Best proximity points of contractive mappings on a metric space with a graph and applications

Asrifa Sultana and V. Vetrivel

Department of Mathematics, Indian Institute of Technology Madras, Chennai-600036, India. (asrifa.iitg@gmail.com, vetri@iitm.ac.in)

Communicated by M. Abbas

ABSTRACT

We establish an existence and uniqueness theorem on best proximity point for contractive mappings on a metric space endowed with a graph. As an application of this theorem, we obtain a result on the existence of unique best proximity point for uniformly locally contractive mappings. Moreover, our theorem subsumes and generalizes many recent fixed point and best proximity point results.

2010 MSC: 54H25; 47H10.

KEYWORDS: Fixed point; best proximity point; contraction; graph; metric space; P-property.

1. INTRODUCTION

Fixed point theory plays an important role for solving equations of the form $Tx = x$ where $T$ is defined on a subset of a metric space, partially ordered metric space, topological vector space or some suitable space. Given two non-empty subsets $A$ and $B$ of a metric space $(X, d)$, consider a non-self mapping $T : A \to B$. If $T(A) \cap A = \emptyset$, there does not exist a solution of the equation $Tx = x$. Then it is interesting to find a point $x \in A$ that is closest to $Tx$ in some sense. Best approximation and best proximity point results have been established in this direction. The well-known best approximation theorem due to Ky Fan [3] states that for a given non-empty compact convex subset $C$ of
a normed linear space $E$ and a continuous mapping $F : C \rightarrow E$, there exists $x^* \in C$ such that $\|x^* - Fx^*\| = d(Fx^*, C) = \inf\{\|Fx^* - x\| : x \in C\}$. Though this result gives the existence of an approximate solution of $Fx = x$, such solution need not be optimal in the sense that $\|x - Fx\|$ is minimum.

Naturally, for the map $T$, one can think of finding an element $x^* \in A$ such that $d(x^*, Tx^*) = \min\{d(x, Tx) : x \in A\}$. Since for all $x \in A$, $d(x, Tx) \geq d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. An optimal solution of $\min\{d(x, Tx) : x \in A\}$ is one for which the value $d(A, B)$ is attained. An element $x^* \in A$ is called a best proximity point for the mapping $T$ if $d(x^*, Tx^*) = d(A, B)$. Hence a best proximity point of the map $T$ is not only an approximate solution of $Tx = x$, but also optimal in the sense that $d(x, Tx)$ is minimum. Clearly, a best proximity point theorem is a natural generalization of a fixed point theorem.

Some interesting best proximity point results can be found in [7, 11, 14] and for applications, one can refer to [5, 6].

Recently, Jachymski [4] established the existence of fixed points for contractive mappings on a metric space endowed with a graph. This result unified various fixed point theorems for contractive mappings on metric spaces and partially ordered metric spaces. For some more fixed point results on a metric space with a graph, one can refer to [1, 13].

1.1. Our contribution. Following Jachymski [4], in this article we prove an existence and uniqueness theorem on best proximity point for non-self contractive mappings on a metric space endowed with a graph. As an application of this result, we obtain a generalization of the fixed point theorem for uniformly locally contractive mappings due to Edelstein [2, Theorem 5.2]. Also, our result enables us to obtain a best proximity point result for non-self mappings on partially ordered metric spaces. Further, our result subsumes a very recent result on existence of a unique best proximity point on a metric space due to V. Sankar Raj [11, Theorem 3.1].

2. Preliminaries

In this section, let us recall some definitions and notations which are needed for our results.

Let $(X, d)$ be a metric space. For given non-empty subsets $A$ and $B$ of $(X, d)$, we denote by $A_0$ and $B_0$ the following sets:

\begin{align*}
A_0 &= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\} \\
B_0 &= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.
\end{align*}

For sufficient conditions which ensure the non-emptiness of $A_0$ and $B_0$, one can refer to [7].

