A uniform approach to normality for topological spaces

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\begin{abstract}
(\(\lambda,\mu\))-regularity and (\(\lambda,\mu\))-normality are defined for generalized topological spaces. Several variants of normality existing in the literature turn out to be particular cases of (\(\lambda,\mu\))-normality. Urysohn’s lemma and Tietze extension theorem are discussed in the light of (\(\lambda,\mu\))-normality.
\end{abstract}

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1. INTRODUCTION

A large amount of research in topology is devoted to the study of classes of subsets of topological spaces, which possess properties similar to those of open sets. In the literature, several such classes are available which include amongst others semi-open sets [10], \(\alpha\)-open sets [12], \(\beta\)-open sets [1], pre-open sets [11], etc. Some other such classes are \(A\)-sets [15], \(B\)-sets [16], \(C\)-sets [7], etc. Since these classes have some features common in them, it is quite natural to enquire if these classes can be obtained by using one common definition? Á. Császár has successfully provided an answer in this regard. The main tool he has used is, the class of mappings \(\gamma: P(X) \to P(X)\) from the power set \(X\) into \(X\) itself possessing the property of monotonicity (that is, for \(A \subseteq B\) implies \(\gamma(A) \subseteq \gamma(B)\)). In a topological space \((X, \tau)\), the operators such as \(\text{int}\), \(\text{cl}\), \(\text{int cl}\), \(\text{cl int}\text{, int cl int, cl int cl}\) etc. are found to belong to this class of mappings. Accordingly, the weaker form of open sets including semi-open sets,

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pre-open sets, \( \alpha \)-open sets, \( \beta \)-open sets are nothing but \( \gamma \)-open sets for different \( \gamma \)'s. All these families form “generalized topologies” on \( X \). In [4], Császár has formulated separation axioms for such spaces. Accordingly, separation axioms using semi-open sets[5], \( \beta \)-open sets[13], etc. become particular cases in [4].

In the same spirit, we introduce and investigate a generalized form of normality called \((\lambda, \mu)\)-normality for generalized topologies in this paper. However, unlike in [4], we use two GT’s simultaneously in our definition. This gives us a more general definition of normality, yet it covers almost all the relevant variants of normality existing in the literature. For example, if \( X \) has a topology, then by taking \( \lambda = \mu = \text{int} \), we get normality for \( X \); \( \lambda = \text{int} \), \( \mu = \text{cl} \) give \( \theta \)-normality; \( \lambda = \text{int} \), \( \mu = \text{cl} \) give \( \Delta \)-normality for \( X \). If \( (X, \tau_1, \tau_2) \) is a bitopological space, then \( \lambda = \text{int}_{\tau_1}, \mu = \text{int}_{\tau_2} \) gives rise to pairwise normality of \( (X, \tau_1, \tau_2) \). Thus our study provides a uniform approach towards various notions of normality existing in the literature. We have shown that the two most important results on normality- the Urysohn’s lemma and Tietze extension theorem are valid for \((\lambda, \mu)\)-normality, although in a milder form. We have also defined and studied \((\lambda, \mu)\)-regularity in the process and provided its characterization.

2. Preliminaries

Á. Császár has defined a generalized topological space [3] in the following way:

**Definition 2.1.** A collection \( \mathcal{G} \) of subsets of \( X \) is called a *generalized topology* (in brief *GT*) [3] on \( X \) if

(i) \( \emptyset \in \mathcal{G} \);
(ii) \( G_i \in \mathcal{G} \) for \( i \in I \neq \emptyset \), implies \( G = \bigcup_{i \in I} G_i \in \mathcal{G} \).

The same has been defined and studied as *semi topological spaces* by Peleg[14].

For a topological space \( (X, \tau) \), each family of semi-open sets, \( \alpha \)-open sets, pre-open sets and \( \beta \)-open sets etc. form a generalized topology on \( X \).

