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Merotopies associated with quasi-uniformities

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Abstract. To an arbitrary quasi-uniformity on the set X, a merotopy on X is assigned. There are results concerning the question whether this merotopy is compatible with the topology induced by the quasi-uniformity and whether the closure operation induced by the merotopy, admits a compatible uniformity. More precise results are obtained in the case of transitive quasi-uniformities.

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1. Introduction

The purpose of the present paper is to establish a relation between two well-known kinds of topological structures, namely quasi-uniformities and merotopies.

Notation and terminology concerning *quasi-uniformities* will be used according to [4]. The concept of a *merotopy* has been introduced in [8], but we shall use according to [3] a more advantageous description of them due to [7]. Thus a merotopy $\mathfrak{C}$ on a set X will mean a non-empty collection of covers of X (we denote by $\Gamma(X)$ the collection of all covers of X) with the properties:

$$\text{If } c \in \mathfrak{C},\ \mathfrak{c}' \in \Gamma(X) \text{ and } c \text{ refines } \mathfrak{c}' \text{ then } \mathfrak{c}' \in \mathfrak{C}, \tag{1.1}$$

$$\mathfrak{c}_1, \mathfrak{c}_2 \in \mathfrak{C} \text{ implies } \mathfrak{c}_1(\cap)\mathfrak{c}_2 \in \mathfrak{C} \tag{1.2}$$

where we say that $\mathfrak{c}$ *refines* $\mathfrak{c}'$ (in symbol $\mathfrak{c} < \mathfrak{c}'$) iff $C \in \mathfrak{c}$ implies the existence of $C' \in \mathfrak{c}'$ satisfying $C \subset C'$, and

$$\mathfrak{c}_1(\cap)\mathfrak{c}_2 = \{C_1 \cap C_2 : C_i \in \mathfrak{c}_i\};$$

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$(\cap)$ is obviously an associative operation. Equivalently, (1.2) may be replaced by

(1.3) $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathfrak{C}$ implies the existence of $\mathfrak{c} \in \mathfrak{C}$ satisfying $\mathfrak{c} < \mathfrak{c}_i$ $(i = 1, 2)$.

The topological category **Qunif** is composed of the objects of *quasi-uniform spaces* $(X,\mathcal{U})$ where $\mathcal{U}$ is a *quasi-uniformity* on X, and of the morphisms of *quasi-uniformly continuous* maps [4]. The category **Mer** contains the objects of *merotopic spaces* $(X, \mathfrak{C})$ where $\mathfrak{C}$ is a merotopy on X and of the morphisms of *merotopically continuous* maps, where $f : X \to X'$ is said to be merotopically continuous or $(\mathfrak{C}, \mathfrak{C}')$-continuous, $\mathfrak{C}$ and $\mathfrak{C}'$ being merotopies on X and X' respectively, iff $\mathfrak{c}' \in \mathfrak{C}'$ implies $f^{-1}(\mathfrak{c}') \in \mathfrak{C}$ (of course, $f^{-1}(\mathfrak{c}') = \{f^{-1}(C') : C' \in \mathfrak{c}'\}$).

We know ([4]) that each quasi-uniformity $\mathcal{U}$ on X induces a topology $\tau(\mathcal{U})$ on X for which the neighbourhood filter of $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$. Similarly, each merotopy $\mathfrak{C}$ on X induces a *closure operation* on X (i.e. a map $c : \exp X \to \exp X$ such that $c(\varnothing) = \varnothing$, $A \subset c(A)$, $c(A \cup B) = c(A) \cup c(B)$ where $\exp X$ is the power set of X) and $c = c(\mathfrak{C})$ is defined by

$$x \in c(A) \Leftrightarrow A \in \sec \mathfrak{v}_c(x)$$

(for $\mathfrak{b} \subset \Sigma(X)$, where $\Sigma(X)$ is the collection of all non-empty subsets of the power set $\exp X$, we write

$$A \in \sec \mathfrak{b} \Leftrightarrow A \subset X A \cap B \neq \varnothing \text{ for each } B \in \mathfrak{b})$$

and the *c-neighborhood filter* $\mathfrak{v}_c(x)$ of $x \in X$ is generated by the filter base $\{\mathrm{st}(x, \mathfrak{c}) : \mathfrak{c} \in \mathfrak{C}\}$. Also each topology τ on X may be considered as a closure $c = c_\tau = \mathrm{cl}_\tau$ special in the sense that $c(c(A)) = c(A)$ for every $A \subset X$.

2. Merotopies associated with quasi-uniformities

Let U be an *entourage* [4] on X. Define $\mathfrak{c}_U = \{U(x) : x \in X\}$. Then $\mathfrak{c}_U$ is a cover on X and, both U and U' being entourages on X with $U \subset U'$, clearly $U(x) \subset U'(x)$ for $x \in X$ so that $\mathfrak{c}_U < \mathfrak{c}_{U'}$. Therefore, if $\mathcal{U}$ is a quasi-uniformity on X, then $\mathfrak{B} = \{\mathfrak{c}_U : U \in \mathcal{U}\}$ is a base [3] for a merotopy $\mathfrak{C}_\mathcal{U}$. More generally, if $\mathcal{B}$ is a base for $\mathcal{U}$ and we set $\mathfrak{B} = \{\mathfrak{c}_U : U \in \mathcal{B}\}$ then $\mathfrak{B}$ is still a base for $\mathfrak{C}_\mathcal{U}$. Moreover, if $(X', \mathcal{U}')$ is another quasi-uniform space and $f : X \to X'$ is quasi-uniformly continuous then f is $(\mathfrak{C}_\mathcal{U}, \mathfrak{C}_{\mathcal{U}'})$-continuous as well: if $U \in \mathcal{U}$, $U' \in \mathcal{U}'$ and $(x, y) \in U$ implies $(f(x), f(y)) \in U'$ then $f(U(x)) \subset U'(f(x))$ so that $\mathfrak{c}_U < f^{-1}(\mathfrak{c}_{U'})$.

