basis of \((X, \tau)\). Observe that for every \(T = V \cup W \in B\), we have \(p \not\in Fr(T)\) (because \(Fr(T) \subseteq Fr(V) \cup Fr(W) \subseteq X \setminus \{p\}\)). This implies that \(B \not\subseteq B\), contradicting the fact that \(B\) is everywhere basic. \(\square\)

Definition 2.12. A topological space \((X, \tau)\) is irreducible if every non-empty open set \(V \in \tau\) is dense in \(X\). Equivalently, \((X, \tau)\) is irreducible if every pair of non-empty open subsets of \(X\) have a non-empty intersection.

Theorem 2.13. Let \(B \neq X\) be an everywhere basic subset of a topological space \((X, \tau)\). If \(X \setminus B\) is irreducible, then \(B\) is compact.

Proof. Let \(U\) be an open cover of \(B\). Let \(B'\) be the family of open sets \(L \in \tau\) which are contained in a finite union of members of \(U\) and let \(B'' = \{\emptyset\} \cup \{M \in \tau: M \setminus B \neq \emptyset\}\). Clearly \(B = \{L \cup M: L \in B'\) y \(M \in B''\}\) is an annular basis of \((X, \tau)\). However, \(B \not\subseteq B\), a contradiction. \(\square\)
Theorems (2.11) and (2.13) have the following consequences:

**Corollary 2.14.** An $\mathcal{R}_1$ topological spaces $(X, \tau)$ is minimally basic iff $(X, \tau)$ is locally compact and 0-dimensional.

**Corollary 2.15.** Let $(X, \tau)$ be an unibasic space and let $x \in X$. Then $X \setminus \{x\}$ is compact. Therefore, if $X$ has a compact, closed and non-empty subspace, then $X$ itself is compact.

**Corollary 2.16.** Every $\mathcal{R}_1$ unibasic space $(X, \tau)$ has a finite topology. In fact, for every $x \in X$, $\{x\}$ is open and $X$ is a finite union of point-closures.

**Definition 2.17.** Let $(X, \tau)$ be an $\mathcal{R}_0$ topological space.

a) $(X, \tau)$ is $\mathcal{R}_1'$ if every compact open subset of $X$ is closed.

b) $(X, \tau)$ is $\mathcal{R}_1''$ every intersection of compact open subspaces of $X$ is compact.

**Remark 2.18.** $\mathcal{R}_1 \Rightarrow \mathcal{R}_1' \Rightarrow \mathcal{R}_1'' \Rightarrow \mathcal{R}_0$.

**Proof.** $(\mathcal{R}_1 \Rightarrow \mathcal{R}_1')$ It enough to observe that if $(X, \tau)$ is $\mathcal{R}_1$, $K \subseteq X$ is compact, $V \subseteq X$ is open and $K \subseteq V$, then $K \subseteq V$.

A subset $S$ of $X$ is a semi-block of a entourage $E$ of $X$ if $S \times S \subseteq E$.

**Lemma 2.19.** Let $R$ be a transitive entourage of a set $X$; let $x \in X$ and let $A \subseteq X$ be a semi-block of $R$ intersecting $R(x)$. Then $A \subseteq R(x)$.

**Proof.** Select a point $y \in A \cap R(x)$ and let $z \in A$. Therefore, $(x, y) \in R$ and $(y, z) \in A \times A \subseteq R$. Since $R$ is transitive, we deduce that $(x, z) \in R$, i.e., $z \in R(x)$.

**Definition 2.20.** Let $\alpha$ be a cover of a set $X$. For $x \in X$, define $\text{Cost}(x, \alpha) = \bigcap \{L: x \in L \in \alpha\}$. The indexed cover $\{\text{Cost}(x, \alpha): x \in X\}$ is denoted as $\alpha^\n$ and is called the cobaricentric cover of $\alpha$. Let $\alpha$ be any cover of a set $X$. Then the entourage $E(\alpha^\n)$ of the cobaricentric cover $\alpha^\n$ is a transitive entourage of $X$.

A cover $\alpha$ of a topological space $(X, \tau)$ is interior-preserving if for each $x \in X$, $\text{Cost}(x, \alpha)$ is a $\tau$-neighborhood of $x$.

**Lemma 2.21.** Let $R$ be a totally bounded transitive entourage on a set $X$. Then the family $\{L: L = R(x) \text{ for some } x \in X\}$ is finite.