Let $(A, B)$ be a pair of non-empty subsets of $(X, d)$ such that $A_0 \neq \emptyset$. Then the pair $(A, B)$ is said to have the $P$-property [11] if and only if
Best proximity points of contractive mappings

\[
\begin{align*}
&d(x_1, y_1) = d(A, B) \\
&d(x_2, y_2) = d(A, B) \\
\Rightarrow &d(x_1, x_2) = d(y_1, y_2),
\end{align*}
\]

where \(x_1, x_2 \in A_0\) and \(y_1, y_2 \in B_0\).

It is easy to verify that for a non-empty subset \(A\) of \((X, d)\), the pair \((A, A)\) has the \(P\)-property. Every pair of non-empty closed convex subsets of a real Hilbert space \(H\) has the \(P\)-property (see [11]).

Consider a directed graph \(G\) where the set \(V(G)\) of its vertices coincides with \(X\), the set \(E(G)\) of its edges is such that \(E(G) \supseteq \Delta\) (where \(\Delta = \{(x, x) : x \in X\}\)) and \(E(G)\) has no parallel edges. We denote by \(\hat{G}\) the undirected graph obtained from \(G\) by ignoring the direction of edges. For given two vertices \(x\) and \(y\), we say that there is a path in \(G\) of length \(N\) (where \(N \in \mathbb{N} \cup \{0\}\)) between them if there exists a sequence \((x^i)_{i=0}^{\infty}\) such that \(x^0 = x\), \(x^N = y\) and \((x^{i-1}, x^i) \in E(G) \forall i = 1, 2, \ldots, N\). The graph \(G\) is called connected if there is a path between any two vertices and weakly connected if \(G\) is connected. For \(x \in V(G) = X\), we denote

\[|x|_G^N = \{y \in X : \text{there is a path in } G \text{ of length } N \text{ from } x \text{ to } y\}.\]

3. Main results

Throughout this section we assume that \((X, d)\) is a metric space endowed with a directed graph \(G\) where \(V(G) = X\), \(E(G) \supseteq \Delta\) and \(G\) has no parallel edges. We now introduce a notion of Banach contraction (for non-self map) with respect to the graph \(G\) for which we prove our main results.

Definition 3.1. Let \(A\) and \(B\) be two non-empty subsets of \((X, d)\). A mapping \(T : A \to B\) is said to be a Banach \(G\)-contraction or simply \(G\)-contraction if for all \(x, y \in A\), \(x \neq y\) with \((x, y) \in E(G)\):

(a) \(d(Tx, Ty) \leq \alpha d(x, y)\) for some \(\alpha \in [0, 1)\);

(b) \[
\begin{align*}
&d(x_1, Ty) = d(A, B) \\
&d(y_1, Ty) = d(A, B) \\
\Rightarrow &(x_1, y_1) \in E(G), \quad \text{for all } x_1, y_1 \in A.
\end{align*}
\]

Theorem 3.2. Let \((X, d)\) be complete metric space, \(A\) and \(B\) be two non-empty closed subsets of \((X, d)\) such that \((A, B)\) has the \(P\)-property. Let \(T : A \to B\) be a \(G\)-contraction such that \(T(A_0) \subseteq B_0\). Assume that for some \(N \in \mathbb{N}\),

(i) there exist \(x_0\) and \(x_1\) in \(A_0\) such that there is a \(N\)-length path \((y_0)_{i=0}^{N-1} \subseteq A_0\) in \(G\) between them and \(d(x_1, Tx_0) = d(A, B)\);

(ii) for any sequence \(\{s_n\}_{n \in \mathbb{N}}\) in \(A\) with \(s_n \to s\) and \(s_{n+1} \in [s_n]_G^N\), there is a subsequence \(\{s_{n_k}\}_{k \in \mathbb{N}}\) such that \((s_{n_k}, s) \in E(G) \forall k \in \mathbb{N}\).