Á. Császár [2] has used a map \( \gamma : P(X) \rightarrow P(X) \) where \( P(X) \) is the power set of \( X \), as his main tool for developing a generalized form of topological spaces. The map \( \gamma \) possesses the property of monotonicity, which says that, if \( A \subseteq B \) then \( \gamma(A) \subseteq \gamma(B) \).

The collection of all such mappings on \( X \) is denoted by \( \Gamma(X) \), or simply by \( \Gamma \).

**Definition 2.2** ([2]). Consider a non empty set \( X \) and a map \( \gamma \in \Gamma(X) \). We say that a subset \( A \) of \( X \) is \( \gamma \)-open if \( A \subseteq \gamma(A) \).

For a topological space \( (X, \tau) \), an open set (resp. semi-open, \( \alpha \)-open, \( \beta \)-open, pre-open) is \( \gamma \)-open for \( \gamma = \text{int} \) (resp. \( \text{cl} \), \( \text{int} \), \( \text{cl} \text{ int} \), \( \text{cl} \text{ cl} \), \( \text{int} \text{ cl} \)). Also for each \( \gamma \in \Gamma(X) \), it may be verified that the \( \gamma \)-open sets form a generalized topology on \( X \).
Definition 2.3 ([2]). Let \( A \) be a subset of \( X \) and \( \gamma \) be a monotonic mapping on \( X \). Then the union of all \( \gamma \)-open sets contained in \( A \) is called the \( \gamma \)-interior of \( A \), and is denoted by \( i_\gamma(A) \).

Proposition 2.4 ([2]). A subset \( A \) of \( X \) is \( \gamma \)-open if and only if \( A = i_\gamma(A) \) if and only if \( A \) is \( i_\gamma \)-open.

Definition 2.5 ([2]). A subset \( A \) of \( X \) is called \( \gamma \)-closed if \( X \setminus A \) is \( \gamma \)-open.

Definition 2.6 ([2]). The intersection of all \( \gamma \)-closed sets containing \( A \) is called \( \gamma \)-closure of \( A \) and is denoted by \( c_\gamma(A) \).

It can be shown that \( c_\gamma(A) \) is the smallest \( \gamma \)-closed set containing \( A \).

Another operator called \( \gamma^* \) is defined, with the help of \( \gamma \) in the following way:

Definition 2.7 ([2]). For any \( A \subseteq X \) and \( \gamma \in \Gamma(X) \), we define
\[
\gamma^*(A) = X \setminus (\gamma(X \setminus A))
\]

Proposition 2.8 ([2]). If \( \gamma \in \Gamma \), then \( \gamma^* \in \Gamma \).

Proposition 2.9 ([2]). A subset \( A \) of \( X \) is \( \gamma^* \)-closed if and only if \( \gamma(A) \subseteq A \).

Definition 2.10 ([19]). Let \( X \) be a topological space and let \( A \subseteq X \). A point \( x \in X \) is in \( \theta \)-closure of \( A \) if every closed neighbourhood of \( x \) intersects \( A \). The \( \theta \)-closure of \( A \) is denoted by \( cl_\theta(A) \). The set \( A \) is called \( \theta \)-closed if \( A = cl_\theta(A) \).

The complement of a \( \theta \)-closed set is called \( \theta \)-open set.

Definition 2.11 ([19]). Let \( X \) be a topological space and let \( A \subseteq X \). A point \( x \in X \) is in \( \delta \)-closure of \( A \) if every regular open neighbourhood of \( x \) intersects \( A \). The \( \delta \)-closure of \( A \) is denoted by \( cl_\delta(A) \). The set \( A \) is called \( \delta \)-closed if \( A = cl_\delta(A) \).

The complement of a \( \delta \)-closed set is called \( \delta \)-open set.