Hence we can state:

Theorem 2.1. *If we associate with each quasi-uniformity $\mathcal{U}$ on the set X the merotopy $\mathfrak{C}_\mathcal{U}$ with base*

(2.4) $$\mathfrak{B} = \{\mathfrak{c}_U : U \in \mathcal{U}\}$$

where

(2.5) $$\mathfrak{c}_U = \{U(x) : x \in X\},$$

*then $\Phi((X,\mathcal{U})) = (X, \mathfrak{C}_\mathcal{U})$, $\Phi(f) = f$ for $f : X \to X'$ define a (covariant) functor $\Phi :$ **Qunif** $\to$ **Mer**.*

It is an interesting question which merotopies can be represented in the form $\mathfrak{C}_{\mathcal{U}}$ with some quasi-uniformity $\mathcal{U}$, or which covers have the form $\mathfrak{c}_U$ for some entourage U. The collection of all covers of the form $\mathfrak{c}_U$ clearly does not coincide with $\Gamma(X)$: if $\mathfrak{c} = \mathfrak{c}_U$ then there is a surjection $f : X \to \mathfrak{c}$ such that $x \in f(x)$ for each $x \in X$, consequently there is a bijection $g : X_0 \to \mathfrak{c}$ for some $X_0 \subset X$ such that $x \in g(x)$ for $x \in X_0$, or equivalently there is an injection $g^{-1} = h : \mathfrak{c} \to X$ such that $h(C) \in C$ for $C \in \mathfrak{c}$, i.e., in the terminology of [8], there is a *transversal* for $\mathfrak{c}$. Now clearly, if $\mathfrak{t} \in \Sigma(X)$ and h is a transversal for $\mathfrak{t}$, then necessarily the following condition must hold:

$$\mathfrak{t}' \subset \mathfrak{t} \text{ implies } \mathfrak{t}'| \leqq | \cup \mathfrak{t}'| \tag{2.6}$$

because $h(\mathfrak{t}') \subset \cup\mathfrak{t}'$. Consequently, if $\mathfrak{c} = \mathfrak{c}_U$ for some entourage U then (2.6) has to be fulfilled for $\mathfrak{t} = \mathfrak{c}$.

According to [6], the condition (2.6) is sufficient for the existence of a transversal for $\mathfrak{t}$ in the case when $\mathfrak{t}$ and each $T \in \mathfrak{t}$ are finite, or even, according to [5], in the case when $\mathfrak{t}$ is infinite but each $T \in \mathfrak{t}$ is finite. However, probably there are no further results on the sufficiency of (2.6) in the general case (if some $T \in \mathfrak{t}$ can be infinite then (2.6) certainly does not guarantee the existence of a transversal, cf. [9]). So we can formulate:

Problem 2.2. *Look for necessary and/or sufficient conditions for a cover $\mathfrak{c}$ of X for the existence of an entourage U satisfying $\mathfrak{c} = \mathfrak{c}_U$.*

Problem 2.3. *Look for necessary and/or sufficient conditions for a merotopy $\mathfrak{C}$ on X for the existence of a quasi-uniformity $\mathcal{U}$ satisfying $\mathfrak{C} = \mathfrak{C}_{\mathcal{U}}$.*

If $\mathcal{U}$ is a quasi-uniformity on X and we look for the closure $c = c(\mathfrak{C}_{\mathcal{U}})$ then it is easy to see:

Lemma 2.4. *$c = c(\mathfrak{C}_{\mathcal{U}})$ is coarser than $c_{\tau(\mathcal{U})}$, i.e.*

$$c_{\tau(\mathcal{U})}(A) \subset c(A) \quad (A \subset X).$$

Proof. Clearly $\mathfrak{v}_c(x)$ is generated by the filter base composed of all sets $\mathrm{st}(x, \mathfrak{c}_U)$ where $U \in \mathcal{U}$, and

$$\mathrm{st}(x, \mathfrak{c}_U) = \bigcup\{U(y) : y \in U(x)\} = U(U^{-1}(x)). \tag{2.7}$$

Obviously $U(x) \subset U(U^{-1}(x))$. □

In general, $c \neq c_{\tau(\mathcal{U})}$; e.g. if $X = \mathbb{R}$ and $\mathcal{U}$ is the *Sorgenfrey quasi-uniformity* generated by the base $\{U_\varepsilon : \varepsilon > 0\}$ where $U_\varepsilon(x) = [x, x+\varepsilon)$ then $U_\varepsilon(U_\varepsilon^{-1}(x)) = (x-\varepsilon, x+\varepsilon)$ so that $c(\mathfrak{C}_{\mathcal{U}})$ is the Euclidean topology on $\mathbb{R}$. It is even possible that the closure $c(\mathfrak{C}_{\mathcal{U}})$ it not a topology:

Example 2.5. Let $X = \{a, b, c\}$ and U be an entourage on X such that $U(a) = \{a\}$, $U(b) = \{a, b\}$, $U(c) = \{a, c\}$. Clearly $U^2 = U$ so that $\{U\}$ is a base for a quasi-uniformity $\mathcal{U}$ on X and $\{\mathfrak{c}_U\}$ is a base for the merotopy $\mathfrak{C}_{\mathcal{U}}$. For $c = c(\mathfrak{C}_{\mathcal{U}})$, we have $c(\{b\}) = \{a, b\}$ and $c(\{a, b\}) = X$.

However, it is not difficult to characterize those quasi-uniformities $\mathcal{U}$ for which $c(\mathfrak{C}_{\mathcal{U}}) = c_{\tau(\mathcal{U})}$. Recall ([4]) that a quasi-uniformity $\mathcal{U}$ on X is said to be *point-symmetric* iff, for each $x \in X$ and $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $V^{-1}(x) \subset U(x)$ or, equivalently, iff $\tau(\mathcal{U})$ is coarser than $\tau(\mathcal{U}^{-1})$.

