APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 16, no. 1(2015), 53-64
doi:10.4995/agt.2015.3141

F-nodec spaces

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Abstract

Following Van Douwen, a topological space is said to be nodec if it satisfies one of the following equivalent conditions:

(i) *every nowhere dense subset of X, is closed;*
(ii) *every nowhere dense subset of X, is closed discrete;*
(iii) *every subset containing a dense open subset is open.*

*This paper deals with a characterization of topological spaces X such that $\mathbf{F}(X)$ is a nodec space for some covariant functor $\boldsymbol{F}$ from the category **Top** to itself. $\boldsymbol{T}_0$, $\boldsymbol{\rho}$ and **FH** functors are completely studied. Secondly, we characterize maps f given by a flow (X, f) in the category **Set** such that $(X, \mathcal{P}(f))$ is nodec (resp., T_0-nodec), where $\mathcal{P}(f)$ is a topology on X whose closed sets are precisely f-invariant sets.*

2010 MSC: *54B30; 54D10; 54G12; 46M15.*

Keywords: *Categories; functors; Nodec spaces; primal Space.*

1. Introduction

Recall that a topological space X is called *submaximal* if every dense subset of X is open. In [8] it was shown that, in a submaximal space without isolated points, every nowhere dense subset (that is the interior of its closure is empty),

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Received 25 June 2014 – Accepted 4 December 2014

is closed and discrete. Hence, a space satisfying the later property is called *nodec*.

Different equivalent conditions for a space to be *submaximal* are given in [1] and the ones for a space to be *nodec* in [14], [12] and [4].

Theorem 1.1. [4, Theorem 2.5] *The following statements are equivalent:*

(1) *X is nodec;*
(2) *Each nowhere dense subset of X, is closed.*
(3) *For each $A \subseteq X$, if $\overline{\mathring{A}} \subseteq A$, then A is closed.*
(4) *For each $A \subseteq X$, if $A \subseteq \mathring{\overline{\mathring{A}}}$, then A is open.*
(5) *For each $A \subseteq X$, $\overline{A} \setminus A \subseteq \overline{\mathring{A}}$.*
(6) *For each $A \subseteq X$, $\overline{A} = A \cup \overline{\mathring{A}}$.*
(7) *For each $A \subseteq X$, $\mathring{A} = A \cap \mathring{\overline{\mathring{A}}}$.*

On the other hand, the theory of category and functors play an enigmatic role in topology, specially the notion of reflective subcategory. So it is of importance to recall the standard notion of reflective subcategory $\boldsymbol{\mathcal{A}}$ of $\boldsymbol{\mathcal{B}}$ that is, a full subcategory such that the embedding $\boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{B}}$ has a left adjoint $\mathbf{F} : \boldsymbol{\mathcal{B}} \longrightarrow \boldsymbol{\mathcal{A}}$ (called reflection). Further, recall that for all $i = 0, 1, 2, 3, 3.\frac{1}{2}$ the subcategory $\mathbf{Top}_i$ of T_i-spaces is reflective in $\mathbf{Top}$, the category of all topological spaces.

$T_{3\frac{1}{2}}$ is also called the Tychonoff-reflection and will be denoted by ρ. In this paper the functors T_0, ρ and FH (the functionally Hausdorff reflection) are studied.

Some authors (see [3],[6], [11]) are interested in separation axioms using the theory of categories and functors as follows.

Definition 1.2. Let $\mathbf{C}$ be a category and $\mathbf{F}$, $\mathbf{G}$ two (covariant) functors from $\mathbf{C}$ to itself.

(1) An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},\mathbf{G})}$-object if $\mathbf{G}(\mathbf{F}(X))$ is isomorphic with $\mathbf{F}(X)$.
(2) Let P be a topological property on the objects of $\mathbf{C}$. An object X of $\mathbf{C}$ is said to be a $T_{(\mathbf{F},P)}$-object if $\mathbf{F}(X)$ satisfies the property P.

Consequently, some new separation axioms $T_{(0,\rho)}$, $T_{(0,FH)}$, $T_{(\rho,FH)}$ are introduced and characterized.

Recently, in [2] a characterization of topological spaces X such that their compactification noted $K(X)$ is a nodec space, is given. In [5], L. Dridi et al characterized topological spaces X such that $\mathbf{F}(X)$ is a submaximal space, for a given covariant functor $\mathbf{F}$.

The first section of this paper is devoted to the characterization of T_0-nodec space.

Second section studies the same problem using the functor ρ (resp., FH).

Finally, in the third section we are interested in the relation between nodec spaces and primal spaces.

Some important results are given.

2. T_0-NODEC SPACES

First let us recall the T_0-reflection of a topolgical space. Let X be a topological space. We define the binary relation $\sim$ on X by $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Then $\sim$ is an equivalence relation on X and the resulting quotient space $\mathbf{T}_0(X) := X/\sim$ is the $\mathbf{T}_0$-reflection of X.