Definition 2.12. A topological space \( X \) is said to be

(1) \([8]\) \( \theta \)-normal if every pair of disjoint closed sets one of which is \( \theta \)-closed are contained in disjoint open sets;
(2) \([8]\) Weakly \( \theta \)-normal if every pair of disjoint \( \theta \)-closed sets are contained in disjoint open sets;
(3) \([6]\) \( \Delta \)-normal if every pair of disjoint closed sets one of which is \( \delta \)-closed are contained in disjoint open sets;
(4) \([6]\) Weakly \( \Delta \)-normal if every pair of disjoint \( \delta \)-closed sets are contained in disjoint open sets.

Definition 2.13 ([9]). A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise normal if given a \( \tau_1 \)-closed set \( A \) and a \( \tau_2 \)-closed set \( B \) with \( A \cap B = \emptyset \), there exist \( \tau_2 \)-open set \( O_2 \) and \( \tau_1 \)-open set \( O_1 \) such that \( A \subseteq O_2 \), \( B \subseteq O_1 \), and \( O_1 \cap O_2 = \emptyset \).
3. $(\lambda, \mu)$-REGULARITY AND $(\lambda, \mu)$-NORMALITY

For defining $(\lambda, \mu)$-regularity and $(\lambda, \mu)$-normality, no topology is required on $X$. It is because, for a non-empty set $X$, $\Gamma(X)$ is also non-empty. However, we may call $X$ a space, once we define some topological property on $X$, such as $(\lambda, \mu)$-regularity, $(\lambda, \mu)$-normality etc.

**Definition 3.1.** Let $X$ be a non-empty set and $\lambda, \mu \in \Gamma(X)$. Then $X$ is said to be $\lambda\text{-regular with respect to } \mu$ if for each point $x \in X$ and each $\lambda$-closed set $P$ such that $x \notin P$, there exist a $\lambda$-open set $U$ and a $\mu$-open set $V$ such that $x \in U$, $P \subseteq V$ and $U \cap V = \emptyset$.

$X$ is said to be $(\lambda, \mu)$-regular if $X$ is $\lambda$-regular with respect to $\mu$ and vice versa.

**Definition 3.2.** A non-empty set $X$ is called $(\lambda, \mu)$-normal if for a given $\lambda$-closed set $A$ and a $\mu$-closed set $B$ with $A \cap B = \emptyset$, there exist a $\mu$-open set $U$ and a $\lambda$-open set $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Below we provide characterizations for $(\lambda, \mu)$-regularity and $(\lambda, \mu)$-normality.

**Theorem 3.3.** Let $X$ be a non-empty set. Then $X$ is $(\lambda, \mu)$-regular if and only if

(i) for a given $x \in X$ and $\lambda$-closed neighbourhood $U$ of $x$, there exists a $\mu$-closed neighbourhood $V$ of $x$ (that is, $x \in V_0 \subseteq V$, for some $V_0 \subseteq X$, where $V_0$ is $\lambda$-open and $V$ is $\mu$-closed) such that $x \in V \subseteq U$.

and

(ii) for a given $y \in X$ and $\mu$-neighbourhood $P$ of $y$, there exists a $\lambda$-closed neighbourhood $Q$ of $y$ such that $y \in Q \subseteq P$.

**Proof.** Let $x \in X$ and $U$ be a $\lambda$-open neighbourhood of $x$. Therefore $x \notin X \setminus U$, a $\lambda$-closed set. Thus, there exists a disjoint pair of $\lambda$-open set $O$ and $\mu$-open set $W$ such that $x \in O$, $X \setminus U \subseteq W$ and $O \cap W = \emptyset$. That is, $X \setminus W \subseteq U$. Hence $x \in O \subseteq X \setminus W \subseteq U$, that is, $x \in V \subseteq U$, where $V = X \setminus W$, a $\mu$-closed set.

Similarly, if $X$ is $\mu$-regular with respect to $\lambda$, then for a given point $x \in X$ and a $\mu$-open neighbourhood $U$ of $x$, there exists a $\lambda$-closed neighbourhood $V$ of $x$ such that $x \in V \subseteq U$.