Theorem 2.6. *The equality $c(\mathfrak{C}_{\mathcal{U}}) = c_{\tau(\mathcal{U})}$ holds iff $\mathcal{U}$ is point-symmetric.*

Proof. By Lemma 2.4, we need, for $x \in X$ and $U \in \mathcal{U}$, the existence of $W \in \mathcal{U}$ such that $W(W^{-1}(x)) \subset U(x)$. Now this condition clearly implies the point-symmetry of $\mathcal{U}$. On the other hand, if, for $U \in \mathcal{U}$, we choose $U_0 \in \mathcal{U}$ satisfying $U_0^2 \subset U$, then, given $x \in X$, $V \in \mathcal{U}$ such that $V^{-1}(x) \subset U_0(x)$, finally we set $W = V \cap U_0 \in \mathcal{U}$, obviously $W(W^{-1})(x) \subset U_0(V^{-1}(x)) \subset U_0^2(x) \subset U(x)$. $\square$

It is easy to find examples of point-symmetric quasi-uniformities. In fact, recall (cf. [1]) that a topology c (i.e. a closure $c = c_\tau$ for a topology τ) is said to be S_1 iff $x \in G$ implies $c(\{x\}) \subset G$ whenever G is c-open. Also recall ([4]) that the *Pervin quasi-uniformity* $\mathcal{P}$ associated with the topology c (and inducing c) is defined by the quasi-uniform subbase $\{U_G : G \text{ is } c\text{-open}\}$ where $U_G(x) = G$ if $x \in G$ and $U_G(x) = X$ if $x \in X - G$. More generally, if $\mathfrak{B}$ is a base for the topology c then the entourages U_B $(B \in \mathfrak{B})$ constitute a subbase for a transitive quasi-uniformity $\mathcal{U}(\mathfrak{B})$ compatible with c (see e.g. [2]). If the topology c is S_1, we can also consider the entourages $U_{x,B} = U_B \cap U_{X-c(\{x\})}$ where $x \in B \in \mathfrak{B}$ to obtain a subbase for a transitive quasi-uniformity $\mathcal{U}_1(\mathfrak{B})$ finer than $\mathcal{U}(\mathfrak{B})$ and coarser than $\mathcal{P}$, hence still compatible with c.

Now we can state:

Proposition 2.7. *If c is an S_1 topology admitting a base $\mathfrak{B}$ then every quasi-uniformity $\mathcal{U}$ finer than $\mathcal{U}_1(\mathfrak{B})$ and compatible with c is point-symmetric.*

Proof. Given $x \in X$ and $U \in \mathcal{U}$, there is a $B \in \mathfrak{B}$ such that $x \in B \subset U(x)$. By S_1, we have $c(\{x\}) \subset B$. Let H denote the c-open set $H = X - c(\{x\})$. Then, for $V = U_B \cap U_H \in \mathcal{U}_1(\mathfrak{B}) \subset \mathcal{U}$, we have $V^{-1}(x) = c(\{x\}) \subset B \subset U(x)$. $\square$

The condition for a quasi-uniformity $\mathcal{U}$ of being point-symmetric has another important consequence for the merotopy $\mathfrak{C}_{\mathcal{U}}$. Recall ([3]) that a merotopy $\mathfrak{C}$ is said to be *Lodato* iff $\mathfrak{c} \in \mathfrak{C}$ implies $\operatorname{int} \mathfrak{c} \in \mathfrak{C}$ where $\operatorname{int} \mathfrak{c} = \{\operatorname{int} C : C \in \mathfrak{c}\}$ and $\operatorname{int} C = X - c(X - C)$, $c = c(\mathfrak{C})$. Now we can state:

Theorem 2.8. *If $\mathcal{U}$ is point-symmetric then $\mathfrak{C}_{\mathcal{U}}$ is a Lodato merotopy.*

Proof. For $\mathfrak{c} \in \mathfrak{C}$, choose $U \in \mathcal{U}$ such that $\mathfrak{c}_U < \mathfrak{c}$ and $U_0 \in \mathcal{U}$ such that $U_0^2 \subset U$. Then, by $U_0(x) \subset \operatorname{int} U(x)$, $\mathfrak{c}_{U_0} < \operatorname{int} \mathfrak{c}_U < \operatorname{int} \mathfrak{c}$ and $\mathfrak{c}_{U_0} \in \mathfrak{C}$ implies $\operatorname{int} \mathfrak{c} \in \mathfrak{C}$. $\square$

3. Semi-symmetric quasi-uniformities

Recall ([3]) that a *semi-uniformity* $\mathcal{U}$ on a set X is a filter on $X \times X$ having a base composed of symmetric entourages; it induces a closure $c(\mathcal{U})$ such that, if $c = c(\mathcal{U})$ and $x \in X$, then $\mathfrak{v}_c(x) = \{U(x) : U \in \mathcal{U}\}$ is the neighborhood filter of x for c.

Now if U is an arbitrary entourage on X then clearly UU^{-1} (we write AB for $A \circ B$ if $A, B \subset X \times X$) is a symmetric entourage on X so that, whenever $\mathcal{U}$ is a quasi-uniformity on X, $\{UU^{-1} : U \in \mathcal{U}\}$ is a base for a semi-uniformity $\mathcal{U}^*$; by Lemma 2.4

$$c(\mathcal{U}^*) = c(\mathfrak{C}_{\mathcal{U}}). \tag{3.8}$$

We look for those quasi-uniformities $\mathcal{U}$ which admit a corresponding semi-uniformity $\mathcal{U}^*$ that is a uniformity. For this purpose, let us say that $\mathcal{U}$ is *semi-symmetric* iff, given $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ satisfying $V^{-1}V \subset UU^{-1}$; the pair (U, V) is said to be *semi-symmetric* in this case and, in particular, the entourage U is said to be *semi-symmetric* iff (U, U) is semi-symmetric. Now it is easy to prove:

Theorem 3.1. *For a quasi-uniformity $\mathcal{U}$, the semi-uniformity $\mathcal{U}^*$ is a uniformity iff $\mathcal{U}$ is semi-symmetric.*

Proof. If $\mathcal{U}^*$ is a uniformity then, for $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $VV^{-1}VV^{-1} \subset UU^{-1}$ whence clearly $V^{-1}V \subset UU^{-1}$. Conversely, if the condition in the statement is fulfilled, let $U \in \mathcal{U}$ and $U_0 \in \mathcal{U}$ be chosen such that $U_0^2 \subset U$, then let $V \in \mathcal{U}$ satisfy $V^{-1}V \subset U_0U_0^{-1}$. Now we can suppose $V \subset U_0$ as V can be replaced by $V \cap U_0$. Then $V(V^{-1}V)V^{-1} \subset U_0U_0U_0^{-1}U_0^{-1} \subset UU^{-1}$. □