The canonical surjection $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is a *quasihomeomorphism*. (A continuous map $q : X \longrightarrow Y$ is said to be a quasihomeomorphism if $U \longmapsto q^{-1}(U)$ (resp., $C \longmapsto q^{-1}(C)$) defines a bijection $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ (resp., $\mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$), where $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$) is the collection of all open sets (resp;, closed sets) of X)[7].

Before giving the main result of this section let us introduce some definitions, notations and remarks.

Notations 2.1. [5, Notations 2.2] Let X be a topological space, $a \in X$ and $A \subseteq X$. We denote by:

(1) $d_0(a) := \{x \in X : \overline{\{x\}} = \overline{\{a\}}\}$
(2) $d_0(A) = \cup[d_0(a); a \in A]$.

Remarks 2.2. [5, Remarks 2.3] Let X be a topological space and A be a subset of X. In [5] the following remarks are given.

(i) $d_0(A) = \mu_X^{-1}(\mu_X(A))$.
(ii) $d_0(d_0(A)) = d_0(A)$.
(iii) $A \subseteq d_0(A) \subseteq \overline{A}$ and consequently $\overline{d_0(A)} = \overline{A}$.
(iv) In particular if A is open (resp., closed), then $d_0(A) = A$.

Definition 2.3. Let X be a topological space. X is called a T_0-*nodec* space if its T_0-reflection is a nodec space.

Now we are in a position to give the characterization of T_0-nodec space.

Theorem 2.4. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is a T_0-nodec space;*
(2) *For any nowhere dense subset A of X, $d_0(A)$ is closed.*
(3) $\forall A \subseteq X$; *if* $\overset{\circ}{\overline{A}} \subseteq d_0(A) \Longrightarrow d_0(A) = \overline{A}$.
(4) $\forall A \subseteq X$; $\overline{A} \backslash d_0(A) \subseteq \overset{\circ}{\overline{A}}$.
(5) $\forall A \subseteq X$; $\overline{A} = d_0(A) \cup \overset{\circ}{\overline{A}}$.

Proof. (1) $\Longrightarrow$ (2) Let A be a nowhere dense subset of X. Then $\overline{\mu_X(A)} \subseteq \mu_X(\overline{A})$, since μ_X is a closed map. Suppose that there exist an open set U of $\mathbf{T}_0(X)$ such that $U \subset \overline{\mu_X(A)}$. So $\mu_X^{-1}(U)$ is an open set and $\mu_X^{-1}(U) \subseteq$

$\mu_X^{-1}(\overline{\mu_X(A)}) \subseteq \mu_X^{-1}(\mu_X(\overline{A})) = d_0(\overline{A}) = \overline{A}$, which contradict the fact that A is a nowhere dense subset of X. Then $\mu_X(A)$ is a nowhere dense subset of $\mathbf{T}_0(X)$. Since the later is nodec, $\mu_X(A)$ is closed. So $\mu_X^{-1}(\mu_X(A)) = d_0(A)$ is closed.

(2) $\Longrightarrow$ (1) Let S be a nowhere dense subset of $T_0(X)$ and $A = \mu_X^{-1}(S)$. Then $d_0(A) = A$. Suppose that there exists $x \in \overset{\circ}{\overline{A}}$. Thus there exists an open set U of X such that $x \in U \subset \overline{A}$. So $\mu_X(U)$ is an open set containing $\mu_X(x)$, since μ_X is open. Moreover, $\mu_X(U) \subseteq \mu_X(\overline{A}) = \mu_X(\overline{\mu_X^{-1}(S)}) \subseteq \mu_X(\mu_X^{-1}(\overline{S})) = \overline{S}$, by [5, Lemma 2.16]. Therefore $\mu_X(x) \in \overset{\circ}{\overline{S}}$, which contradicts the fact that S is a nowhere dense subset of $T_0(X)$. Then $\overset{\circ}{\overline{A}} = \varnothing$ and consequently A is a nowhere dense subset of X. Thus $d_0(A) = A$ is closed, by hypothesis. So S is a closed set of $T_0(X)$.

(2) $\Longrightarrow$ (4) Let A be a subset of X. Since X is T_0-nodec, $\overline{\mu_X(A)} \backslash \mu_X(A) \subset \overline{\overset{\circ}{\overline{\mu_X(A)}}}$, by Theorem 1.1. Then $\mu_X^{-1}(\overline{\mu_X(A)} \backslash \mu_X(A)) = \overline{\mu_X^{-1}(\mu_X(A))} \backslash \mu_X^{-1}(\mu_X(A)) = \overline{d_0(A)} \backslash d_0(A) = \overline{A} \backslash d_0(A) \subseteq \mu_X^{-1}(\overline{\overset{\circ}{\overline{\mu_X(A)}}}) = \overline{\overset{\circ}{\overline{\mu_X^{-1}(\mu_X(A))}}} = \overline{\overset{\circ}{\overline{d_0(A)}}} = \overline{\overset{\circ}{\overline{A}}}$.