Conversely, let $x \in X$ and $F$ be a $\lambda$-closed set such that $x \notin F$. Then $x \in X \setminus F$ and $X \setminus F$ is $\lambda$-open. Hence by $(i)$, there exists a $\lambda$-open set $V_0$ and a $\mu$-closed set $V$ such that $x \in V_0 \subseteq V \subseteq X \setminus F$. Then by $(ii)$, there exists a $\lambda$-open set $Q_0$ and a $\mu$-closed set $Q$ such that $x \in Q_0 \subseteq Q \subseteq V_0$. Thus we have, $x \in Q_0$, $F \subseteq P_0$, where $P_0 = X \setminus V$, such that $Q_0$ is $\lambda$-open, $P_0$ is $\mu$-open and $P_0 \cap Q_0 = \emptyset$. Hence $X$ is $(\lambda, \mu)$-regular. 

**Theorem 3.4.** Let $X$ be a non-empty set. Then $X$ is $(\lambda, \mu)$-normal if and only if for a given $\mu$-closed set $C$ and a $\lambda$-open set $D$ such that $C \subseteq D$, there are a $\lambda$-open set $G$ and a $\mu$-closed set $F$ such that $C \subseteq G \subseteq F \subseteq D$. 

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Theorem 4.3. Let $\lambda f R D$ of $D$ given hypothesis, there exist a type (lower semi-continuous or g.l.s.c. in brief) if for each $f$ that $C$ Let Example 4.2. in generalized topology. and generalized lower semi-continuous, may fail to be generalized continuous Definition 4.1. Let us consider, $G$ Let $\lambda, \mu$ such that $X \lambda \mu$ be a $(\lambda, \mu)$-normal space and $\lambda, \mu \lambda, \mu$ be the $\lambda, \mu$-open set such that $Y = X \lambda \mu$ is $G$-closed and $\lambda, \mu$-open set. Hence $X$ is $(\lambda, \mu)$-normal. □

In our next section, we provide generalized versions of Uryshon’s lemma and Tietze extension theorem, which holds for $(\lambda, \mu)$-normality.

4. Uryshon’s lemma and Tietze extension theorem for $(\lambda, \mu)$-normality

**Definition 4.1** ([18]). Let $(X, \lambda)$ be a generalized topological space and $\mathbb{R}$ be the real line with the usual topology. A mapping $f : X \rightarrow \mathbb{R}$ is said to be generalized upper semi-continuous or g.u.s.c. in brief (resp. generalized lower semi-continuous or g.l.s.c. in brief) if for each $a \in \mathbb{R}$, the set $\{ x \in X : f(x) < a \}$ (resp. $\{ x \in X : f(x) > a \}$) is $\lambda$-open.

Unlike in topology, a mapping which is both generalized upper semi-continuous and generalized lower semi-continuous, may fail to be generalized continuous in generalized topology.

**Example 4.2.** Let $X = [0, 1]$, $\lambda$ consist of the unions of the members of the type $[0, a], (b, 1], a, b \in [0, 1]$. Let $Y = X$, under the usual subspace topology of $\mathbb{R}$. Then the identity mapping $I : X \rightarrow Y$ is g.u.s.c. and g.l.s.c. but not generalized continuous as $f^{-1}(a, b) \notin \lambda$.

**Theorem 4.3.** Let $X$ be a $(\lambda, \mu)$-normal space. Then for any disjoint pair of $\lambda$-closed set $H$ and $\mu$-closed set $F$, there exists a real valued function $g$ on $X$ such that

(i) $g(x) = 0$ for $x \in F$, $g(x) = 1$ for $x \in H$, $0 \leq g(x) \leq 1$, for all $x \in X$;

(ii) $g$ is $\lambda$-upper semi-continuous and $\mu$-lower semi-continuous.