Then there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) with \(d(x_{n+1}, Tx_n) = d(A, B)\) for \(n \in \mathbb{N}\), converging to a best proximity point of \(T\). Furthermore, \(T\) has a unique best proximity point if for any two elements \(x\) and \(y\) in \(A_0\), there exists a path \((y')_{i=0}^{l} \subseteq A_0\) in \(\hat{G}\) between them.
Proof. By (i), there exist two points \( x_0, x_1 \in A_0 \) such that \( d(x_1, Tx_0) = d(A, B) \) and a sequence \( (y_i)_{i=0}^{N} \) containing points of \( A_0 \) such that \( y_0^0 = x_0, y_0^N = x_1, \) and \( (y_i^{i-1}, y_i^i) \in E(G) \) \( \forall i \in \mathbb{N} \). As \( y_0^i \in A_0 \) and \( T(A_0) \subseteq B_0 \), there exists \( y_1^1 \in A_0 \) such that \( d(y_1^1, Ty_0^0) = d(A, B) \). Similarly, for \( i = 2, \ldots, N \), there exists \( y_i^1 \in A_0 \) such that \( d(y_i^1, Ty_i^0) = d(A, B) \).

As \( (y_0^0 = x_0, y_0^0) \in E(G) \) and \( T \) is a \( G \)-contraction, it follows from the above that \( (x_1, y_1^1) \in E(G) \). In a similar way, it follows that \( (y_1^{i-1}, y_1^i) \in E(G) \) for \( i = 2, \ldots, N \). Let \( x_2 = y_1^N \). Thus \( (y_1^i)_{i=0}^{N} \) is a path from \( x_1 = y_1^0 \) to \( x_2 = y_1^N \).

Again, for each \( i = 1, 2, \ldots, N \), since \( y_i^i \in A_0 \) and \( Ty_i^i \in T(A_0) \subseteq B_0 \), there exists \( y_2^2 \in A_0 \) such that \( d(y_2^2, Ty_i^i) = d(A, B) \). Also, we have \( d(x_2, Tx_1) = d(A, B) \). As shown in the previous paragraph, it follows that \( (x_2, y_2^2) \in E(G) \) and \( (y_2^{i-1}, y_2^i) \in E(G) \) \( \forall i = 2, \ldots, N \). Set \( x_3 = y_2^N \). Thus \( (y_2^i)_{i=0}^{N} \) is a path from \( x_2 = y_2^0 \) to \( x_3 = y_2^N \).

Continuing in this manner for all \( n \in \mathbb{N} \), we obtain a sequence \( \{x_n\}_{n \in \mathbb{N}} \) where \( x_{n+1} \in [x_n]_G \) and \( d(x_{n+1}, Tx_n) = d(A, B) \) by producing a path \( (y_n^i)_{i=0}^{N} \) from \( x_n = y_n^0 \) to \( x_{n+1} = y_n^N \) in such way that

\[
d(y_{n+1}^i, Ty_n^i) = d(A, B) \quad \forall i = 0, \ldots, N.
\]

Using the \( P \)-property of \((A, B)\), it follows from equation (3.1) that for each \( n \in \mathbb{N} \),

\[
d(y_{n-1}^{i-1}, y_n^i) = d(Ty_{n-1}^{i-1}, Ty_n^i) \quad \forall 1 \leq i \leq N.
\]

Now for any positive integer \( n \),

\[
d(x_n, x_{n+1}) = d(y_0^0, y_N^N) \leq d(y_0^0, y_1^0) + d(y_1^0, y_1^1) + \cdots + d(y_N^{N-1}, y_N^N) = \sum_{i=1}^{N} d(y_1^{i-1}, y_1^i) = \sum_{i=1}^{N} d(Ty_{n-1}^{i-1}, Ty_n^i).
\]

Since for all \( n \in \mathbb{N} \) and \( 1 \leq i \leq N \), \((y_{n-1}^{i-1}, y_n^i) \in E(G) \) and \( T \) is a \( G \)-contraction, it follows from the above inequalities that for \( n \in \mathbb{N} \),

\[
d(x_n, x_{n+1}) \leq \alpha \sum_{i=1}^{N} d(y_{n-1}^{i-1}, y_n^i) \quad \text{for some } \alpha \in [0, 1).
\]

Repeating the process, it follows that for all \( n \in \mathbb{N} \),

\[
d(x_n, x_{n+1}) \leq \alpha^n \sum_{i=1}^{N} d(y_0^{i-1}, y_0^i) = M \alpha^n \quad \text{where } M = \sum_{i=1}^{N} d(y_0^{i-1}, y_0^i).
\]