(4) $\Longrightarrow$ (5) Let A be a subset of X. As $\overline{\overset{\circ}{\overline{A}}} \subseteq \overline{A}$ and $d_0(A) \subseteq \overline{A}$, it is clear that $d_0(A) \cup \overline{\overset{\circ}{\overline{A}}} \subset \overline{A}$. Conversely, $\overline{A} = d_0(A) \cup (\overline{A} \backslash d_0(A)) \subseteq d_0(A) \cup \overline{\overset{\circ}{\overline{A}}}$, by (4). Thus, the equality holds.

(5) $\Longrightarrow$ (2) Let A be a nowhere dense subset of X. Then $\overline{A} = d_0(A) \cup \overline{\overset{\circ}{\overline{A}}} = d_0(A)$.

(2) $\Longrightarrow$ (3) Straightforward.

(3) $\Longrightarrow$ (2) If A is a nowhere dense subset of X, then $\overline{\overset{\circ}{\overline{A}}} = \varnothing \subseteq d_0(A)$. Thus $d_0(A) = \overline{A}$, by (3).

□

Remark 2.5. Clearly every nodec space is a $\mathbf{T}_0$-nodec space. The converse does not hold:

Indeed, given a set $X = \{0, 1, 2\}$ equipped with the topology $\tau = \{\varnothing, \{2\}, X\}$. We can easily see that $\mathbf{T}_0(X)$ is a nodec space. However $\{0\}$ is a nowhere dense subset of X and not closed.

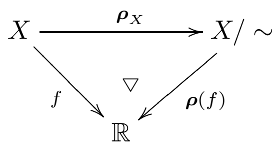
3. $\boldsymbol{\rho}$-nodec spaces and $\boldsymbol{FH}$-nodec spaces

Let X be a topological space, F a subset of X and $x \in X$. x and F are said to be *completely separated* if there exists a continuous map $f : X \longrightarrow \mathbb{R}$ such that $f(x) = 0$ and $f(F) = \{1\}$. Now, two distinct points x and y in X are called completely separated if x and $\{y\}$ are completely separated.

A space X is said to be *completely regular* if every closed subset F of X is completely separated from any point x not in F. Recall that a topological space X is called a T_1*-space* if each singleton of X is closed. A completely regular T_1-space is called a *Tychonoff space* [13].

A *functionally Hausdorff* space is a topological space in which any two distinct points of this space are completely separated. Remark here that a Tychonoff space is a functionally Hausdorff space and consequently a Hausdorff space (T_2-space).

Now, for a given topological space X, we define the equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in \mathbf{C}(X)$ (where $\mathbf{C}(X)$ design the family of all continuous maps from X to $\mathbb{R}$). Let us denote by $X/\sim$ the set of equivalence classes and let $\boldsymbol{\rho}_X : X \longrightarrow X/\sim$ be the canonical surjection map assigning to each point of X its equivalence class. Since every f in $\mathbf{C}(X)$ is constant on each equivalence class, we can define $\boldsymbol{\rho}(f) : X/\sim \longrightarrow \mathbb{R}$ by $\boldsymbol{\rho}(f)(\boldsymbol{\rho}_X(x)) = f(x)$. One may illustrate this situation by the following commutative diagram.



Now, equip $X/\sim$ with the topology whose closed sets are of the form $\cap[\boldsymbol{\rho}(f_\alpha)^{-1}(F_\alpha) : \alpha \in I]$, where $f_\alpha : X \longrightarrow \mathbb{R}$ (resp., F_α) is a continuous map (resp., a closed subset of $\mathbb{R}$). It is well known that, with this topology, $X/\sim$ is a Tychonoff space (see for instance [15]) and its denoted by $\boldsymbol{\rho}(X)$.

The construction of $\boldsymbol{\rho}(X)$ satisfies some categorical properties:

For each Tychonoff space Y and each continuous map $f : X \longrightarrow Y$, there exists a unique continuous map $\widetilde{f} : \boldsymbol{\rho}(X) \longrightarrow Y$ such that $\widetilde{f} \circ \boldsymbol{\rho}_X = f$. We will say that $\boldsymbol{\rho}(X)$ is the $\boldsymbol{\rho}$*-reflection* (or Tychonoff-reflection) of X.

From the above properties, it is clear that $\boldsymbol{\rho}$ is a covariant functor from the category of topological spaces **Top** into the full subcategory **Tych** of **Top** whose objects are Tychonoff spaces.