**Proof.** Let $X$ be a $(\lambda, \mu)$-normal space and $G$ and $H$ be two disjoint subsets of $X$ such that $G$ is $\mu$-closed and $H$ is $\lambda$-closed. Let us consider, $G_0 = G$ and $K_1 = X \setminus H$. Then $G_0$ is $\mu$-closed and $K_1$ is $\lambda$-open set such that $G_0 \subseteq K_1$. Since $X$ is $(\lambda, \mu)$-normal, therefore there exist a $\lambda$-open set $K_1/2$ and a $\mu$-closed set $G_1/2$ such that $G_0 \subseteq K_1/2 \subseteq G_1/2 \subseteq K_1$. Again applying the hypothesis to each pair of sets $(G_0$ and $K_1/2$ and $(G_1/2$ and $K_1)$, we obtain $\lambda$-open sets $K_1/4, K_3/4$ and $\mu$-closed sets $G_1/4, G_3/4$ such that...
Continuing this process, we obtain two families \( \{G_i\} \) and \( \{K_i\} \), where \( i = p/2^q \), where \( \{ p = 1, 2, \ldots, 2^q - 1, q = 1, 2, \ldots \} \). If \( i \) is any other dyadic rational number other than \( p/2^q \), then let \( K_i = \emptyset \), whenever \( i \leq 0 \) and \( K_i = X \), for \( i > 1 \). Similarly, \( G_i = \emptyset \) for \( i < 0 \) and \( G_i = X \) for \( i \geq 1 \). Thus, for every \( r \leq s \leq t \), we have

\[
K_r \subseteq K_s \subseteq G_s \subseteq G_t \quad \text{and for } s < t, \text{ we have } G_s \subseteq K_t.
\]

Now, we define a function \( g : X \to [0, 1] \) such that

\[
g(x) = \inf \{ t \mid x \in K_t \}
\]

Clearly, \( g(x) \in [0, 1] \). If \( x \in G \), then \( x \in K_i \) for all \( i \), therefore \( g(x) = 0 \), when \( x \in H = X \setminus K \), then \( x \notin K_i \) for all \( i \in [0, 1] \), hence \( g(x) = 1 \). Now, we have to show that \( g \) is \( \lambda \)-upper semi-continuous and \( \mu \)-lower semi-continuous.

First we show that

(i) if \( x \in G_p \) then \( g(x) \leq p \)
(ii) if \( x \notin K_p \), then \( g(x) \geq p \).

Let \( x \in G_p \), then \( x \in K_s \) for every \( s > p \). Therefore \( g(x) \leq p \). Similarly, if \( x \notin K_p \), then \( x \notin K_s \) for any \( s < p \), hence \( g(x) \geq p \).

Thus we can say that whenever \( g(x) > p \), we have \( x \notin G_p \) and \( g(x) < p \), we have \( x \in K_p \).

Now, we consider, \( x \in g^{-1}([0, a)) \), then \( g(x) \in [0, a) \), that is, there exists \( t < a \) such that \( g(x) < t \) and hence \( x \in K_t \), therefore \( g^{-1}([0, a)) \subseteq \bigcup_{t < a} K_t \).

Conversely, let \( x \in \bigcup_{t < a} K_t \), then \( x \in K_i \) for some \( i < a \), thus \( g(x) \leq i < a \). Therefore \( g^{-1}([0, a)) = \bigcup_{t < a} K_t \), a \( \lambda \)-open set. Hence \( g \) is \( \lambda \)-upper semi-continuous function.

Again, consider, \( x \in g^{-1}((a, 1]) \), then \( g(x) \in (a, 1] \), that is, there exists \( t \) with \( a < t \) such that \( g(x) > t \) and hence \( x \notin G_t \), that is, \( x \in X \setminus G_t \), therefore \( g^{-1}((a, 1]) \subseteq \bigcup_{t > a} (X \setminus G_t) \).

Conversely, let \( x \in \bigcup_{t > a} (X \setminus G_t) \), then \( x \in X \setminus G_t \) for some \( i > a \), thus \( g(x) \geq i > a \) and \( x \in g^{-1}((a, 1]) \). Therefore \( g^{-1}((a, 1]) = \bigcup_{t > a} (X \setminus G_t) \), a \( \mu \)-open set. Hence \( g \) is \( \mu \)-lower semi-continuous function.