Of course, each uniformity is an example of a semi-symmetric quasi-uniformity. But it is easy to find non-symmetric examples, too. E.g. if $\mathcal{U}$ is the Sorgenfrey quasi-uniformity on $X = \mathbb{R}$ whose base is composed of the entourages $U_\varepsilon = \{(x, y) : x \leqq y < x + \varepsilon\}$ ($\varepsilon > 0$) then $U_\varepsilon U_\varepsilon^{-1} = U_\varepsilon^{-1}U_\varepsilon = \{(x, y) : |x - y| < \varepsilon\}$. Similarly if $\mathcal{U}$ is the *Michael quasi-uniformity* on $X = \mathbb{R}$, i.e. the base is composed of $\{U_\varepsilon : \varepsilon > 0\}$ where $U_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ if $x \in \mathbb{Q}$ and $U_\varepsilon(x) = \{x\}$ if $x \in \mathbb{R} - \mathbb{Q}$, then $U_\varepsilon U_\varepsilon^{-1}(x) = (x - 2\varepsilon, x + 2\varepsilon)$, while clearly $U_\varepsilon(x) \subset (x - \varepsilon, x + \varepsilon)$ and $U_\varepsilon^{-1}(x) \subset (x - \varepsilon, x + \varepsilon)$ so that $U_\varepsilon^{-1}(U_\varepsilon(x)) \subset U_\varepsilon(U_\varepsilon^{-1}(x))$. On the other hand, e.g. Example 2.5 is not semi-symmetric: $U(U^{-1}(b)) = \{a, b\}$ and $U^{-1}(U(b)) = X$.

Corollary 3.2. *If a quasi-uniformity $\mathcal{U}$ is both semi-symmetric and point-symmetric then the topology $\tau(\mathcal{U})$ is completely regular.*

Proof. By Theorem 2.6 $c_{\tau(\mathcal{U})} = c(\mathfrak{C}_{\mathcal{U}})$, by (3.8) and Theorem 3.1 the latter is a topology induced by a uniformity. □

It is easy to see that point-symmetry and semi-symmetry are properties of a quasi-uniformity independent of each other. In fact, the Sorgenfrey quasi-uniformity is semi-symmetric without being point-symmetric, while if c is an S_1 topology that is not completely regular then its Pervin quasi-uniformity is point-symmetric by Proposition 2.7 but not semi-symmetric by Corollary 3.2.

Semi-symmetric quasi-uniformities have rather good invariance properties. Recall that, if $f : X \to Y$, then the inverse image $f^{-1}(\mathcal{U})$ of a quasi-uniformity $\mathcal{U}$ on Y is generated by the entourages $\hat{f}^{-1}(U)$ for $U \in \mathcal{U}$ where $\hat{f}(x, y) = (f(x), f(y))$.

Lemma 3.3. *If $f : X \to Y$ is surjective and $\mathcal{U}$ is a semi-symmetric quasi-uniformity on Y then $f^{-1}(\mathcal{U})$ is semi-symmetric.*

Proof. If $U, V \in \mathcal{U}$ and $V^{-1}V \subset UU^{-1}$, further $(f(x), f(y)) \in V$, $(f(y), f(z)) \in V^{-1}$ then $(f(x), f(z)) \in V^{-1}V \subset UU^{-1}$ so that there is some $w \in Y$ satisfying $(f(x), w) \in U^{-1}$, $(w, f(z)) \in U$, and choosing $u \in X$ such that $w = f(u)$, we get $(f(x), f(u)) \in U^{-1}$, $(f(u), f(z)) \in U$, i.e. $(x, u) \in \hat{f}^{-1}(U^{-1})$, $(u, z) \in \hat{f}^{-1}(U)$. □

The condition of surjectivity cannot be dropped as semi-symmetry is not hereditary:

Example 3.4. Let $X = \{a, b, c, d\}$, $U(a) = \{a\}$, $U(b) = \{a, b\}$, $U(c) = \{a, c\}$, $U(d) = X$. Then $U^2 = U$, so that $\{U\}$ is a base for a quasi-uniformity $\mathcal{U}$ on X. The semi-symmetry of $\mathcal{U}$ is easily checked using the formulas for $U(x)$ and those $U^{-1}(a) = X$, $U^{-1}(b) = \{b, d\}$, $U^{-1}(c) = \{c, d\}$, $U^{-1}(d) = \{d\}$. Define $X_0 = \{a, b, c\}$, $U_0 = U \cap (X_0 \times X_0)$. Then $\mathcal{U}|X_0$ coincides with the quasi-uniformity in Example 2.5 which fails to be semi-symmetric.

Lemma 3.5. *If $\mathcal{U}_i$ is a semi-symmetric quasi-uniformity on X_i $(i \in I)$ and $X = \prod\{X_i : i \in I\}$ then $\mathcal{U} = \prod \mathcal{U}_i$ is semi-symmetric on X.*

Proof. Let $U \in \mathcal{U}$ be given. We can suppose $U = \prod U_i$ where $U_i \in \mathcal{U}_i$ for $i \in F$ and a finite $F \subset I$, $U_i = X_i \times X_i$ otherwise. Choose $V_i \in \mathcal{U}_i$ such that $V_i^{-1}V_i \subset U_iU_i^{-1}$ for $i \in F$ and $V_i = X_i \times X_i$ otherwise. For $V = \prod V_i$, we have $V^{-1}V \subset UU^{-1}$. □

Some partial results concerning heredity may be obtained by introducing the following definition: let us say that $\mathcal{U}$ is *strongly semi-symmetric* iff, given $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $V^{-1}V \subset U \cup U^{-1}$; in this case (U, V) is *strongly semi-symmetric* and, in particular, $U \in \mathcal{U}$ is *strongly semi-symmetric* iff so is (U, U).

Lemma 3.6. *A strongly semi-symmetric quasi-uniformity is semi-symmetric as well.*

Proof. If $V^{-1}V \subset U \cup U^{-1}$ and $(x, y) \in V^{-1}V$ then either $(x, y) \in U$ or $(x, y) \in U^{-1}$. In the first case, let $(x, x) \in U^{-1}$, in the second one let $(y, y) \in U$. In both cases, $(x, y) \in UU^{-1}$. □