On the other hand the quotient space $X/\sim$ which is denoted by $\mathbf{FH}(X)$ is a functionally Hausdorff space.

The construction $\mathbf{FH}(X)$ satisfies some categorical properties:

For each functionally Hausdorff space Y and each continuous map $f : X \longrightarrow Y$, there exists a unique continuous map $\widetilde{f} : \mathbf{FH}(X) \longrightarrow Y$ such that $\widetilde{f} \circ \boldsymbol{\rho}_X = f$. We will say that $\mathbf{FH}(X)$ is the *functionally Hausdorff-reflection* of X (or the **FH**-reflection of X).

Consequently, it is clear that **FH** is a covariant functor from the category of topological spaces **Top** into the full subcategory **FunHaus** of **Top** whose objects are functionally Hausdorff spaces.

We need to introduce and recall some definitions, notations and results.

Notations 3.1. [5, Notation 3.1] Let X be a topological space, $a \in X$ and A a subset of X. We denote by:

(1) $d_{\boldsymbol{\rho}}(a) := \cap[f^{-1}(f(\{a\})) : f \in \mathbf{C}(X)]$.

(2) $d_{\boldsymbol{\rho}}(A) := \cup[d_{\boldsymbol{\rho}}(a) : a \in A]$.

The following results are given in [5].

Proposition 3.2. [5, Proposition 3.2] *Let X be a topological space, $a \in X$ and A a subset of X. Then:*

(1) $d_\rho(A) = \rho_X^{-1}(\rho_X(A))$.
(2) $d_\rho(a)$ *is a closed subset of* X.
(3) $A \subseteq d_\rho(A) \subseteq \cap[f^{-1}(f(A)) : f \in \mathbf{C}(X)]$.
(4) $\forall f \in \mathbf{C}(X)$, $f(A) = f(d_\rho(A))$.

Now, we introduce the following definition.

Definition 3.3. Let X be a topological space. X is called a *ρ-nodec* (resp., *FH-nodec*) space if its ρ-reflection (resp., FH-reflection) is a nodec space.

In order to give a characterization of ρ-nodec spaces, we need to recall some elementary proporties which characterize Tychonoff spaces in terms of *zero-set* (resp., *cozero-sets*). Consider a topological space X and $A \subseteq X$. A is called a zero-set if there exists $f \in C(X)$ such that $A = f^{-1}(\{0\})$. The complement of a zero-set is called a cozero-set.

According to [15, Proposition 1.7], a space is Tychonoff if and only if the family of zero-sets of the space is a base for the closed sets (equivalently, the family of cozero-sets of the space is a base for the open sets). In [5] it is shown that a closed (resp., open) subset of $\boldsymbol{\rho}(X)$ is of the form $\cap[\boldsymbol{\rho}(f)^{-1}(\{0\}) : f \in H]$ (resp., $\cup[\boldsymbol{\rho}(f)^{-1}(\mathbb{R}^\star) : f \in H]$) , where H is a collection of continuous maps from X to $\mathbb{R}$.

Recall that a subset V of a topological space X is called a *functionally open* subset of X (for short F-open) if and only if $d_\rho(V)$ is open in X (see [5]). Now in order to characterize ρ-nodec spaces and FH-nodec spaces, we introduce the following definitions:

Definition 3.4. Let X be a topological space and V a set of X.

V is called a Functionally nowhere dense subset of X (for short *F-nowhere dense*) if and only if for any F-open subset W of X, $d_\rho(W) \nsubseteq \overline{d_\rho(V)}$.

Definition 3.5. Let X be a topological space and A a nonempty subset of X. A is said to be a ρ-nowhere dense subset of X if for any nonzero continuous map f from X to $\mathbb{R}$ there exists a continuous map g from X to $\mathbb{R}$ satisfying $fg \neq 0$ and $g(A) = \{0\}$.

Remark 3.6. V is an F-open set of X if and only if $\rho_X(V)$ is an open set of $FH(X)$.

Proposition 3.7. *Let X be a topological space and A a subset of X. Then the following statements are equivalent:*

(i) *A is a ρ-nowhere dense subset of X;*
(ii) *$\rho_X(A)$ is a nowhere dense subset of $\rho(X)$.*

Proof. $(i) \Longrightarrow (ii)$ Let A be a ρ-nowhere dense subset of X. According to the universal property of $\rho(X)$, each continuous map $g : \rho(X) \longrightarrow \mathbb{R}$ may be written as $g = \rho(f)$ with $f = g \circ \rho_X$. Now suppose that there exists a nonzero continuous map f from X to $\mathbb{R}$ such that $\rho(f)^{-1}(\mathbb{R}^\star) \subset \overline{\rho_X(A)}$. Then there exists a continuous map g such that $fg \neq 0$ and $g(A) = \{0\}$. Thus $\rho_X(A) \subset \rho(g)^{-1}\{0\}$ and consequently $\overline{\rho_X(A)} \subset \rho(g)^{-1}\{0\}$. So $\rho(f)^{-1}(\mathbb{R}^\star) \subset \overline{\rho_X(A)} \subset \rho(g)^{-1}\{0\}$. Therefore $\rho(f)^{-1}(\mathbb{R}^\star) \cap \rho(g)^{-1}(\mathbb{R}^\star) = \varnothing$ which contradict the fact that $fg \neq 0$.