Our next theorem resembles with the classical Tietze extension theorem. But before that, we quote a result which will be used in our main theorem.
Theorem 4.4 ([3]). Let \((X, \lambda)\) be a generalized topological space and \((Y, \lambda_y)\) be a generalized topological subspace of \(X\). Then a subset \(A\) of \(Y\) is \(\lambda\)-closed in \(Y\) if and only if it is the intersection of \(Y\) with a \(\lambda\)-closed set in \(X\).

Now we come to our main proposed result.

Theorem 4.5. Let \(X\) be a \((\lambda, \mu)\)-normal space. Let \(A \subseteq X\) be a \(\lambda\)-closed as well as \(\mu\)-closed set and \(f\) be a real valued function defined on \(A\) which is \(\lambda\)-upper semi-continuous as well as \(\mu\)-lower semi-continuous function. Then there exists an extension \(F\) of \(f\) to the whole of \(X\) such that \(F\) is \(\lambda\)-upper semi-continuous and \(\mu\)-lower semi-continuous in \(X\).

Proof. Let \(X\) be a \((\lambda, \mu)\)-normal space and \(A\) be a \(\lambda\)-closed and \(\mu\)-closed subset of \(X\). Suppose \(f\) is a real valued function on \(A\) which is \(\lambda\)-upper semi-continuous and \(\mu\)-lower semi-continuous.

Let \(n\) be a positive integer, then for each integer \(k\), let

\[
U^n_k = \{x : f(x) \geq k/n\} \text{ and } L^n_k = \{x : f(x) \leq (k - 1)/n\}
\]

Then, for every integer \(k\), \(U^n_k\) and \(L^n_k\) are \(\lambda\)-closed and \(\mu\)-closed subsets of \(A\) respectively. Therefore \(U^n_k\) and \(L^n_k\) are \(\lambda\)-closed and \(\mu\)-closed subsets of \(X\) also and \(U^n_k \cap L^n_k = \emptyset\).

By Theorem 4.3, for each \(k = 0, 1, 2, \ldots\), there is a function \(U_k\) defined on \(X\), which is \(\lambda\)-upper semi-continuous and \(\mu\)-lower semi-continuous, such that

\[
U_k(x) = 0, \text{ for } x \in L^n_k \quad U_k(x) = 1/n, \text{ for } x \in U^n_k
\]
and \(0 \leq U_k(x) \leq 1/n\) for all \(x \in X\).

Also, for each \(k = 0, -1, -2, \ldots\), there is a function \(V_k\) defined on \(X\) which is \(\lambda\)-upper semi-continuous and \(\mu\)-lower semi-continuous function, such that

\[
V_k(x) = -1/n, \text{ for } x \in L^n_k \quad V_k(x) = 0, \text{ for } x \in U^n_k
\]
and \(-1/n \leq V_k(x) \leq 0\) for all \(x \in X\).

Now we know that \(L^n_k \subseteq L^n_{k+1}\) and \(U^n_{k+1} \subseteq U^n_k\). Thus for \(k = 0, -1, -2, \ldots\), if \(f(x) \geq 0\), then \(x \in U^n_k\) and hence \(V_k(x) = 0\). Similarly, for \(k = 1, 2, \ldots\), if \(f(x) \leq 0\), then \(x \in L^n_k\) and hence \(U_k(x) = 0\).

Now, we define a real valued function \(f_n\) on \(X\) as follows:

\[
f_n(x) = \sum_{k=1}^{\infty} U_k(x) + \sum_{k=-\infty}^{0} V_k(x) \text{ for } x \in X
\]

Since \(U_k, V_k\) are \(\lambda\)-upper semi continuous and \(\mu\)-lower semi-continuous function therefore \(f_n\) is also a \(\lambda\)-upper semi continuous and \(\mu\)-lower semi-continuous function.

