Fell type topologies of quasi-pseudo-metric spaces and the Kuratowski-Painlevé convergence

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Abstract. We study the double Fell topology when this hypertopology is constructed over a quasi-pseudo-metric space. In particular, its relationship with the Wijsman hypertopology is studied. We also propose an extension of the Kuratowski-Painlevé convergence in the bitopological setting.

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1. Introduction and preliminaries

Recently, the study of nonsymmetric structures has received a new drive as a consequence of its applications to Computer Science. This theory began with Smyth (see [20, 21]). He tried to find a convenient category for computation and he proposed the quasi-uniform spaces as the suitable context. Continuing the work of Smyth, other authors have applied the nonsymmetric topology to this area (see [17, 18, 19]). Furthermore, some hypertopologies have been successfully applied to several areas of Computer Science (see [21, 23]). All these facts motivate our interest in the nonsymmetric study of several hypertopologies. In this paper, we continue the work developed by the author in [15].

The Fell topology was introduced by Fell in [8]. In [15] it is introduced a definition for the Fell hypertopology in the nonsymmetric situation. Some satisfactory results about the relationship of some hypertopologies with the Fell topology are obtained in the quasi-uniform setting. We continue this work and obtain extensions of well-known results in the symmetric case about the relationship between the Fell and the Wijsman hypertopologies. We also study a definition for the Kuratowski-Painlevé convergence in the bitopological setting and obtain extensions of interesting results as the Mrowka’s Theorem.

Our basic references for quasi-uniform and quasi-pseudo-metric spaces are [9] and [12]. Terms and undefined concepts may be found in such references.
A quasi-pseudo-metric on a set $X$ is a nonnegative real valued function $d$ on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$ and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If, in addition, $d$ satisfies the condition: (iii) $d(x, y) = 0 \Rightarrow x = y$, then $d$ is said to be a quasi-metric on $X$.

A quasi-(pseudo-)metric space is a pair $(X, d)$ such that $X$ is a nonempty set and $d$ is a quasi-(pseudo-)metric on $X$.

If $d$ is a quasi-(pseudo-)metric on $X$, then the function $d^{-1}$ defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-(pseudo-)metric on $X$, called the conjugate quasi-pseudo-metric of $d$, and the function $d^s$ defined on $X \times X$ by $d^s(x, y) = \max \{d(x, y), d^{-1}(x, y)\}$ for all $x, y \in X$, is a (pseudo-)metric on $X$.

Each quasi-pseudo-metric $d$ on $X$ generates a topology $\mathcal{T}(d)$ on $X$ which has as a base the family of balls of the form $B_d(x, r) = \{y \in X : d(x, y) < r\}$, where $x \in X$ and $r > 0$. Note that if $d$ is a quasi-metric, then $\mathcal{T}(d)$ is a $T_1$ topology on $X$. We denote $B_d(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}$.

The quasi-pseudo-metric $\ell$ on $\mathbb{R}$ defined by $\ell(x, y) = \max \{x - y, 0\}$ for all $x, y \in \mathbb{R}$, is called the lower quasi-pseudo-metric on $\mathbb{R}$. Its conjugate quasi-pseudo-metric $\ell^{-1}$ is denoted by $u$ and is called the upper quasi-pseudo-metric on $\mathbb{R}$. Note that $\ell^s = \ell \vee u$ is the usual metric on $\mathbb{R}$. A function from a topological space $(X, \mathcal{T})$ to $\mathbb{R}$ is said to be lower semicontinuous (resp. upper semicontinuous) if it is continuous when we consider the topology generated by the lower (resp. upper) quasi-pseudo-metric on $\mathbb{R}$.

A quasi-uniformity on a set $X$ is a filter $\mathcal{U}$ on $X \times X$ which satisfies: (i) $\Delta \subseteq U$ for all $U \in \mathcal{U}$ and (ii) given $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $\Delta = \{(x, x) : x \in X\}$ and $V^2 = \{(x, z) \in X \times X : \exists y \in X \text{ such that } (x, y) \in V, (y, z) \in V\}$. The elements of $\mathcal{U}$ are called entourages.

The filter $\mathcal{U}^{-1}$, formed by all sets of the form $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$ where $U \in \mathcal{U}$, is a quasi-uniformity on $X$ called the conjugate quasi-uniformity of $\mathcal{U}$.

If $\mathcal{U}$ is a quasi-uniformity on $X$, then the family $\{U^n = U \cap U^{-1} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}^*$ (in fact, it is a uniformity), which is the coarsest uniformity containing $\mathcal{U}$. This uniformity is called the supremum of the quasi-uniformities $\mathcal{U}$ and $\mathcal{U}^{-1}$.

Every quasi-uniformity $\mathcal{U}$ generates a topology $\mathcal{T}(\mathcal{U})$ on $X$. A neighborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$ where $U(x) = \{y \in X : (x, y) \in U\}$.