$(ii) \Longrightarrow (i)$ Let A be a subset of X such that $\rho_X(A)$ is a nowhere dense subset of $\rho(X)$. Then for any nonzero continuous map f from X to $\mathbb{R}$, $\rho(f)^{-1}(\mathbb{R}^\star) \nsubseteq \overline{\rho_X(A)}$. Thus there exists $x \in X$ such that $\rho(x) \in \rho(f)^{-1}(\mathbb{R}^\star)$ and $\rho(x) \notin \overline{\rho_X(A)}$. Therefore there exists a nonzero continuous map g such that $\rho(x) \in \rho(g)^{-1}(\mathbb{R}^\star)$ and $\rho(g)^{-1}(\mathbb{R}^\star) \cap \rho_X(A) = \varnothing$. So $g(A) = \{0\}$. Hence A is a ρ-nowhere dense subset of X.

□

Now, we are in a position to give the characterization of ρ-nodec spaces.

Theorem 3.8. *Let X be a topological space. Then the following statements are equivalent:*

- (i) *X is a ρ-nodec space;*
- (ii) *For any ρ-nowhere dense set A of X, we have $d_\rho(A)$ is an intersection of zero sets of X.*

Proof. $(i) \Longrightarrow (ii)$ Let A be a ρ-nowhere dense subset of X. According to the Proposition 3.7 $\rho_X(A)$ is a nowhere dense subset of $\rho(X)$. Since X is a ρ-nodec space, then $\rho_X(A)$ is a closed set of $\rho(X)$ and thus $\rho_X(A) = \bigcap[\rho(f)^{-1}\{0\} : f \in H]$ (where H is a subfamily of $\mathbf{C}(X)$). So that $\rho_X^{-1}(\rho_X(A)) = \bigcap[\rho_X^{-1}(\rho(f)^{-1}\{0\}) : f \in H]$. Therefore $d_\rho(A) = \bigcap[f^{-1}\{0\} : f \in H]$ is an intersection of zero sets of X.

$(ii) \Longrightarrow (i)$ Conversely, let A a subset of X such that $\rho_X(A)$ is a nowhere dense subset of $\rho(X)$. Then, by Proposition 3.7, A is a ρ-nowhere dense subset of X and consequently $d_\rho(A)$ is an intersection of zero sets of X. Hence there exists a subfamily $\{f_i : i \in I\}$ of $\mathbf{C}(X)$ satisfying $\rho_X^{-1}(\rho_X(A)) = \bigcap[f_i^{-1}\{0\} : i \in I]$. Then:

$\rho_X(A) = \rho_X(\bigcap[f_i^{-1}\{0\} : i \in I]) = \rho_X(\bigcap[\rho_X^{-1}(\rho(f_i)^{-1}\{0\}) : i \in I]) = \rho_X(\rho_X^{-1}(\bigcap[\rho(f_i)^{-1}\{0\} : i \in I])) = \bigcap[\rho(f_i)^{-1}\{0\} : i \in I]$

Finally, $\rho_X(A)$ is a closed subset of $\rho(X)$. □

Theorem 3.9. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is FH-nodec.

(ii) For any F-nowhere dense subset A of X, $d_\rho(A)$ is closed.

Proof. $(i) \Longrightarrow (ii)$

Let A be an F-nowhere dense subset of X. First let us show that $\rho_X(A)$ is a nowhere dense subset of $FH(X)$.

Suppose that $\overline{\rho_X(A)}^{\circ} \neq \varnothing$. Then there exist $\rho_X(U)$ an open subset of $FH(X)$ such that: $\varnothing \neq \rho_X(U) \subset \overline{\rho_X(A)}$. Thus $\rho_X^{-1}(\rho_X(U)) \subset \rho_X^{-1}(\overline{\rho_X(A)}) \subset \overline{\rho_X^{-1}(\rho_X(A))} = \overline{d_\rho(A)}$. So $d_\rho(U) = \rho_X^{-1}(\rho_X(U))$ is open in X that is U is an F-open subset and $d_\rho(U) \subset \overline{d_\rho(A)}$, which contradict the fact that A is F-nowhere dense. Hence $\rho_X(A)$ is a nowhere dense subset of $FH(X)$ and consequently a closed set of $FH(X)$. Therefore $d_\rho(A) = \rho_X^{-1}(\rho_X(A))$ is closed.