E.g. the Sorgenfrey quasi-uniformity is strongly semi-symmetric because $\{(x, y) : |x - y| < \varepsilon\} = \{(x, y) : x \leqq y < x + \varepsilon\} \cup \{(x, y) : x - \varepsilon < y \leqq x\}$. The same holds for the Michael quasi-uniformity: $U_\varepsilon(x) \cup U_\varepsilon^{-1}(x) = (x - \varepsilon, x + \varepsilon)$ if $x \in \mathbb{Q}$ and $= \{x\} \cup ((x - \varepsilon, x + \varepsilon) \cap \mathbb{Q})$ if $x \in \mathbb{R} - \mathbb{Q}$, while $U_\delta^{-1}(U_\delta(x)) \subset (x - 2\delta, x + 2\delta)$ if $x \in \mathbb{Q}$ and $= \{x\} \cup ((x - \delta, x + \delta) \cap \mathbb{Q})$ if $x \in \mathbb{R} - \mathbb{Q}$. In Example 3.4, we find a semi-symmetric but not strongly semi-symmetric quasi-uniformity; in fact strong semi-symmetry is hereditary:

Lemma 3.7. *If $f : X \to Y$ and $\mathcal{U}$ is strongly semi-symmetric on Y then $f^{-1}(\mathcal{U})$ is strongly semi-symmetric on X.*

Proof. Assume $U, V \in \mathcal{U}$ and $V^{-1}V \subset U \cup U^{-1}$. If $(x,y) \in \hat{f}^{-1}(V^{-1})\ \hat{f}^{-1}(V)$ then $(f(x), f(y)) \in V^{-1}V \subset U \cup U^{-1}$, so $(x,y) \in \hat{f}^{-1}(U) \cup \hat{f}^{-1}(U^{-1})$. □

However, the analogue of Lemma 3.5 is not valid for strongly semi-symmetric quasi-uniformities:

Example 3.8. Let $X = \mathbb{R}^2$, $\mathcal{U}$ be the Sorgenfrey quasi-uniformity, and consider $\mathcal{U} \times \mathcal{U}$. We know that both factors are strongly semi-symmetric. For $U = U_1 \times U_1$, no $V_\delta = U_\delta \times U_\delta$ is suitable: $(0, \frac{3}{4}\delta) \in U_\delta$, $(\frac{3}{4}\delta, \frac{1}{2}\delta) \in U_\delta^{-1}$, $(0, \frac{1}{4}\delta) \in U_\delta$, $(\frac{1}{4}\delta, -\frac{1}{2}\delta) \in U_\delta^{-1}$, so $((0,0), (\frac{1}{2}\delta, -\frac{1}{2}\delta)) \in V_\delta^{-1}V_\delta$ but $((0,0), (\frac{1}{2}\delta, -\frac{1}{2}\delta)) \notin U \cup U^{-1} = (U_1 \times U_1) \cup (U_1^{-1} \times U_1^{-1})$ because $(0, \frac{1}{2}\delta) \notin U_1^{-1}$ and $(0, -\frac{1}{2}\delta) \notin U_1$.

4. The transitive case

Problems 2.2 and 2.3 have partial solution in the case of *transitive* entourages and quasi-uniformities, respectively. In order to see this, consider a system $\mathfrak{t} \in \Sigma(X)$ and define an operation $\mu : \Sigma(X) \to \Sigma(X)$ by

$$\mu(\mathfrak{t}) = \{T(x) : x \in X\} \tag{4.9}$$

where

$$T(x) = \bigcap \{T \in \mathfrak{t} : x \in T\} \tag{4.10}$$

and we define $\cap\varnothing = X$. Clearly $x \in T(x)$, hence $\mu(\mathfrak{t})$ is always a cover of X so that $\mu : \Sigma(X) \to \Gamma(X)$.

Lemma 4.1. *The operation μ is idempotent.*

Proof. Let $\mathfrak{t} \in \Sigma(X)$ and $\mathfrak{t}' = \mu(\mathfrak{t})$. For $x, y \in X$ and $x \in T(y)$ we have $\{T \in \mathfrak{t} : y \in T\} \subset \{T \in \mathfrak{t} : x \in T\}$, consequently $T(x) \subset T(y)$, so that $\bigcap\{T' \in \mathfrak{t}' : x \in T'\} = \bigcap\{T(y) \in \mathfrak{t}' : x \in T(y)\} \supset T(x)$ while obviously $T(x) \in \mathfrak{t}'$, $x \in T(x)$ imply $\bigcap\{T' \in \mathfrak{t}' : x \in T'\} \subset T(x)$. By this, $\bigcap\{T' \in \mathfrak{t}' : x \in T'\} = T(x)$ and $\mu(\mathfrak{t}') = \mu(\mu(\mathfrak{t})) = \mu(\mathfrak{t})$. □

Let us say that a system $\mathfrak{t} \in \Sigma(X)$ is *point-true* iff $\mu(\mathfrak{t}) = \mathfrak{t}$; hence a point-true system is always a cover of X. In other words,

Lemma 4.2. *A system $\mathfrak{t}$ is point-true iff a) $\bigcap\{T \in \mathfrak{t} : x \in T\} \in \mathfrak{t}$ if $x \in X$ and b) if $T \in \mathfrak{t}$, there is $x \in T$ such that $x \in T' \in \mathfrak{t}$ implies $T \subset T'$.*

Now let U be a transitive (i.e. such that $U^2 = U$) entourage on X. As $x \in U(y)$ implies $U(x) \subset U(y)$ (because $(x,z) \in U$ and $(y,x) \in U$ imply $(y,z) \in U$), we have $U(x) = \bigcap\{U(y) : x \in U(y)\}$, so that:

Lemma 4.3. *If U is a transitive entourage on X then the cover $\mathfrak{c}_U$ is point-true.*

Conversely:

Lemma 4.4. *If $\mathfrak{c}$ is a point-true cover of X then there is a transitive entourage U on X such that $\mathfrak{c} = \mathfrak{c}_U$.*

Proof. Define $(x,y) \in U \subset X \times X$ iff $x \in C \in \mathfrak{c}$ implies $y \in C$. Then $(x,x) \in U$ for $x \in X$ and $(x,y) \in U$, $(y,z) \in U$ imply $(x,z) \in U$ so that U is a transitive entourage on X. By definition, $U(x) = \bigcap\{C \in \mathfrak{c} : x \in C\} \in \mathfrak{c}$ by Lemma 4.2 a), and, if $C \in \mathfrak{c}$, there is by Lemma 4.2 b) an $x \in C$ such that $C = U(x)$. Consequently $\mathfrak{c} = \{U(x) : x \in X\}$. □

Lemma 4.5. *The transitive entourage U in the above lemma is uniquely determined by $\mathfrak{c}$.*