Each quasi-pseudo-metric $d$ on $X$ induces a quasi-uniformity $\mathcal{U}_d$ on $X$ which has as a base the family of entourages of the form $\{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$, $n \in \mathbb{N}$. Moreover, $\mathcal{T}(\mathcal{U}_d) = \mathcal{T}(d)$.

In addition of a quasi-uniformity and a quasi-pseudo-metric, we can define on a space another structure which makes precise the concept of nearness. This structure is a relation $\delta$ in $P_0(X)$. We write $A \notin B$ for $(A, B) \in \delta$ and $A \notin B$ instead of $(A, B) \notin \delta$. 
Definition 1.1. Let $X$ be a nonempty set. A relation $\delta$ in $\mathcal{P}_0(X)$ is a quasi-proximity for $X$ if it satisfies the following conditions:

i) $X\delta\emptyset$ and $\emptyset\delta X$.

ii) $C\delta(A \cup B)$ if and only if $C\delta A$ or $C\delta B$ and $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.

iii) $\{x\}\delta\{x\}$ for each $x \in X$.

iv) If $A\delta B$, there exists $C \in \mathcal{P}_0(X)$ such that $A\delta C$ and $(X\setminus C)\delta B$.

The pair $(X, \delta)$ is called a quasi-proximity space.

Obviously, if $\delta$ is a quasi-proximity on $X$, then so is the opposite relation $\delta^{-1}$. This quasi-proximity is called the conjugate quasi-proximity of $\delta$. A quasi-proximity $\delta$ is a proximity if $\delta = \delta^{-1}$.

Let $A$ and $B$ be subsets of a quasi-proximity space $(X, \delta)$. If $A\delta B$, then $A$ is said to be near $B$ and if $A\delta B$, then $A$ is said to be far from $B$. A set $B$ is said to be a $\delta$-neighborhood of a set $A$ if $A\delta(X\setminus B)$.

Every quasi-proximity $\delta$ on a space $X$ induces in a natural way a topology on $X$. If $x \in X$, the neighborhoods of $x$ are the $\delta$-neighborhoods of $x$.

Furthermore, if $(X, \mathcal{U})$ is a quasi-uniform space, then $\mathcal{U}$ induces a quasi-proximity $\delta_{\mathcal{U}}$ such that $A\delta_{\mathcal{U}} B$ if and only if $(A \times B) \cap U \neq \emptyset$ for all $U \in \mathcal{U}$.

A bitopological space (see [10, 13]) is a triple $(X, P, Q)$ where $X$ is a set and $P$ and $Q$ are topologies on $X$. A bitopological space is said to be quasi-pseudometrizable (resp. quasi-uniformizable) if there exists a quasi-pseudo-metric $d$ (resp. a quasi-uniformity $\mathcal{U}$) on $X$ such that $T(d) = P$ and $T(d^{-1}) = Q$ (resp. $T(\mathcal{U}) = P$ and $T(\mathcal{U}^{-1}) = Q$). In this case we say that $d$ (resp. $\mathcal{U}$) is a quasi-pseudo-metric (resp. quasi-uniformity) compatible with the bitopological space $(X, P, Q)$.

Given a topological space $(X, T)$ we denote by $\mathcal{P}_0(X)$ the family of nonempty subsets of $X$ and by $CL_0(X)$ we denote the family of nonempty closed subsets of $X$. We also shall use $\mathcal{P}(X) = \mathcal{P}_0(X) \cup \emptyset$. If $(X, P, Q)$ is a bitopological space we denote by $CL_0^P(X)$ (resp. $CL_0^Q(X)$, $CL_0^d(X)$) the family of nonempty $P$-closed (resp. $Q$-closed, $P \lor Q$-closed) subsets of $X$.

2. Fell type topologies of quasi-pseudo-metric spaces

In [15] it can be found a discussion about the definition of the Fell hyper-topology in the nonsymmetric case. We propose the use of double topological spaces rather than bitopological spaces. We recall some definitions.

Definition 2.1 ([15]). A double topological space is simply a pair of topological spaces $((X, \tau), (Y, v))$.

In the following, we will also use double space.

Definition 2.2 ([15]). Let $(X, P, Q)$ be a bitopological space. We define the double upper Fell topological space as the double topological space $((CL_0^Q(X), F_P^+), (CL_0^Q(X), F_Q^+))$ where $F_P^+$ is the topology generated by all sets of the form $G^+ = \{A \in CL_0^Q(X) : A \subseteq G\}$ where $G$ is a $P$-open set and $X\setminus G$ is
that \( X \) on \( F \) C L

Then Proposition 2.4. Let \((X, P, Q)\) be a quasi-pseudo-metrizable bitopological space. Then \( F_P^- = T^-(W_d) \), \( F_Q^- = T^-(W_{d-1}) \) on \( P_0(X) \) and \( F_P^+ \subseteq T^+(W_d) \) on \( C L^Q_0(X) \) and \( F_Q^+ \subseteq T^+(W_{d-1}) \) on \( C L^P_0(X) \) where \( d \) is a quasi-pseudo-metric compatible with the bitopological space.