$(ii) \Longrightarrow (i)$

Let $\rho_X(A)$ be a nowhere dense subset of $FH(X)$, where A is a subset of X. We prove that A is F-nowhere dense in X. Suppose that there exists V an F-open subset of X, such that $d_\rho(V) \subset \overline{d_\rho(A)}$. So $\rho_X(d_\rho(V)) \subset \rho_X(\overline{d_\rho(A)}) \subset \overline{\rho_X(d_\rho(A))}$. Thus $\rho_X(V) \subset \overline{\rho_X(A)}$.

V is an F-open set of X, that is $d_\rho(V)$ an open subset of X, and thus $\rho_X(V)$ is open in $FH(X)$. Then $\overline{\rho_X(A)}^{\circ} \neq \varnothing$, which contradict the fact that $\rho_X(A)$ is nowhere dense in $FH(X)$. Hence A is F-nowhere dense in X. By (ii), $d_\rho(A)$ is closed in X and consequently $\rho_X(A)$ is closed in $FH(X)$. Therefore $FH(X)$ is nodec. □

4. Alexandroff Topology

According to Kennisson, a flow in a category **C** is a couple (X, f), where X is an object of C and $f : X \longrightarrow X$ is a morphism, called the iterator (see [9] and [10]).

Recall that the topology $\mathcal{P}(f)$ defined on a flow (X, f) of the category **Set**, is a topology such that closed sets are exactly those A which are f-invariant (i.e., $f(A) \subseteq A$) and consequently open sets are those which are f^{-1}-invariant. It is clearly seen that for any subset A of X, the topological closure $\overline{A}$ is exactly $\cup[f^n(A) : n \in \mathbb{N}]$. In particular for any point $x \in X$, $\overline{\{x\}} = \mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{N}\}$ called the orbit of x by f. One can see easily that the family $\{V_f(x) : x \in X\}$ is a basis of open sets of $\mathcal{P}(f)$, where $V_f(x) := \{y \in X : f^n(y) = x, \text{for some integer n}\}$.

Clearly, $\mathcal{P}(f)$ is an Alexandroff topology on X.

An element x of X is said to be a periodic point if $f^n(x) = x$ for some positive integer n.

A characterization of maps f such that $(X, \mathcal{P}(f))$ is nodec, is given by the following result.

Proposition 4.1. *Let (X, f) be a flow in **Set**. Then the following statements are equivalent.*

(i) $(X, \mathcal{P}(f))$ is a nodec space;

(ii) $\forall\, x \in X$ we have x is either a periodic point or $f(x)$ is a fixed point.

Proof. $(i) \Longrightarrow (ii)$. Let $x \in X$. If x is not a periodic point, then $x \notin \overline{\{f(x)\}}$. Suppose that there exists $y \in \overline{\{f(x)\}}^{\circ}$, then $V_f(y) \subseteq \overline{\{f(x)\}}^{\circ} \subseteq \overline{\{f(x)\}}$. Hence $y = f^n(f(x)) = f^{n+1}(x)$ for some integer n so that $x \in V_f(y) \subseteq \overline{\{f(x)\}}$, which contradict the fact that x is not a periodic point. Thus $\overset{\circ}{\overline{\{f(x)\}}} = \varnothing$. Since $(X, \mathcal{P}(f))$ is a nodec space, we get $\{f(x)\}$ is closed and consequently $f(x)$ is a fixed point.

$(ii) \Longrightarrow (i)$. Let A be a subset of X such that $\overset{\circ}{\overline{A}} = \varnothing$ and $x \in A$. Then we have $f^{-1}(\overline{\{x\}}) \nsubseteq \overline{\{x\}} \subset \overline{A}$, otherwise $\overline{\{x\}}$ is an open set. Let $y \in f^{-1}(\overline{\{x\}}) \setminus \overline{\{x\}}$, it is clear that y is not a periodic point. Indeed, if y is a periodic point, then $\overline{\{y\}} = \overline{\{f(y)\}}$. But $f(y) \in \overline{\{x\}}$, so we get $y \in \overline{\{x\}}$, a contradiction. Thus $f(y)$ is a fixed point. Since $\overset{\circ}{\overline{A}} = \varnothing$, then we have $\{x\}$ is not open which means that $f^{-1}(\{x\}) \setminus \{x\} \neq \varnothing$. Let $z \in f^{-1}(\{x\}) \setminus \{x\}$, it is clear that z is not a periodic point. Hence $f(z) = x$ is a fixed point and consequently $\{x\}$ is a closed subset. Therefore A is a closed set. $\square$

Example 4.2. Consider the map $f\colon \begin{array}{l} \mathbb{N} \longrightarrow \mathbb{N} \\ n \longmapsto n+1 \end{array}$ where $\mathbb{N}$ is the set of all natural numbers including 0.
Let p be a positive integer and $n \in \mathbb{N}$, then we have $f^p(n) = n + p$. Hence n is not a periodic point and $f(n)$ is not a fixed point for every $n \in \mathbb{N}$. Now, consider the topological space $(\mathbb{N}, \mathcal{P}(f))$ and set $A = 2\mathbb{N} + 1$. Since every open subset of $(\mathbb{N}, \mathcal{P}(f))$ must contain 0, then $\overset{\circ}{\overline{A}} = \varnothing$. However, A is not closed ($\overline{A} = \mathbb{N} \setminus \{0\}$).