Now, consider \(C^n_k = U^n_{k-1} \cap L^n_{k+1}\) for \(k \in \mathbb{Z}\). Thus \(C^n_k \subseteq U^n_{k-1}\) and \(C^n_k \subseteq L^n_{k+1}\), that is, \(C^n_k = \{x : k/n \leq f(x) \leq k/n + 1\}\). Therefore \(A = \bigcup\{C^n_k, k \in \mathbb{Z}\}\).

Now, let \(k \geq 1\), and \(x \in C^n_k\), that is, \(0 \leq f(x) \leq k/n\). Since \(x \in L^n_{k+1}\), therefore \(U_j(x) = 0\) for \(j \geq k+1\) and \(x \in U^n_{k-1}\), therefore \(U_j(x) = 1/n\) for \(1 \leq j \leq k-1\). Also \(0 \leq U_k(x) \leq 1/n\). Thus

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\[ f_n(x) = \mathcal{U}_1(x) + \mathcal{U}_2(x) + \ldots + \mathcal{U}_{k-1}(x) + \mathcal{U}_k(x) + \mathcal{U}_{k+1}(x) + \ldots \]
\[ f_n(x) = \frac{k-1}{n} + \mathcal{U}_k(x) \]

Therefore
\[ |f(x) - f_n(x)| \leq \left| \frac{k}{n} - \frac{(k-1)}{n} - \mathcal{U}_k(x) \right| \]
\[ \leq \frac{1}{n} + |\mathcal{U}_k(x)| \]
\[ \leq \frac{2}{n}, \quad \text{for } x \in C^n_k. \]

Now, let \( k \leq 0 \), and \( x \in C^n_k \), that is, \( -(k+1)/n \leq f(x) \leq 0 \). Since \( x \in L^n_{k+1} \), therefore \( \mathcal{V}_j(x) = -1/n \) for \( (k-1) \leq j \leq 0 \) and \( x \in U^n_{k-1} \), therefore \( \mathcal{V}_j(x) = 0 \) for \( j \leq (k+1) \). Thus
\[ f_n(x) = \mathcal{V}_0(x) + \mathcal{V}_{-1}(x) + \mathcal{V}_{-2}(x) + \ldots + \mathcal{V}_{k-1}(x) + \mathcal{V}_k(x) + \mathcal{V}_{k+1}(x) + \ldots \]
\[ f_n(x) = \frac{k}{n} + \mathcal{V}_k(x), \text{ for a non positive integer } k \]

Therefore
\[ |f(x) - f_n(x)| \leq 2/n \]

Hence \( |f(x) - f_n(x)| \leq 2/n \) for all \( x \in A \). We recall that \( g \)-nets defined in [17] behave almost the same way in generalized topology as the nets do in topology and a sequence is just a particular case of \( g \)-nets in generalized topology. Due to this fact, \( f_n|A \) for \( n = 1, 2, \ldots \) converge uniformly to \( f \) on \( A \). Also \( (f_n|A) \) forms a Cauchy sequence with respect to the uniform norm on \( A \). As a result, as in [9], \( f \) has an extension \( F \) to \( X \). It may be easily shown that \( F \) is \( \lambda \)-upper semi-continuous and \( \mu \)-lower semi-continuous function.

\[ \square \]

5. Conclusion

Under different set of conditions, we get different variants of normality. If we take

(i) \( \lambda = \mu = \text{interior operator of a topology on } X \), then the \( \lambda \)-closed sets and \( \mu \)-closed sets are nothing but the closed sets of \( X \). Therefore \((\lambda, \mu)\)-normality just becomes normality.