**Proof.** It is easy to show that \( d(x, \cdot)^{-1}(-\infty, \alpha) = B_d(x, \alpha)^- \) so we obtain that \( F_P^- = T^-(W_d) \) on \( P_0(X) \). In a similar way, it can be proved \( F_Q^- = T^-(W_{d-1}) \) on \( P_0(X) \).

Now, we show that \( F_P^+ \subseteq T^+(W_d) \) on \( C L^Q_0(X) \). Let \( G \) be a \( P \)-open set such that \( X \setminus G \) is \( P \lor Q \)-compact, and \( A \in G^+ \). For all \( x \in X \setminus G \), let us consider
\[ \varepsilon_x = d(A, x). \] Since \( A \) is a \( Q \)-closed set then \( \varepsilon_x > 0 \). Choose \( 0 < \alpha_x < \varepsilon_x \). Thus \( \{ B_{d^{-1}}(x, \alpha_x) : x \in X \setminus G \} \) is a \( Q \)-open cover of \( X \setminus G \) so there exists \( \{ x_1, \ldots, x_n \} \subseteq X \setminus G \) such that
\[ X \setminus G \subseteq \bigcup_{i=1}^n B_{d^{-1}}(x_i, \alpha_{x_i}). \]

Consider the \( T^+(W_d) \)-open set \( C = \cap_{i=1}^n d(\cdot, x_i)^{-1}(\alpha_{x_i}, +\infty) \). Clearly, \( A \subseteq C \). Let us see that \( C \subseteq G^+ \). Let \( B \in C \) and suppose that \( B \cap (X \setminus G) \neq \emptyset \). Given \( b \in B \cap (X \setminus G) \) there exists \( i \in \{1, \ldots, n \} \) such that \( d^{-1}(x_i, b) < \alpha_{x_i} \). A contradiction with \( d(B, x_i) > \alpha_{x_i} \). Thus, \( B \subseteq C \subseteq G^+ \), i.e. \( G^+ \in T^+(W_d) \). Similarly, we prove \( F^+_Q \subseteq T^+(W_{d-1}) \) on \( CL^Q_0(X) \).

**Remark 2.5.** We give an example showing that the above Proposition is not true when we define \( F^+_P \) and \( T^+(W_d) \) on \( CL^P_0(X) \) and \( F^+_Q \) and \( T^+(W_{d-1}) \) on \( CL^Q_0(X) \).

We consider the set \( \mathbb{N} \cup \{ \infty \} \) with the following quasi-metric:
\[
\begin{align*}
    d(n, m) &= 1 & \text{if } n \neq m \\
    d(n, n) &= 0 & \text{for all } n \in \mathbb{N} \\
    d(n, \infty) &= \frac{1}{n} & \text{for all } n \in \mathbb{N} \\
    d(\infty, n) &= 1 & \text{for all } n \in \mathbb{N} \\
    d(\infty, \infty) &= 0
\end{align*}
\]

It is evident that \( T(d) = P \) is the discrete topology. Therefore \( \mathbb{N} \) is a \( T(d) \)-clopen set, and its complement is obviously a \( T(d^{-1}) \)-compact set, so we consider the \( F^+_Q \)-open set \( \mathbb{N}^+ \). We shall prove that the set \( \mathbb{N} \in \mathbb{N}^+ \) has not a \( T^+(W_d) \)-neighborhood contained in \( \mathbb{N}^+ \). If \( n \in \mathbb{N} \) and \( N \in d(\cdot, n)^{-1}(\alpha, +\infty) \) where \( \alpha \in \mathbb{R} \) we deduce that \( \alpha < 0 \) but \( \{ \infty \} \) is a \( T(d) \)-closed set which belongs to \( d(\cdot, n)^{-1}(\alpha, +\infty) = CL^P_0(X) \) so this set is not contained in \( \mathbb{N}^+ \). On the other hand, if \( \mathbb{N} \in d(\cdot, \infty)^{-1}(\alpha, +\infty) \) we obtain the same contradiction.

It is natural to wonder when the Wijsman and Fell hypertopologies agree.

The following extension of a concept introduced by Beer in [1] and reformulated in [2], gives us the answer.

**Definition 2.6.** Let \( (X, d) \) be a quasi-pseudo-metric space. We say that it has nice closed balls if the proper closed \( d \)-balls and the proper closed \( d^{-1} \)-balls are \( T(d^\circ) \)-compact.

Now, we can extend a result which can be found in [3].

**Theorem 2.7.** Let \( (X, d) \) be a quasi-pseudo-metric space, \( P = T(d) \) and \( Q = T(d^{-1}) \). Then \( F_P = T(W_d) \) on \( CL^P_0(X) \) and \( F_Q = T(W_{d^{-1}}) \) on \( CL^Q_0(X) \) if and only if \( (X, d) \) has nice closed balls.