Remark 4.3. Let (X, f) be a flow in **Set**, we equip X with the topology $\mathcal{P}(f)$. Then for every $a \in X$ we have $d_0(a)$ is closed if and only if a is a periodic point.

Indeed, suppose that $d_0(a)$ is a closed subset of $(X, \mathcal{P}(f))$, that is $d_0(a)$ is a f-invariant set. On the other hand, we know that $a \in d_0(a)$ then $f(a) \in f(d_0(a)) \subseteq d_0(a)$ and consequently $\overline{\{a\}} = \overline{\{f(a)\}}$. Hence $a \in \overline{\{f(a)\}}$ which implies that $a = f^n(f(a)) = f^{n+1}(a)$ for some n in $\mathbb{N}$. Therefore a is a periodic point.

For the converse, suppose that a is a periodic point which means that $f^n(a) = a$ for some positive integer n.
Let $x \in d_0(a)$, then $\overline{\{x\}} = \overline{\{a\}}$ and so $x \in \overline{\{a\}}$. Hence $d_0(a) \subseteq \overline{\{a\}}$.
For the reverse inclusion, let $x \in \overline{\{a\}}$ then $x = f^m(a)$ for some m in $\mathbb{N}$. One can easily see that $f^{pn}(a) = a$ for every $p \in \mathbb{N}$, thus we get $a = f^{nm}(a) = f^{nm-m+m}(a) = f^{nm-m}(f^m(a)) = f^{nm-m}(x)$. It follows that $a \in \overline{\{x\}}$ and so $\overline{\{x\}} = \overline{\{a\}}$. Therefore $x \in d_0(a)$. Finally $d_0(a) = \overline{\{a\}}$ is closed.

It is easily to see that every submaximal space (resp., $\mathbf{T}_0$-submaximal space) is a nodec space (resp., $\mathbf{T}_0$-Nodec space). The converse does not hold. as showen in the following examples.

Example 4.4. given a set $X = \{a, b, c\}$ equipped with the trivial topology. Clearly X is a nodec space which is not $\mathbf{T}_0$ and consequently not submaximal.

Example 4.5. Let L be an infinite set, $a, b \notin L$ and $X = L \cup \{a, b\}$. We equip X with the topology whose open sets are $\varnothing, X, L$, co-finite subset of L and co-finite subset of X containing $\{a, b\}$. Then $\overline{\{a\}} = \overline{\{b\}}$ and $\mathbf{T}_0(X) = L \cup \{\mu_X(a)\}$. Thus $\mathbf{T}_0(X)$ is nodec. Indeed, it is straightforward to see that the nowhere dense sets are finite subsets of $\mathbf{T}_0(X)$, which are closed. However, X is not $\mathbf{T}_0$-submaximal, since for an infinite subset of L such that $L \setminus S$ is infinite, we have $\overline{S} \setminus S$ is not closed.

Remark 4.6. A primal nodec space is not always a submaximal space as shows the following example.
Let $X = \{a, b, c\}$ and $f : X \longrightarrow X$ the map defined by:

$$\begin{cases} f(a) = b \\ f(b) = c \\ f(c) = a. \end{cases}$$

It is clear that $(X, \mathcal{P}(f))$ is nodec but not submaximal, since $f^2 \neq f$.

The following Proposition shows that for a given primal space $(X, \mathcal{P}(f))$, there is an equivalence between $\mathbf{T}_0$-submaximal and $\mathbf{T}_0$-nodec.

Proposition 4.7. *Let (X, f) be a flow in **Set**. Then the following statements are equivalent.*
(i) $(X, \mathcal{P}(f))$ is a $\mathbf{T}_0$-submaximal space;
(ii) $(X, \mathcal{P}(f))$ is a $\mathbf{T}_0$-nodec space;
(iii) $f(x)$ is a periodic point for every $x \in X$.