Now for \( m \geq n, n \in \mathbb{N} \),

\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \leq M \alpha^n + \cdots + M \alpha^{m-n} = M \frac{\alpha^n [1 + \cdots + \alpha^{m-n-1}]}{1 - \alpha} \leq M \frac{\alpha^n}{1 - \alpha}.
\]
Best proximity points of contractive mappings

Hence \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence. Therefore \( \{x_n\}_{n \in \mathbb{N}} \) converges to some point \( x^* \in A \) as \( n \to \infty \). By (ii), there is a subsequence \( \{x_{n_k}\}_{k \in \mathbb{N}} \) such that \( (x_{n_k}, x^*) \in E(G) \) \( \forall k \in \mathbb{N} \). Hence,

\[
d(Tx_{n_k}, Tx^*) \leq ad(x_{n_k}, x^*) \quad \text{for } k \in \mathbb{N}.
\]

Thus taking \( k \to \infty \), \( Tx_{n_k} \to Tx^* \). Using the continuity of the metric function, we get \( d(x_{n_k+1}, Tx_{n_k}) \to d(x^*, Tx^*) \) as \( k \to \infty \). Now \( \{d(x_{n_k+1}, Tx_{n_k})\} \) is nothing but a constant sequence with value \( d(A, B) \). Therefore \( d(x^*, Tx^*) = d(A, B) \).

Suppose that \( p \) and \( q \) are two best proximity points of \( T \). Consider two sequences \( \{p_n\}_{n \in \mathbb{N}} \) and \( \{q_n\}_{n \in \mathbb{N}} \) where \( p_n = p \) and \( q_n = q \) for all \( n \geq 1 \).

Clearly, \( d(p_{n+1}, Tp_n) = d(A, B) \) and \( d(q_{n+1}, Tq_n) = d(A, B) \) for all \( n \geq 1 \). As \( p, q \in A_0 \), it follows from the hypothesis that there is a path \( (y^i)_{i=0}^M \subseteq A_0 \) in \( G \) between \( p_1 = p \) and \( q_1 = q \). For each \( i = 1, 2, \ldots, M - 1 \), since \( y^i_1 \in A_0 \) and \( T(y^i) \in T(A_0) \subseteq B_0 \), we can obtain \( \{y^i_n\}_{n \in \mathbb{N}} \) such that \( d(y^i_{n+1}, Ty^i_n) = d(A, B) \) \( \forall n \in \mathbb{N} \). It is easy to verify that \( T \) is also a \( G \)-contraction. Also, we have \( \{y^i_1, y^i_2\} = E(G) \) for \( 1 \leq i \leq M \). Thus it follows that \( (y^i_n)_{i=0}^M \) is a path in \( G \) between \( p_2 = y^0_2 \) and \( q_2 = y^M_2 \). Similarly, it follows that \( \forall n \in \mathbb{N} \), \( (y^i_n)_{i=0}^M \) is a path in \( G \) from \( p_n = y^n_0 \) to \( q_n = y^n_M \). Now for \( n \in \mathbb{N} \),

\[
d(p, q) = d(p_{n+1}, q_{n+1}) \leq \sum_{i=1}^{M} d(y^i_{n+1}, y^i_{n+1}) = \sum_{i=1}^{M} d(Ty^i_{n+1}, Ty^i_n) \\ \leq \alpha \sum_{i=1}^{M} d(y^i_n, y^i_{n+1}) \leq \cdots \leq \alpha^n \sum_{i=1}^{M} d(y^i_1, y^i_2). \quad [\text{where } \alpha \in (0, 1)]
\]

This implies that \( p = q \) and this completes the proof.

**Remark 3.3.** Theorem 3.2 still holds true if we replace the condition (ii) by the continuity of the function \( T \) on the set \( A \).

The above Theorem 3.2 yields the following result due to Jachymski [4].