**Proof.** Let us suppose that there exists a proper closed \( d^{-1} \)-ball \( \overline{B_{d^{-1}}}(x, \alpha) \) which is not \( P \lor Q \)-compact. Therefore, there is \( y_0 \in X \) such that \( d(y_0, x) > \alpha \).
and there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq B_{d^{-1}}(x, \alpha) \) which does not admit a \( P \lor Q \)-cluster point in \( X \) since \( \overline{B}_{d^{-1}}(x, \alpha) \) is a \( P \)-closed set. Let \( A_n = \{x_n\}^Q \cup \overline{\{y_0\}}^Q \) for all \( n \in \mathbb{N} \). Let us prove that the sequence \( \{A_n\}_{n \in \mathbb{N}} \) \( F_P \)-converges to \( \overline{\{y_0\}}^Q \). Let \( V^+ \cap V^+_1 \cap \ldots \cap V^+_n \) be an \( F_P \)-open set containing \( \overline{\{y_0\}}^Q \). Clearly \( A_n \in V^+_1 \cap \ldots \cap V^+_n \). Suppose now, to obtain a contradiction, that \( \{A_n\}_{n \in \mathbb{N}} \) can find \( n_k \geq k \) such that \( A_{n_k} \not\subseteq V^+ \). Choose \( y_{n_k} \in A_{n_k} \) such that \( y_{n_k} \not\in V \) for all \( k \in \mathbb{N} \). Since \( A_{n_k} = \{x_{n_k}\}^Q \cup \overline{\{y_0\}}^Q \) and \( \overline{\{y_0\}}^Q \in V^+ \), it is easy to show that \( x_{n_k} \not\in V \). Hence, since \( X \setminus V \) is a \( P \lor Q \)-compact set, \( \{x_{n_k}\}_{k \in \mathbb{N}} \) admits a \( P \lor Q \)-cluster point \( z \). A contradiction, so \( \{A_n\}_{n \in \mathbb{N}} \) is \( F_P \)-convergent to \( \overline{\{y_0\}}^Q \).

Let us show now that \( \{d(A_n, x)\}_{n \in \mathbb{N}} \) does not converge to \( d(\overline{\{y_0\}}^Q, x) \) in the lower topology of \( \mathbb{R} \), i.e. \( \{A_n\}_{n \in \mathbb{N}} \) is not \( T(W_d) \)-convergent to \( \overline{\{y_0\}}^Q \). We have that \( d(A_n, x) \leq d(x_n, x) \leq \alpha \). Moreover, \( d(\overline{\{y_0\}}^Q, x) > \alpha \) since if \( z \in \overline{\{y_0\}}^Q \) then \( d(y_0, z) = 0, \) so \( \alpha < d(y_0, x) \leq d(y_0, z) + d(z, x) = d(z, x) \). Therefore
\[
d(\overline{\{y_0\}}^Q, x) - d(A_n, x) \geq d(\overline{\{y_0\}}^Q, x) - \alpha > 0.
\]
Consequently, \( F_P \neq T(W_d) \) on \( CL^Q_0(X) \). A contradiction.

If there is a proper closed \( d \)-ball \( \overline{B}_d(x, \alpha) \) which is not \( T(d^P) \)-compact, we can prove the statement in a similar way.

Suppose now that \((X, d)\) has nice closed balls. By Proposition 2.4, we only have to show that \( T^+(W_d) \subseteq F^+_P \) on \( CL^Q_0(X) \) and \( T^+(W_{d^{-1}}) \subseteq F^+_Q \) on \( CL^P_0(X) \). Let \( x \in X \) and \( \alpha \geq 0 \). Let us consider the \( T^+(W_d) \)-open set \( d(\cdot, x)^{-1}[\alpha, +\infty[) \). We first suppose that \( \overline{B}_{d^{-1}}(x, \beta) \) is not equal to \( X \). Then \( \overline{B}_{d^{-1}}(x, \beta) \) is \( P \lor Q \)-compact. If \( A \subseteq CL^Q_0(X) \) and \( d(A, x) = \alpha \), we can find a sequence \( \{a_n\}_{n \in \mathbb{N}} \subseteq \overline{B}_{d^{-1}}(x, \beta) \cap A \) such that \( \{d(a_n, x)\}_{n \in \mathbb{N}} \) converges to \( d(A, x) \). Since \( \overline{B}_{d^{-1}}(x, \beta) \) is \( P \lor Q \)-compact, there is a \( P \lor Q \)-convergent subsequence \( \{a_{n_k}\}_{k \in \mathbb{N}} \) of \( \{a_n\}_{n \in \mathbb{N}} \). If we denote by \( a \) its limit we obtain that \( a \in A \) and \( d(a, x) = \alpha \). Therefore, \( d(\cdot, x)^{-1}[\alpha, +\infty[ = (X \setminus \overline{B}_{d^{-1}}(x, \alpha)^P \) which is a \( F^+_P \)-open set.

On the other hand, if \( \overline{B}_{d^{-1}}(x, \alpha) = X \) then \( d(\cdot, x)^{-1}[\alpha, +\infty[ = \emptyset \subseteq F^+_P \).