Proof. $(i) \Longrightarrow (ii)$. Straightforward.
$(ii) \Longrightarrow (iii)$. Let $x \in X$. Suppose that $f(x)$ is not a periodic point, then x is not a periodic point and $x \neq f(x)$. For the set $A = \{f(x)\}$, we have $\overset{\circ}{\overline{A}} = \varnothing$. Indeed, if there exists $y \in \overset{\circ}{\overline{A}} \subseteq \overline{A}$ then $y = f^n(x)$ for some positive integer n. Hence $x \in \mathcal{V}_f(y) \subseteq \overset{\circ}{\overline{A}} \subseteq \overline{A} = \overline{\{f(x)\}}$ which implies that $f(x)$ is a periodic point, a contradiction. Therefore $\overset{\circ}{\overline{A}} = \varnothing$. By Theorem 1.1 we get $d_0(A)$ is closed and finally, by Remark 4.3, we conclude that $f(x)$ is a periodic point.
$(iii) \Longrightarrow (i)$. First, remark that if x is not a periodic point then $\{x\}$ is open. In fact, if $\{x\}$ is not open, then there exists $y \in f^{-1}(\{x\}) \setminus \{x\}$ which implies that $x = f(y)$ is a periodic point.
Now, let A be a dense subset of X. Then every non periodic point of X belongs to A which means that all points of A^c are periodic points. Since $[d_0(A)]^c \subseteq A^c$, then all points of $[d_0(A)]^c$ are also periodic points.
Let $x \in [d_0(A)]^c$ and $y \in \overline{\{x\}}$. So y is a periodic point and $\overline{\{y\}} = \overline{\{x\}}$. Therefore $y \in [d_0(A)]^c$ and consequently $\overline{\{x\}} \subseteq [d_0(A)]^c$ for each $x \in [d_0(A)]^c$. Since $(X, \mathcal{P}(f))$ is an Alexandroff space, then $[d_0(A)]^c$ is a closed subset of X and $[d_0(A)]$ is open.

□

Remarks 4.8. Let (X, f) be a flow in **Set**.

(1) $(X, \mathcal{P}(f))$ is FH-nodec and ρ-nodec, since $\mathcal{P}(f)$ is an Alexandroff topology and for every $a \in X$, $d_\rho(a)$ is a closed subset of X.
(2) Let A be a subset of X and denote $B = X \setminus d_\rho(A)$. Then $d_\rho(B) = B$. Indeed suppose that there exists $x \in d_\rho(B) \cap d_\rho(A)$. Thus there exist $a \in A$ and $b \in B$ such that $f(x) = f(a) = f(b)$, for all $f \in \mathbf{C}(X)$. Therefore $b \in d_\rho(A)$ which contradict the fact that $B = X \setminus d_\rho(A)$. On the other hand, $d_\rho(B) == \cup[d_{\boldsymbol{\rho}}(b) : b \in B]$ is a closed set of $(X, \mathcal{P}(f))$, since the latter is Alexandroff. Hence, $d_\rho(A)$ is an open subset of X which implies that $(X, \mathcal{P}(f))$ is FH-submaximal and ρ-submaximal.

As example of primal space which is $\mathbf{T}_0$-Nodec but not nodec, we give the following:

Example 4.9. Consider the map $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by:

$$\begin{cases} f(0) = 0 \\ f(n) = n + 1 \text{ if } n \notin 3\mathbb{N} \\ f(n) = n - 1 \text{ if } n \in 3\mathbb{N} \setminus \{0\}. \end{cases}$$

Let $n \in \mathbb{N} \setminus \{0\}$.

$-$ If $n \equiv 0[3]$ then $n - 1 \notin 3\mathbb{N}$. Hence

$$f^2(n) = f(f(n)) = f(n-1) = n - 1 + 1 = n,$$

that is n is a periodic point.

$-$ If $n \equiv 2[3]$ then $n + 1 \equiv 0[3]$. Hence

$$f^2(n) = f(f(n)) = f(n+1) = n + 1 - 1 = n,$$

that is n is a periodic point.

$-$ If $n \equiv 1[3]$ then $f(n) = n + 1 \equiv 2[3]$ that is $f(n)$ is a periodic point.

Therefore $f(n)$ is a periodic point for every $n \in \mathbb{N}$.

Now, consider the topological space $(\mathbb{N}, \mathcal{P}(f))$. We can easily check that for each $n \in \mathbb{N}$ we have $f(n) \equiv 0[3]$ or $f(n) \equiv 2[3]$. Then $\{n\}$ is open if and only if $n = 0$ or $n \equiv 1[3]$. Let $A \subseteq \mathbb{N}$ such that $\overset{\circ}{\overline{A}} = \varnothing$, then a is a periodic point for each $a \in A$. Therefore $d_0(A)$ is a closed subset. Therefore, $(\mathbb{N}, \mathcal{P}(f))$ is a $\mathbf{T}_0$-Nodec space which is not a nodec space.

Acknowledgements. *The authors thank the referee for his useful comments and suggestions to improve the presentation and the mathematical content of this paper.*

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