**Theorem 3.4** (see [4]). Let \( (X, d) \) be complete and \( f : X \to X \) be a map such that for all \( x, y \in X \) with \( (x, y) \in E(G) \), \( (fx, fy) \in E(G) \) and \( d(fx, fy) \leq kd(x, y) \) where \( k \in [0, 1) \). Assume that for any \( \{y_n\}_{n \in \mathbb{N}} \) in \( X \) with \( y_n \to y^* \) and \( (y_{n+1}, y_n) \in E(G) \) \( \forall n \geq 1 \), there exists a subsequence \( \{y_{n_p}\}_{p \in \mathbb{N}} \) such that \( (y_{n_p}, y^*) \in E(G) \) for all \( p \in \mathbb{N} \). Then the following statements hold:

(i) \( \{f^n(x)\}_{n \in \mathbb{N}} \) converges to a fixed point of \( f \) if \( (x, fx) \in E(G) \);

(ii) if \( G \) is weakly connected and there exists \( x_0 \in X \) such that \( (x_0, fx_0) \in E(G) \), then \( \forall x \in X \), \( \{f^n(x)\}_{n \in \mathbb{N}} \) converges to a unique fixed point of \( f \).

Further, we get the following result due to V. Sankar Raj [11] as a corollary to the Theorem 3.2 by taking \( E(G) = X \times X \).

**Corollary 3.5** ([11, Theorem 3.1]). Let \( (X, d) \) be a complete metric space, \( A \) and \( B \) be two non-empty closed subsets of \( (X, d) \) such that \( A_0 \neq \emptyset \) and \( (A, B) \)
satisfies $P$-property. Suppose that $T : A \to B$ is such that $T(A_0) \subseteq B_0$ and
\begin{equation}
(3.3) \quad d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in A \text{ and for some } k \in [0, 1).
\end{equation}
Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$. Further, for any fixed $x_0 \in A_0$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $d(x_n, Tx_{n-1}) = d(A, B)$ for $n \in \mathbb{N}$, converging to $x^*$.

The following example shows that our Theorem 3.2 is an extension of the above result due to V. Sankar Raj [11].

**Example 3.6.** Consider $X = \mathbb{R}^2$ with usual metric and suppose that
\begin{align*}
A &= \left\{ \left( 0, \frac{1}{n} \right) : n \in \mathbb{N} \right\} \cup \{(0, 0)\}, \\
B &= \left\{ \left( 1, \frac{1}{n} \right) : n \in \mathbb{N} \right\} \cup \{(1, 0)\}.
\end{align*}
It is easy to check that the pair $(A, B)$ has the $P$-property. Suppose that a map $T : A \to B$ is defined as follows:
\begin{align*}
T((0, x)) &= \left( 1, \frac{x}{2} \right), \quad \text{for all } (0, x) \in A \text{ with } x \neq 1, \\
T((0, 1)) &= (1, 1).
\end{align*}
Consider a graph $G$ with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : d(x, y) < \frac{1}{2}\}$. Let $x = (0, x')$ and $y = (0, y')$ be two elements in $A$ with $(x, y) \in E(G)$. Then,
\begin{equation}
(3.3) \quad d(T(x), T(y)) = d \left( \left( 1, \frac{x'}{2} \right), \left( 1, \frac{y'}{2} \right) \right) \leq \frac{1}{2} d(x, y).
\end{equation}
If $x_1 = (0, x'_1)$ and $y_1 = (0, y'_1)$ are two elements in $A$ such that
\begin{equation}
\frac{1}{2} d(x_1, T(x)) = d(y_1, T(y)) = d(x_1, y_1).
\end{equation}
Then by using the $P$-property of $(A, B)$, it follows from the above equation that $d(x_1, y_1) = d(T(x), T(y)) \leq \frac{1}{2} d(x, y) < \frac{1}{2}$. Hence the pair $(x_1, y_1) \in E(G)$. This proves that $T$ is a non-self $G$-contraction with $\alpha = \frac{1}{2}$. Clearly, $(X, d)$ is complete and $A$ and $B$ are closed subsets of $X$. Also, note that in this case $A_0 = A$, $B_0 = B$ and $T(A_0) = T(A) \subseteq B = B_0$. Let $x_0 = (0, \frac{1}{2})$, $x_1 = (0, \frac{1}{4})$ and $N = 1$. Then $d(x_1, T(x_0)) = d(x_1, B) = 1$ and the pair $(x_1, x_0) \in E(G)$. Hence, the condition (i) of Theorem 3.2 holds. Also, let $\{s_n\}_{n \in \mathbb{N}}$ be a sequence in $A$ such that $s_n \to s$ as $n \to \infty$. Then there exists a positive integer $M$ such that $d(s_n, s) < \frac{1}{2}$ $\forall n \geq M$. Let $n_k = M + k$ for $k \geq 1$. Consequently, $\{s_{n_k}\}_{k \in \mathbb{N}}$ is a subsequence of the sequence $\{s_n\}_{n \in \mathbb{N}}$ such that $(s_{n_k}, s) \in E(G) \forall k \in \mathbb{N}$. This implies that the condition (ii) of Theorem 3.2 is also satisfied. Therefore Theorem 3.2 guarantees the existence of a best proximity point of $T$. Note that $(0, 0)$ and $(0, 1)$ are two best proximity points. However,
\begin{equation}
\frac{1}{2} d(T(0, 0), (1, 1)) = d((1, 0), (1, 1)) = 1 > kd((0, 0), (0, 1)),
\end{equation}
for any $k \in [0, 1)$. This proves that $T$ does not satisfy the contractive condition (3.3).
4. Applications