In a similar way it can be proved \( T^+(W_{d^{-1}}) \subseteq F^+_Q \) on \( CL^P_0(X) \). \( \square \)

**Remark 2.8.** We observe that by using the above proof, it can be shown:

\( F^+_P = T^+(W_d) \) on \( CL^Q_0(X) \) and \( F^+_Q = T(W_{d^{-1}}) \) on \( CL^P_0(X) \) if and only if \((X, d)\) has nice closed balls. Therefore, we deduce that the double Fell topology agrees with the double Wijsman topology if and only if the double upper Fell topology agrees with the double upper Wijsman topology.

In the following Remark, we give an example where the above theorem does not work if we change either the definition of the double Fell topology or the definition of nice closed balls. We will use the following definition.

**Definition 2.9.** Let \((X, P, Q)\) be a bitopological space. We define the double upper Vietoris topological space as the double topological space \((CL^Q_0(X), V^+_P)\), \((CL^P_0(X), V^+_Q)\) where \( V^+_P \) is the topology generated by all sets of the form...
\( G^+ = \{ A \in \text{CL}_0^Q(X) : A \subseteq G \} \) where \( G \) is a \( P \)-open set; \( V_Q^+ \) is defined in a similar way by writing \( P \) instead of \( Q \) and \( Q \) instead of \( P \).

The double lower Vietoris topological space is defined as the double topological space \((\text{CL}_0^Q(X), V_P^-), (\text{CL}_0^P(X), V_Q^-)\) where \( V_P^- \) is generated by all sets of the form \( G^- = \{ A \in \text{CL}_0^Q(X) : A \cap G \neq \emptyset \} \) where \( G \) is \( P \)-open; \( V_Q^- \) is defined in a similar way by writing \( P \) instead of \( Q \) and \( Q \) instead of \( P \).

The double Vietoris topological space is defined as the double topological space \((\text{CL}_0^Q(X), V_P^+), (\text{CL}_0^P(X), V_Q^+)\) where \( V_P = V_P^+ \lor V_P^- \) and \( V_Q = V_Q^+ \lor V_Q^- \).

**Remark 2.10.** Now we motivate one fact about the definition of the double Fell topology. We think that, maybe, the natural definition for \( F_P \) is to be the topology generated by the sets of the form \( G^+ \) and \( V^- \) where \( G \) and \( V \) are \( P \)-open sets and \( X \backslash G \) is \( Q \)-compact. In a similar way, we define the \( F_Q \) hypertopology. We give an example where, with this definition, Theorem 2.7 is not true.

Let \( d \) be the quasi-metric on \( \mathbb{N} \) given by

\[
d(n, m) = \begin{cases} 
\frac{1}{m} & \text{if } n < m \\
1 & \text{if } n = m \\
0 & \text{if } n > m
\end{cases}
\]

We consider the quasi-metric space \((\mathbb{N}, d)\). Let \( P = \mathcal{T}(d) \) and \( Q = \mathcal{T}(d^{-1}) \). We claim that \( F_P = V_P \) on \( \mathcal{P}_0(X) \) and \( F_Q = V_Q \) on \( \mathcal{P}_0(X) \). Since Proposition 2.4 is also true with this definition for the double Fell topology, we can deduce, using that \( \mathcal{T}(W_d) \subseteq V_P \) and \( \mathcal{T}(W_{d^{-1}}) \subseteq V_Q \) on \( \mathcal{P}_0(X) \) (see [16]), that \( F_P = \mathcal{T}(W_d) \) on \( \text{CL}_0^Q(X) \) and \( F_Q = \mathcal{T}(W_{d^{-1}}) \) on \( \text{CL}_0^P(X) \) but \( (X, d) \) has not nice closed balls. We only have to prove that \( V_P^+ \subseteq F_P^+ \) and \( V_Q^+ \subseteq F_Q^+ \).

Let \( G \in P \) and we consider the \( V_P^+ \)-open set \( G^+ \). Since \( G \) is a \( \mathcal{T}(d) \)-open set, it easy to prove that \( \mathbb{N} \backslash G \) is a finite set, so it is \( Q \)-compact. Therefore, \( G^+ \in F_P^+ \) so \( V_P^+ = F_P^+ \) on \( \mathcal{P}_0(X) \).

On the other hand, let us suppose that \( G \in Q \) and we consider the \( V_Q^+ \)-open set \( G^+ \). It is clear that every subset of \( X \) is \( P \)-compact. Hence, \( F_Q^+ = V_Q^+ \) on \( \mathcal{P}_0(X) \). We observe that this statement is not true if we consider the topology \( P \lor Q \), since it is the discrete topology.

We consider the closed ball \( B_d(n, 1/n) = \{ n, n+1, \ldots \} \). It is evident that this set is not \( P \lor Q \)-compact.

We notice that if we change the definition of a quasi-pseudo-metric space having nice closed balls by saying that a quasi-pseudo-metric space has this property if the proper closed \( d \)-balls are \( \mathcal{T}(d) \)-compact and the proper closed \( d^{-1} \)-balls are \( \mathcal{T}(d^{-1}) \)-compact the result is not true either. The preceding example shows that. The above ball is not \( Q \)-compact, since it is an infinite set and \( Q \) is the discrete topology.