**Proof.** Let $\{A_1, A_2, \ldots, A_n\}$ be a finite cover of $X$ consisting of semi-blocks of $R$. By Lemma (2.19), each $R(x)$ is the union of the sets $A_i$ which intersect $R(x)$. Hence the family $\{L: L = R(x) \text{ for some } x \in X\}$ has at most $2^n$ elements.

**Theorem 2.22.** Let $(X, \tau)$ be a topological space. Consider the following properties:

1. $\tau$ is finite.
2. $\mathcal{P}$ is the only quasi-uniformity on $X$ which induces $\tau$.
3. Every interior-preserving cover of $X$ is finite.
(4) \((X, \tau)\) is hereditarily compact.
(5) \(\delta_P\) is the only quasi-proximity on X which induces \(\tau\).
(6) \(\delta_P\) is the only transitive quasi-proximity on X which induces \(\tau\).
(7) \((X, \tau)\) is unibasic.

Then 1) \(\Rightarrow\) 2) \(\Rightarrow\) 3) \(\Rightarrow\) 4) \(\Rightarrow\) 5) \(\Rightarrow\) 6) \(\Rightarrow\) 7); if \((X, \tau)\) is \(R''_1\), 7) \(\Rightarrow\) 4) and if \((X, \tau)\) is \(R'_1\), 7) \(\Rightarrow\) 1).

Proof. The proofs of the implications 1) \(\Rightarrow\) 2) \(\Rightarrow\) 3) \(\Rightarrow\) 4) \(\Rightarrow\) 5) appear in ([1]). However, using Lemma (2.21) we obtain a quick proof of the implication 2) \(\Rightarrow\) 3). Assuming 2), we deduce that \(P = FT\). Hence, if \(\alpha\) is an interior-preserving cover of \(X\), the entourage \(R = E(\alpha)\) is totally bounded and transitive. Therefore, by Lemma (2.21), the family \(\{L: L = R(x)\text{ for some }x \in X\}\) is finite. This, in turn, implies that \(\alpha\) is finite. Indeed, consider the topology of \(X\) whose closed sets are arbitrary unions of arbitrary intersections of elements of \(\alpha\). The point-closures in this topology are precisely the sets \(Cost(x, \alpha)\), where \(x \in X\). Since every closed set in this topology is finite we conclude that this topology is finite and hence, \(\alpha\) is finite. The implication 5) \(\Rightarrow\) 6) is evident and 6) \(\Rightarrow\) 7) is a consequence of Theorem (2.8). If \((X, \tau)\) is \(R''_1\) and \(V \in \tau\), \(V \neq X\), clearly \(V\) is the intersection of all the compact open sets \(X \setminus \{x\}\), where \(x \in X \setminus V\). By hypothesis, \(V\) must be compact. We have proved then that 7) \(\Rightarrow\) 4) when \((X, \tau)\) is \(R''_1\)-space. Finally, if \((X, \tau)\) is \(R'_1\), each set \(X \setminus \{x\}\) is compact and open and, hence, it is also closed. Therefore, each point-closure is open. Since \(X\) is compact, \(X\) is the closure of a finite subset of \(X\). Since \((X, \tau)\) is \(R_0\), the topology \(\tau\) must be finite.

H.-P. Künzi has proved that properties 3), 4), 5), 6), 7) and 2') \(P\) is the only totally bounded quasi-uniformity on \(X\) which induces \(\tau\) are equivalent (see [4]).

The validity of the implication 7) \(\Rightarrow\) 2) is still open.

Typical examples of topological spaces admitting a unique totally bounded quasi-uniformity are the hereditarily compact spaces and set \(\omega_0\) equipped with the lower topology \(\{[0, n]: n \in \omega_0\} \cup \{\emptyset, \omega_0\}\).

The space with carrier set \(\omega_0 + 2\) and topology \(\{[0, n]: n \in \omega_0\} \cup \{(\omega_0 + 2) \setminus \{\omega_0 + 1\}, \omega_0 + 2, (\omega_0 + 2) \setminus \{\omega_0\} = \emptyset\}\) admits a unique totally bounded quasi-uniformity, while this is not true for its subspace \((\omega_0 + 2) \setminus \{\omega_0\}\) (see example page 148 [4]).

**Example 2.23** (see example 1 in [5]). Let \(N\) be the set of the positive integers equipped with the topology \(\tau = \{\{1, \ldots, n\}: n \in N\} \cup \{\emptyset, N\}\). Obviously, every proper open subset of \(N\) is compact, but \(N\) is not compact. This example shows that a topological space that admits a unique compatible quasi-proximity need not be compact.

**Question:** If \((X, \tau)\) is an unibasic space is equivalently to say the \(P\) is the only compatible quasi-uniformity?
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