Let $A$ and $B$ be two non-empty subsets of a metric space $(X,d)$. A mapping $f : A \to B$ is called $(\epsilon,k)$-uniformly locally contractive [2] (where $k \in [0,1)$ and $\epsilon > 0$) if $d(fx,fy) \leq kd(x,y)$ for all $x,y \in A$ with $d(x,y) < \epsilon$. An $(\epsilon,k)$-uniformly locally contractive mapping need not be a contraction, for example one can refer to [2, 8]. As an application of Theorem 3.2, we now establish the following result for uniformly locally contractive mappings.

**Theorem 4.1.** Let $(X,d)$ be complete metric space, $A$ and $B$ be closed subsets of $(X,d)$ such that $A_0 \neq \emptyset$ and $(A,B)$ satisfies P-property. Suppose that $T : A \to B$ is an $(\epsilon,k)$-uniformly locally contractive mapping satisfying $T(A_0) \subseteq B_0$. Then $T$ has a unique best proximity point if the space $(A_0,d)$ is $\epsilon$-chainable, that is, given $a,b \in A_0$, there exist $N \in \mathbb{N}$ and a sequence $(y^i)_{i=0}^{N}$ in $A_0$ such that $y^0 = a$, $y^N = b$ and $d(y^{i-1},y^i) < \epsilon$ for each $i = 1, 2, \ldots, N$.

**Proof.** Consider the graph $G$ where $V(G) = X$ and $E(G)$ as follows:

$$E(G) = \{(x,y) \in X \times X : d(x,y) < \epsilon\}.$$ 

It is clear that $E(G) \supseteq \Delta$ and $G$ has no parallel edges. Also, in this case $G = \bar{G}$. Let $x,y \in A$ be such that $(x,y) \in E(G)$ and for all $x_1,y_1 \in A$,