**Remark 2.11.** We claim that if \( (X, d) \) is a quasi-metric space having nice closed balls then \( \mathcal{T}(d) = \mathcal{T}(d^{-1}) \). Let us show this.
Let \( \{x_n\}_{n \in \mathbb{N}} \) be a \( \mathcal{T}(d^{-1}) \)-convergent sequence to \( x \). Then, if \( m \in \mathbb{N} \) there exists \( n_0 \in \mathbb{N} \) such that \( d(x_n, x) < 1/2m \) for all \( n \geq n_0 \). In addition, we can find a proper closed \( d^{-1} \)-ball with center \( x \) (otherwise, since \( (X, \mathcal{T}(d^{-1})) \) is a \( T_1 \) space, we would have that \( X = \{x\} \) and the result is obvious). Let \( \overline{B}_{d^{-1}}(x, \alpha) \) be such a ball. Then, \( x_n \in \overline{B}_{d^{-1}}(x, \alpha) \) if \( n \) is greater or equal than a certain natural number \( n_1 \). Hence, \( \{x_n\}_{n \in \mathbb{N}} \) admits a \( \mathcal{T}(d) \)-cluster point \( y \), and, furthermore for each \( m \in \mathbb{N} \)

\[
d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{m}
\]

for a sufficient large \( n \), so \( x = y \) and, therefore, \( \mathcal{T}(d) \subseteq \mathcal{T}(d^{-1}) \).

The other inclusion is similar.

In general, the equality \( \mathcal{T}(d) = \mathcal{T}(d^{-1}) \) is not true in a quasi-pseudo-metric space \( (X, d) \) having nice closed balls. Let \( \mathbb{Z} \) be the set of integers. The Khalimsky line consists of \( \mathbb{Z} \) with the topology generated by all sets of the form \( \{2n - 1, 2n, 2n + 1\}, n \in \mathbb{Z} \). It is introduced in image processing in [11]. Then the quasi-pseudo-metric \( d \) defined on \( \mathbb{Z} \) by \( d(2n, 2n - 1) = d(2n, 2n + 1) = d(n, n) = 0 \) for all \( n \in \mathbb{N} \) and \( d(x, y) = 1 \) otherwise, generates the topology of the Khalimsky line. It is clear that the proper closed \( d \)-balls and the proper closed \( d^{-1} \)-balls are finite so they are \( \mathcal{T}(d^*) \)-compact. Furthermore, it is obvious that \( \mathcal{T}(d) \neq \mathcal{T}(d^{-1}) \).

When we consider a quasi-metric space we obtain the following result.

**Corollary 2.12.** Let \( (X, d) \) be a quasi-metric space and \( \mathcal{T}(d) = P, \mathcal{T}(d^{-1}) = Q \). The following statements are equivalent.

i) \( F_P = \mathcal{T}(W_d) \) and \( F_Q = \mathcal{T}(W_{d^{-1}}) \) on \( CL_0^P(X) \).

ii) \( F_P = \mathcal{T}(W_d) \) on \( CL_0^Q(X) \) and \( F_Q = \mathcal{T}(W_{d^{-1}}) \) on \( CL_0^Q(X) \).

iii) \( F_P = \mathcal{T}(W_d) \) on \( CL_0^Q(X) \) and \( F_Q = \mathcal{T}(W_{d^{-1}}) \) on \( CL_0^P(X) \).

iv) \( (X, d) \) has nice closed balls.

v) \( P = Q \) and \( (X, d) \) has nice closed balls.

**Proof.** i) \( \Rightarrow \) ii) and i) \( \Rightarrow \) iii) are obvious. ii) implies iv) can be shown as above, taking into account that \( (X, P) \) and \( (X, Q) \) are \( T_1 \) spaces. By the above Theorem we obtain iii) \( \Rightarrow \) iv). iv) \( \Rightarrow \) v) is the above Remark. The implication v) \( \Rightarrow \) i) is [3, Theorem 5.1.10]. \( \square \)

**Remark 2.13.** Let us observe that the above Corollary is not true when we consider a quasi-pseudo-metric space. Let us show that ii) \( \Rightarrow \) iv) fails. Consider the quasi-pseudo-metric space \( (\mathbb{R}, \ell) \) where \( \ell \) denotes the lower quasi-pseudo-metric. Clearly, we have that \( F_P = \mathcal{T}(W_{\ell}) \) on \( CL_0^P(X) \) and \( F_Q = \mathcal{T}(W_u) \) on \( CL_0^Q(X) \) where \( P = \mathcal{T}(\ell) \) and \( Q = \mathcal{T}(u) \). However, \( (\mathbb{R}, \ell) \) does not have nice closed balls, since the closed \( \ell \)-balls and closed \( u \)-balls are not bounded.

### 3. Other Fell type topologies

As we have already observed, we have various possibilities in order to define the Fell hypertopology in the nonsymmetric situation. This section is devoted
to describe the advantages and disadvantages of our definition compared with other ones.