(ii) \( \lambda = \operatorname{int}_{\tau_1} \) and \( \mu = \operatorname{int}_{\tau_2} \), two different interior operators over two different topologies \( \tau_1 \) and \( \tau_2 \), then \((\lambda, \mu)\)-normality becomes pairwise normality [9] of \((X, \tau_1, \tau_2)\)

(iii) \( \lambda = \text{interior operator and } \mu = \operatorname{cl}_\theta \) operator, then \((\lambda, \mu)\)-normality becomes \( \theta \)-normality [8]. This is due to fact that \( \operatorname{cl}_\theta \in \Gamma \), that is, \( \operatorname{cl}_\theta \) operator is monotonic. First we verify that for \( A, B \subseteq X \) such that \( A \subseteq B \), we have \( \operatorname{cl}_\theta(A) \subseteq \operatorname{cl}_\theta(B) \). Let \( x \in \operatorname{cl}_\theta(A) \), then every closed neighbourhood of \( x \) intersects \( A \). Since \( A \subseteq B \), therefore every closed neighbourhood of \( x \) intersects \( B \) also. Hence \( x \in \operatorname{cl}_\theta(B) \). Thus \( \operatorname{cl}_\theta(A) \subseteq \operatorname{cl}_\theta(B) \). Therefore \( \operatorname{cl}_\theta \in \Gamma \). Hence by Proposition 2.8, \( \operatorname{cl}_\theta \in \Gamma \).

Now, let \( A \) be \( \mu \)-closed, that is, \( \operatorname{cl}_\theta \)-closed set. Then by Proposition 2.9 \( \operatorname{cl}_\theta(A) = (\operatorname{cl}_\theta)^*(A) \) and by Proposition 2.8 \( \operatorname{cl}_\theta(A) = (\operatorname{cl}_\theta)^*(A) \subseteq A \), that is, \( A \) is \( \theta \)-closed. Since every \( \theta \)-open set is open therefore in the light of \((\lambda, \mu)\)-normality, we have disjoint pair of sets in which one is \( \lambda \)-closed, that is, closed set and the other is \( \mu \)-closed, that is, \( \theta \)-closed set separated.
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by disjoint $\mu$-open, that is, $\theta$-open set and hence open set and $\lambda$-open set, that is, open set respectively.

(iv) $\lambda=$–interior operator and $\mu=cl^*_\delta$ operator, then $(\lambda,\mu)$-normality becomes $\Delta$-normality [6]. Because $cl^*_\delta \in \Gamma$, that is, $cl^*_\delta$ operator is monotonic

As, consider $A,B \subseteq X$ such that $A \subseteq B$. Then $cl^*_\delta(A) \subseteq cl^*_\delta(B)$. Since $x \in cl^*_\delta(A)$, then every regular open neighbourhood of $x$ intersects $A$. Since $A \subseteq B$, therefore every regular open neighbourhood of $x$ intersects $B$ also. Hence $x \in cl^*_\delta(B)$. Thus $cl^*_\delta(A) \subseteq cl^*_\delta(B)$. Therefore $cl^*_\delta \in \Gamma$. From the Proposition 2.8 $cl^*_\theta \in \Gamma$ also.

Now, a set $A$ is $\mu$-closed set, that is, $cl^*_\delta$-closed implies that $cl^*_\delta(A) = (cl^*_\delta)^*(A) \subseteq A$, that is, $A$ is $\delta$-closed. As every $\delta$-open set is open therefore in the light of $(\lambda,\mu)$-normality, we have disjoint pair of sets in which one is $\lambda$-closed, that is, closed set and the other is $\mu$-closed, that is, $\delta$-closed sets separated by disjoint $\mu$-open, that is, $\delta$-open set and hence open set respectively.

(v) $\lambda=\mu=cl^*_\delta$ operator, we have disjoint pair of $\theta$-closed sets separated by disjoint pair of $\theta$-open sets which is again open sets. Therefore $(\lambda,\mu)$-normality becomes weakly $\theta$-normality [8].

(vi) $\lambda=\mu=cl^*_\delta$ operator, we have disjoint pair of $\delta$-closed sets separated by disjoint pair of $\delta$-open sets which is again open sets. Therefore $(\lambda,\mu)$-normality becomes weakly $\Delta$-normality [6].

(vii) $\lambda=\mu=\text{closure operator}$, then space is always $(\lambda,\mu)$-normal. Because here every set is $\lambda$-open as well as $\mu$-open.

References

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