Proof. Let U_1 and U_2 be transitive entourages on X such that $\mathfrak{c}_{U_1} = \mathfrak{c}_{U_2}$. Given $x \in X$, there is $y \in X$ satisfying $U_1(x) = U_2(y)$. Then $x \in U_1(x)$ implies $x \in U_2(y)$, hence $U_2(x) \subset U_2(y) = U_1(x)$ and $U_2(x) \subset U_1(x)$. Therefore $U_2 \subset U_1$. Similarly $U_1 \subset U_2$. □

Theorem 4.6. *There is a bijection from the set of all transitive entourages on X to the set of all point-true covers of X given by the formulas*

$$U \mapsto \mathfrak{c}_U, \tag{4.11}$$

$$\mathfrak{c} \mapsto U_{\mathfrak{c}}, U_{\mathfrak{c}}(x) = \bigcap\{C \in \mathfrak{c} : x \in C\}(x \in X). \tag{4.12}$$

Concerning the behaviour of transitive quasi-uniformities, let us first remark:

Lemma 4.7. *Let U_i be transitive entourages on X for $i = 1, ..., n$ and $U = \bigcap_1^n U_i$. Then $\mathfrak{c}_U = \mu((\bigcap)_1^n \mathfrak{c}_{U_i})$.*

Proof. Let us denote $\mathfrak{c}_{U_i} = \mathfrak{c}_i$, $\mathfrak{c}_U = \mathfrak{c}$. Then, for $x \in X$, we have by (4.12), for the element of $\mathfrak{c}$ corresponding to x, $U(x) = \bigcap_1^n U_i(x) = \bigcap_{i=1}^n \bigcap\{C_i \in \mathfrak{c}_i : x \in C_i\} = \bigcap\{C_i \in \mathfrak{c}_i : x \in C_i, i = 1, ..., n\} = \bigcap\{C \in (\bigcap)_1^n \mathfrak{c}_i : x \in C\}$; the latter $\bigcap$ is the element of $\mu((\bigcap)_1^n \mathfrak{c}_i)$ corresponding to x. □

Observe that μ cannot be omitted because $\mathfrak{c}_1(\cap)\mathfrak{c}_2$ may fail to be point-true for point-true covers $\mathfrak{c}_i$ $(i = 1, 2)$.

Example 4.8. Let $X = \mathbb{R}$, $\mathfrak{c}_1 = \{(2n, 2n+2) : n \in \mathbb{Z}\} \cup \{(2n-2, 2n+2) : n \in \mathbb{Z}\}$ and $\mathfrak{c}_2 = \{(2n-1, 2n+1) : n \in \mathbb{Z}\} \cup \{(2n-1, 2n+3) : n \in \mathbb{Z}\}$. It is easy to check using Lemma 4.2 that both $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are point-true covers. Now $\mathfrak{c}_1(\cap)\mathfrak{c}_2 = \{(n, n+1) : n \in \mathbb{Z}\} \cup \{(n, n+2) : n \in \mathbb{Z}\} \cup \{(n, n+3) : n \in \mathbb{Z}\} \cup \{\varnothing\}$ is not point-true since neither $(n, n+3)$ nor $\{\varnothing\}$ does fulfil Lemma 4.2 b).

Now we can prove:

Theorem 4.9. *If $\mathcal{U}$ is a transitive quasi-uniformity then the merotopy $\mathfrak{C} = \mathfrak{C}_{\mathcal{U}}$ fulfils*

$$\mathfrak{C} \text{ has a base } \mathfrak{B} \text{ composed of point-true covers} \tag{4.13}$$

such that

$$\text{if } \mathfrak{c}_i \in \mathfrak{B} \text{ for } i = 1, \ldots, n \text{ then } \mu((\bigcap)_1^n \mathfrak{c}_i) \in \mathfrak{B}. \tag{4.14}$$

Conversely if $\mathfrak{C}$ is a merotopy satisfying (4.13) *and* (4.14) *then there exists a transitive quasi-uniformity $\mathcal{U}$ such that $\mathfrak{C} = \mathfrak{C}_{\mathcal{U}}$.*

Proof. (4.13) is obvious if $\mathfrak{B} = \{\mathfrak{c}_U : U \in \mathcal{U} \text{ is transitive}\}$. If $\mathfrak{c}_i \in \mathfrak{B}$ $(i = 1, ..., n)$ then there are transitive entourages $U_i \in \mathcal{U}$ such that $\mathfrak{c}_i = \mathfrak{c}_{U_i}$. By Lemma 4.7, $\mu((\bigcap)_1^n \mathfrak{c}_{U_i}) = \mathfrak{c}_U \in \mathfrak{B}$ for $U = \bigcap_1^n U_i \in \mathcal{U}$ and $\mathfrak{B}$ fulfils (4.14).

Conversely, if the merotopy $\mathfrak{C}$ satisfies (4.13) and (4.14), let $\mathfrak{B}$ denote the base for $\mathfrak{C}$ occurring in (4.13). By Lemma 4.4, there are transitive entourages U such that $\mathfrak{c} = \mathfrak{c}_U$ for each $\mathfrak{c} \in \mathfrak{B}$. Denote by $\mathcal{B}$ the set of all these U. By Lemma 4.7 and (4.14), $\mathcal{B}$ is a filter base on $X \times X$ and by $U^2 = U$, it is a base for a transitive quasi-uniformity $\mathcal{U}$. Clearly $\mathfrak{C}_{\mathcal{U}} = \mathfrak{C}$. □

In contrast to Lemma 4.5, there is no uniqueness in the above theorem:

Example 4.10. Let $X = \mathbb{R}$, $\mathfrak{c} = \{[2n, 2n+2) : n \in \mathbb{Z}\}$ and $\mathfrak{c}_1 = \mathfrak{c} \cup \{[0,1)\}$, $\mathfrak{c}_2 = \mathfrak{c} \cup \{[1,2)\}$. Each of the point-true covers $\mathfrak{c}$ and $\mathfrak{c}_i$ $(i = 1, 2)$ define merotopic bases $\{\mathfrak{c}\}$, $\{\mathfrak{c}_i\}$ for the same merotopy $\mathfrak{C}$ (observe $\mathfrak{c}_i < \mathfrak{c} < \mathfrak{c}_i$). However, if we choose transitive entourages U_i such that $\mathfrak{c}_i = \mathfrak{c}_{U_i}$ (cf. Lemma 4.4) then $\{U_i\}$ is a base for a quasi-uniformity $\mathcal{U}_i$ and $\mathfrak{C}_{\mathcal{U}_i} = \mathfrak{C}$ while $U_2 \nsubseteqq U_1$ (e.g. $1 \in U_2(0) - U_1(0)$), so $\mathcal{U}_1 \neq \mathcal{U}_2$.