We begin giving the definition that we think is more natural.

**Definition 3.1.** Let \((X, P, Q)\) be a bitopological space. We define the double upper fine Fell space as the double space \(\left((CL^Q(X), FF^+_P), (CL^P_0(X), FF^+_Q)\right)\) where \(FF^+_P\) is the topology generated by all sets of the form \(G^+\) where \(G\) is a \(P\)-open set and \(X\setminus G\) is \(Q\)-compact; the topology \(FF^+_Q\) is defined in the corresponding natural way.

The double fine Fell space is the double space \(\left((CL^Q(X), FF_P), (CL^P_0(X), FF_Q)\right)\) where \(FF_P = FF^+_P \vee FF^-_Q\) and \(FF_Q = FF^+_Q \vee FF^-_P\).

With this definition, not all the results proved in the previous section work (see Remark 2.10).

Another possible definition is suggested by Burdick’s investigations ([5, 6, 7]). He looked for a context in which he considered separately the upper and lower Vietoris topologies on a hyperspace and explored the interactions between them.

**Definition 3.2.** Let \((X, P, Q)\) be a bitopological space. The double mixed Fell space is the double space \(\left((CL^Q(X), MF_P), (CL^P_0(X), MF_Q)\right)\) where \(MF_P = FF^+_P \vee FF^-_Q\) and \(MF_Q = FF^+_Q \vee FF^-_P\).

We call this hypertopology mixed, because we interchange the natural lower hypertopologies between the two hyperspaces that we construct. We notice that all results obtained in the previous section are true using this definition whenever we change the definition of the Wijsman lower hypertopology. Let us observe that our main results only use the upper hypertopologies since the double lower Wijsman topology always coincides with the double lower Fell topology. However, we think that is not a natural definition, although it provides a nontrivial topology on the bitopological space \((\mathbb{R}, \mathcal{T}(\ell), \mathcal{T}(u))\).

Furthermore, we can give another definition.

**Definition 3.3.** Let \((X, P, Q)\) be a bitopological space. The double mixed fine Fell space is the double space \(\left((CL^Q_0(X), MFF_P), (CL^P_0(X), MFF_Q)\right)\) where \(MFF_P = FF^+_P \vee FF^-_Q\) and \(MFF_Q = FF^+_Q \vee FF^-_P\).

Unfortunately, the double mixed fine Fell space has the same problems of generalization as the double fine Fell space. However, it is an appropriate Fell type topology to study epiconvergence of lower semicontinuous functions in the double setting, which will be discussed elsewhere.

4. **The Kuratowski-Painlevé convergence**

In this section, we propose a definition for the Kuratowski-Painlevé convergence in the nonsymmetric case and obtain some results about the relationships of this type of convergence and some hypertopologies.

The Kuratowski-Painlevé convergence was introduced to describe the limit of a net in terms of the members of the net itself. We propose the following definitions.
Definition 4.1. Let \((X, P, Q)\) be a bitopological space and \(\{A_\lambda\}_{\lambda \in \Lambda}\) a net of subsets of \(X\).

i) A point \(x_0\) belongs to \(P\)-Li\(A_\lambda\) (resp. \(Q\)-Li\(A_\lambda\)) and we say that \(x_0\) is a \(P\)-limit point (resp. \(Q\)-limit point) of \(\{A_\lambda\}_{\lambda \in \Lambda}\) if each \(Q\)-neighborhood (resp. \(P\)-neighborhood) of \(x_0\) intersects \(A_\lambda\) for all \(\lambda\) in some residual subset of \(\Lambda\).

ii) A point \(x_0\) belongs to \(P\)-Ls\(A_\lambda\) (resp. \(Q\)-Ls\(A_\lambda\)) and we say that \(x_0\) is a \(P\)-cluster point (resp. \(Q\)-cluster point) of \(\{A_\lambda\}_{\lambda \in \Lambda}\) if each \(Q\)-neighborhood (resp. \(P\)-neighborhood) of \(x_0\) intersects \(A_\lambda\) for all \(\lambda\) in some cofinal subset of \(\Lambda\).

The proof of the following proposition is straightforward.

Proposition 4.2. Let \((X, P, Q)\) be a bitopological space. If \(\{A_\lambda\}_{\lambda \in \Lambda}\) is a net of subsets of \(X\) then \(P\)-Li\(A_\lambda\) (resp. \(Q\)-Li\(A_\lambda\)) and \(P\)-Ls\(A_\lambda\) (resp. \(Q\)-Ls\(A_\lambda\)) are \(Q\)-closed sets (resp. \(P\)-closed sets).

Definition 4.3. Let \((X, P, Q)\) be a bitopological space and let \(\{A_\lambda\}_{\lambda \in \Lambda}\) be a net of subsets of \(X\) and \(A \in \mathcal{P}(X)\).