$$d(x_1,Tx) = d(A,B) \text{ and } d(y_1,Ty) = d(A,B).$$

Since $(x,y) \in E(G)$, $d(Tx,Ty) \leq kd(x,y)$ where $k \in [0,1)$. Hence and by the P-property of $(A,B)$, we have $d(x_1,y_1) < \epsilon$. Therefore $T$ is a $G$-contraction. Since $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$, there exist $x_0$ and $x_1$ in $A_0$ such that $d(x_1,Tx_0) = d(A,B)$. The $\epsilon$-chainability of $(A_0,d)$ implies that there exist a natural number $N$ and a sequence $(y^i)_{i=0}^{N}$ containing points of $A_0$ such that $y^0 = x_0$, $y^N = x_1$ and $d(y^{i-1},y^i) < \epsilon$ for $i = 1, 2, \ldots, N$. Thus $(y^i)_{i=0}^{N} \subseteq A_0$ is a path in $G$ between $x_0$ and $x_1$. If $\{s_n\}_{n \in \mathbb{N}}$ is a sequence in $A$ such that $s_n \to s$, then there exists $M \in \mathbb{N}$ such that $d(s_n,s) < \epsilon \forall n \geq M$. Hence we can obtain a subsequence $\{s_{n_p}\}_{p \in \mathbb{N}}$ such that $(s_{n_p},s) \in E(G)$ $\forall p \in \mathbb{N}$. Also, it is clear from the $\epsilon$-chainability of $(A_0,d)$ that for every $x,y \in A_0$, there is a path $(q^i)_{i=0}^{l} \subseteq A_0$ in $\bar{G}$ (i.e., $G$) between them. Thus $T$ has a unique best proximity point by Theorem 3.2. \hfill \Box

As a corollary to the above theorem, we get the following theorem due to Edelstein [2] by considering $A = B = X$.

**Theorem 4.2 ([2, Theorem 5.2]).** Let $(X,d)$ be a complete metric space. An $(\epsilon,k)$-uniformly locally contractive mapping $f : X \to X$ has a unique fixed point if $(X,d)$ is $\epsilon$-chainable.

In the last part of this section we establish the following result for non-self contractive mapping on a partially ordered metric space.

Let $(X,d)$ be a metric space endowed with a partial order $\preceq$ and $A$ and $B$ be two non-empty subsets of $(X,d)$. By $X_{\preceq}$, we denote the following set:

$$X_{\preceq} = \{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$
Following [10], we say that a mapping \( T : A \to B \) is a proximally monotone mapping if for all \( x_1, x_2 \in A \) with \( x_1 \preceq x_2 \):

\[
\begin{align*}
    d(y_1, Tx_1) &= d(A, B) \\
    d(y_2, Tx_2) &= d(A, B)
\end{align*}
\]

\( \Rightarrow \) \( (y_1, y_2) \in X \preceq \), for all \( y_1, y_2 \in A \).

**Theorem 4.3.** Let \( (X, d) \) be complete metric space, \( A \) and \( B \) be two closed subsets of \( (X, d) \) such that \( (A, B) \) has the \( P \)-property. Let \( T : A \to B \) be a proximally monotone map such that \( T(A_0) \subseteq B_0 \) and \( d(Tx, Ty) \leq kd(x, y) \) for all \( x \preceq y \) and for some \( k \in [0, 1) \).

Assume that either \( T \) is continuous on \( A \) or for any \( \{y_n\}_{n \in \mathbb{N}} \) in \( A \) with \( y_n \to y^* \) and \( (y_n, y_{n+1}) \in X \preceq \) for \( n \in \mathbb{N} \), there exists \( (y_{n_p})_{p \in \mathbb{N}} \) such that \( (y_{n_p}, y^*) \in X \preceq \) for \( p \in \mathbb{N} \). Then \( T \) has a best proximity point if there exist \( x_0 \) and \( x_1 \) in \( A_0 \) such that \( d(x_1, Tx_0) = d(A, B) \) and \( (x_0, x_1) \in X \preceq \). Moreover, the best proximity point of \( T \) is unique if for \( x, y \in A_0 \), there exists \( z \in A_0 \) such that \( (x, z), (y, z) \in X \preceq \).

**Proof.** By considering the graph \( G \) where \( V(G) = X \) and

\[
E(G) := \{(x, y) \in X \times X : x \preceq y \lor y \preceq x\},
\]

the proof follows by Theorem 3.2 and Remark 3.3. \( \square \)

The above result includes the fixed point results for mappings on a partially ordered metric space due to Ran and Reurings [12] and J. J. Nieto and R. R. López [9].

**Acknowledgements.** The authors are grateful to the referees for their valuable comments and suggestions to improve this manuscript. The first author is thankful to University Grants Commission (F.2 – 12/2002(SA – I)), New Delhi, India for the financial support.

**References**


Best proximity points of contractive mappings