Observe that this Example shows: if U_i $(i = 1, 2)$ are transitive entourages and $\mathfrak{c}_{U_1} < \mathfrak{c}_{U_2} < \mathfrak{c}_{U_1}$ then $U_1 = U_2$ need not hold. Also $\{\mathfrak{c}_1, \mathfrak{c}_2\}$ is a base for $\mathfrak{C}$ but $\{U_1, U_2\}$ is not a quasi-uniform base at all as $U_1 \nsubseteqq U_2 \nsubseteqq U_1$. Certainly, it is a quasi-uniform subbase; however, if $U = U_1 \cap U_2$, then $\{U\}$ is a base for a quasi-uniformity $\mathcal{U}$ but, since by Lemma 4.7 $\mathfrak{c}_U = \{[2n, 2n+2) : n \in \mathbb{Z} - \{0\}\} \cup \{[0,1), [1,2)\}$, we have $\mathfrak{C} \neq \mathfrak{C}_{\mathcal{U}}$ as $\mathfrak{c}_U < \mathfrak{c}$ and $\mathfrak{c} \not< \mathfrak{c}_U$.

Example 4.10 contains a merotopy and quasi-uniformities inducing very bad topologies. However, it is possible the find a better example:

Example 4.11. Let $X = \mathbb{R} - \mathbb{Z} = \bigcup_{n \in \mathbb{Z}} I_n$ where $I_n = (n, n+1)$. Let τ denote the subspace topology on X of the Euclidean one on $\mathbb{R}$. Denote by $\mathfrak{B}$ the base for τ composed of all (τ)-open sets B contained in some I_n. Consider the (point-true) covers of X $\mathfrak{c}_{x,B} = \{\{x\}, B - \{x\}, X - \{x\}\}$; clearly $\mathfrak{c}_{x,B} = \mathfrak{c}_{U_{x,B}}$. Denote also $\mathfrak{c}' = \{X\} \cup \{I_{2k-1} : k \in \mathbb{Z}\}$, $\mathfrak{c}'' = \{X\} \cup \{I_{2k} : k \in \mathbb{Z}\}$. Clearly both $\mathfrak{c}'$ and $\mathfrak{c}''$ are point-finite, point-true covers of X. We write $\mathfrak{c}' = \mathfrak{c}_{U'}$, $\mathfrak{c}'' = \mathfrak{c}_{U''}$ with transitive entourages U', U''. Let $\mathcal{U}'$ be the transitive quasi-uniformity defined by the subbase $\{U_{x,B} : x \in B \in \mathfrak{B}\} \cup \{U'\}$, and similarly define $\mathcal{U}''$ with the help of the subbase $\{U_{x,B} : x \in B \in \mathfrak{B}\} \cup \{U''\}$.

We have $\mathcal{U}' \neq \mathcal{U}''$. In fact, assume the contrary; then $U' \supset U = \bigcap_1^n U_{x_i,B_i} \cap U''$ for suitable $x_i \in B_i$, $1 \leqq i \leqq n$. There is a $k \in \mathbb{Z}$ such that I_{2k-1} is disjoint from all sets $B_1, ..., B_n$ so that $U'(x) = I_{2k-1}$ for $x \in I_{2k-1}$ while $U(x)$ is cofinite as $U_{x_i,B_i}(x) = X - \{x_i\}$ and $U''(x) = X$.

Let us write $\mathfrak{C}' = \mathfrak{C}_{\mathcal{U}'}$, $\mathfrak{C}'' = \mathfrak{C}_{\mathcal{U}''}$. For an arbitrary cover $\mathfrak{c} \in \mathfrak{C}'$, we can find, according to Lemma 4.7, $x_i \in B_i \in \mathfrak{B}$ such that $\mu((\bigcap)_1^n \mathfrak{c}_{x_i,B_i}(\cap)U') < \mathfrak{c}$. We claim

$$\mu((\bigcap)_1^n \mathfrak{c}_{x_i,B_i}) < \mu((\bigcap)_1^n \mathfrak{c}_{x_i,B_i}(\cap)U').$$

In fact, if $x \in B_i$ for some i then the member containing x of the left hand side is contained either in $B_i \cap I_{2k-1} = B_i$ for some k or in $B_i \cap X = B_i$; both sets belong to the right hand side. If $x \notin B_i$ for each $i = 1, ..., n$, then there is a k such that I_{2k} is disjoint from all sets B_i occurring on the left hand side and then the member of the left hand side containing some $y \in I_{2k}$ is the same as the one containing x; therefore this member is the one containing y of the right hand side. Thus the left hand side, belonging to $\mathfrak{C}''$, refines $\mathfrak{c}$ and $\mathfrak{c} \in \mathfrak{C}''$, $\mathfrak{C}' \subset \mathfrak{C}''$. A similar argument furnishes $\mathfrak{C}'' \subset \mathfrak{C}'$ so that finally $\mathfrak{C}' = \mathfrak{C}'' = \mathfrak{C}$.

Clearly both $\mathcal{U}'$ and $\mathcal{U}''$ induce the (very good) topology τ. According to Proposition 2.7, they are point-symmetric, so that the merotopy $\mathfrak{C}$ induces τ as well (see Theorem 2.6).

Example 3.4 shows that the invariance properties of semi-symmetry are essentially the same in the transitive case as in the general one. However, we can establish useful criteria guaranteeing the symmetry of a transitive entourage or the semi-symmetry of a transitive quasi-uniformity.

Lemma 4.12. *If $\mathfrak{c}$ is a point-true cover of X, $U = U_{\mathfrak{c}}$ is the corresponding transitive entourage, then $\mathfrak{c}_{U^{-1}} = \mu(\mathfrak{c}^c)$ where $\mathfrak{c}^c = \{X - C : C \in \mathfrak{c}\}$.*

Proof. Let $V = U^{-1}$, $x \in X$. Now $y \in V(x)$ iff $x \in U(y) = \bigcap\{C \in \mathfrak{c} : y \in C\}$ iff $y \in C \in \mathfrak{c} \Rightarrow x \in C$ iff $x \notin C \in \mathfrak{c} \Rightarrow y \notin C$ iff $x \in X - C$, $C \in \mathfrak{c} \Rightarrow y \in X - C$ iff $y \in \bigcap\{X - C : C \in \mathfrak{c},\ x \in X - C\}$ and the latter $\bigcap$ is the element corresponding to x of $\mu(\mathfrak{c}^c)$. □

Observe that μ cannot be dropped: let $X = [0,1] \subset \mathbb{R}$, $\mathfrak{c} = \{[0,x] : 0 \leqq x < 1\} \cup \{1\}$; now $\mathfrak{c}^c = \{(x,1] : 0 \leqq x < 1\} \cup [0,1)$ is not point-true.

Theorem 4.13. *Let $\mathfrak{c}$ be a point-true cover of X and $U = U_{\mathfrak{c}}$. U is symmetric iff $\mathfrak{c}$ is a partition of X.*

Proof. Necessity: Suppose $U(x) \cap U(y) \neq \varnothing$, say, $z \in U(x) \cap U(y)$. Then $U(z) \subset U(x) \cap U(y)$ by the transitivity, $x \in U(z)$ and $y \in U(z)$ by the symmetry, and $U(x) \cup U(y) \subset U(z)$ by the transitivity again. Hence $U(x) = U(z) = U(y)$.

Sufficiency: If $U(x) = C_0$ then $U^{-1}(x) = \bigcap\{X - C : C \in \mathfrak{c},\ x \notin C\}$ by Lemma 4.12, hence $U^{-1}(x) = C_0$ provided $\mathfrak{c}$ is a partition. □

Theorem 4.14. *Let $\mathfrak{C} = \mathfrak{C}_{\mathcal{U}}$ for a transitive quasi-uniformity $\mathcal{U}$. The latter is semi-symmetric iff there is a base $\mathfrak{B}$ for $\mathfrak{C}$ composed of covers $\mathfrak{c}_U$ with transitive $U \in \mathcal{U}$ and such that these U constitute a base for $\mathcal{U}$, further, if $\mathfrak{c} \in \mathfrak{B}$, there is a $\mathfrak{c}' \in \mathfrak{B}$ such that, whenever $C_i' \in \mathfrak{c}'$ and $C_1' \cap C_2' \neq \varnothing$, there is $C \in \mathfrak{c}$ satisfying $C_1' \cup C_2' \subset C$.*

Proof. Necessity: Let $\mathfrak{B} = \{\mathfrak{c}_U : U \in \mathcal{U}$ is transitive $\}$. Given $\mathfrak{c} = \mathfrak{c}_U \in \mathfrak{B}$, $U \in \mathcal{U}$ transitive, choose a transitive $V_0 \in \mathcal{U}$ such that $V_0^{-1}V_0 \subset UU^{-1}$ and set $V = V_0 \cap U \in \mathcal{U}$. Finally let $\mathfrak{c}' = \mathfrak{c}_V$. Now if $C_1' = V(x)$, $C_2' = V(y)$ and $C_1' \cap C_2' \neq \varnothing$, we have some z such that $z \in V(x) \cap V(y)$, hence $y \in V^{-1}(z) \subset V^{-1}(V(x)) \subset U(U^{-1}(x))$. Consequently there is some u satisfying $u \in U^{-1}(x)$, $y \in U(u)$, i.e. $x, y \in U(u)$, therefore $C_1' \cup C_2' = V(x) \cup V(y) \subset U(x) \cup U(y) \subset U(u)$ by the transitivity of U. For $C = U(u) \in \mathfrak{c}$ we obtain $C_1' \cup C_2' \subset C$.

Sufficiency: Given $U \in \mathcal{U}$, choose a transitive $U_0 \in \mathcal{U}$ such that $U_0 \subset U$ and $\mathfrak{c}_{U_0}$ belongs to the base $\mathfrak{B}$ in the hypothesis. Set $\mathfrak{c} = \mathfrak{c}_{U_0}$, then choose $\mathfrak{c}' \in \mathfrak{B}$ satisfying $C_1' \cup C_2' \subset C \in \mathfrak{c}$ whenever $C_i' \in \mathfrak{c}'$ and $C_1' \cap C_2' \neq \varnothing$, and let $\mathfrak{c}' = \mathfrak{c}_V$ for some transitive $V \in \mathcal{U}$. If $x \in X$ and $y \in V^{-1}(V(x))$, then $V(x), V(y) \in \mathfrak{c}'$ and $V(x) \cap V(y) \neq \varnothing$ so that $V(x) \cup V(y) \subset C = U_0(z) \subset U(z)$ for a suitable $z \in X$. Then $x, y \in U(z)$, hence $z \in U^{-1}(x)$ and $y \in U(U^{-1}(x))$. From $V^{-1}(V(x)) \subset U(U^{-1}(x))$ we obtain $V^{-1}V \subset UU^{-1}$. □

A similar (but simpler) argument furnishes:

Corollary 4.15. *Let $\mathfrak{c} = \mathfrak{c}_U$ for a transitive entourage U. The latter is semi-symmetric iff, whenever $C_i \in \mathfrak{c}$ and $C_1 \cap C_2 \neq \varnothing$, there exists $C \in \mathfrak{c}$ satisfying $C_1 \cup C_2 \subset C$.*

Semi-symmetry and point-symmetry are independent concepts also for transitive quasi-uniformities. In fact, the example given above for a point-symmetric but not semi-symmetric quasi-uniformity was a Pervin quasi-uniformity, hence transitive. For a semi-symmetric but not point-symmetric, transitive quasi-uniformity, consider:

Example 4.16. Let $X = \{a, b\}$, c be the closure associated with the Sierpiński topology $\{\varnothing, \{a\}, X\}$, $\mathcal{U}$ the (transitive) Pervin quasi-uniformity of c generated by the base $\{U\}$ where $U = U_{\{a\}}$ and $\mathfrak{c}_U = \{\{a\}, X\}$. Then $U(a) = \{a\}$, $U(b) = X$, $U^{-1}(a) = X$, $U^{-1}(b) = \{b\}$. Clearly $U^{-1}(U(a)) = U^{-1}(U(b)) = U(U^{-1}(a)) = U(U^{-1}(b)) = X$ so that $\mathcal{U}$ is semi-symmetric, but it is not point-symmetric because $U^{-1}(a) \subsetneqq U(a)$.

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