We say that \(\{A_\lambda\}_{\lambda \in \Lambda}\) is \(P\)-Kuratowski-Painlevé upper convergent (resp. \(Q\)-Kuratowski-Painlevé upper convergent) to \(A\) if \(P\)-Ls\(A_\lambda\) \(\subseteq\) \(A\) (resp. \(Q\)-Ls\(A_\lambda\) \(\subseteq\) \(A\)). We write \(A = K^+_P\text{-lim}\ A_\lambda\) (resp. \(A = K^+_Q\text{-lim}\ A_\lambda\)).

We say that \(\{A_\lambda\}_{\lambda \in \Lambda}\) is \(P\)-Kuratowski-Painlevé lower convergent (resp. \(Q\)-Kuratowski-Painlevé lower convergent) to \(A\) if \(A \subseteq P\)-Li\(A_\lambda\) (resp. \(A \subseteq Q\)-Li\(A_\lambda\)). We write \(A = K^-_P\text{-lim}\ A_\lambda\) (resp. \(A = K^-_Q\text{-lim}\ A_\lambda\)).

We say that \(\{A_\lambda\}_{\lambda \in \Lambda}\) is \(P\)-Kuratowski-Painlevé convergent (resp. \(Q\)-Kuratowski-Painlevé convergent) to \(A\) if \(A = P\)-Li\(A_\lambda\) = \(P\)-Ls\(A_\lambda\) (resp. \(A = Q\)-Li\(A_\lambda\) = \(Q\)-Ls\(A_\lambda\)). We write \(A = K_P\text{-lim}\ A_\lambda\) (resp. \(A = K_Q\text{-lim}\ A_\lambda\)).

With these definitions, we can extend a classical result which gives a relationship between the Kuratowski-Painlevé convergence and the convergence in the Fell topology. We need the following definition.

Definition 4.4 ([15]). A bitopological space \((X, P, Q)\) is said to be locally bicom pact if every point has a neighborhood base in \(P\) and a neighborhood base in \(Q\) whose elements are \(P \lor Q\text{-compact sets.}\)

Theorem 4.5. Let \((X, P, Q)\) be a bitopological space, \(A \in CL^Q_0(X)\) (resp. \(A \in CL^P_0(X)\)) and \(\{A_\lambda\}_{\lambda \in \Lambda}\) a net in \(CL^Q_0(X)\) (resp. \(CL^P_0(X)\)).

i) \(A = K^-_P\text{-lim}\ A_\lambda\) (resp. \(A = K^-_Q\text{-lim}\ A_\lambda\)) if and only if \(A = F^-_Q\text{-lim}\ A_\lambda\) (resp. \(A = F^-_P\text{-lim}\ A_\lambda\)).

ii) If \(A = K^+_P\text{-lim}\ A_\lambda\) (resp. \(A = K^+_Q\text{-lim}\ A_\lambda\)) then \(A = F^+_P\text{-lim}\ A_\lambda\) (resp. \(A = F^+_Q\text{-lim}\ A_\lambda\)).

iii) \(F_P\text{-lim}\ A_\lambda\) is a locally bicom pact quasi-uniformizable bitopological space and \(A = F^+_P\text{-lim}\ A_\lambda\) (resp. \(A = F^+_Q\text{-lim}\ A_\lambda\)) then \(A = K^+_P\text{-lim}\ A_\lambda\) (resp. \(A = K^+_Q\text{-lim}\ A_\lambda\)).
Proof. i) This statement is straightforward.

ii) Let us suppose that \( A = K_f^+ \)-lim \( A_\lambda \). Then

\[
P - Ls(K \cap A_\lambda) \subseteq P - LsA_\lambda \subseteq A
\]

for all \( P \cap Q \)-compact set \( K \). Let \( K_0 \) be a \( P \cap Q \)-compact and \( P \)-closed set such that \( K \cap A_\lambda \neq \emptyset \) for a cofinal subset of \( \Lambda \). Let us prove that \( P - LsA_\lambda \neq \emptyset \). Choose \( k_\lambda \in K \cap A_\lambda \) for all \( \lambda \) belonging to a cofinal subset \( \Lambda_0 \) of \( \Lambda \). We can obtain that \( \{k_\lambda\} \subseteq A_0 \) admits a \( P \cap Q \)-cluster point \( k \in K \). It is evident that \( k \in P - LsA_\lambda \cap K \subseteq A \) so \( A \cap K \neq \emptyset \). Therefore, if \( A \cap K = \emptyset \) then \( A_\lambda \cap K = \emptyset \) eventually. Hence, \( A = F_{P^+}^+ \)-lim \( A_\lambda \).

The other statement can be proved in a similar way.

iii) Let us suppose that \( P - LsA_\lambda \not\subseteq A \). Let \( x \in (P - LsA_\lambda) \setminus A \). Since \((X, P, Q)\) is a locally bicompact quasi-uniformizable bitopological space, we can find a \( P \)-closed and \( P \cap Q \)-compact \( Q \)-neighborhood \( V \) of \( x \) such that \( V \cap A = \emptyset \) but \( A_\lambda \cap V \neq \emptyset \) frequently. Therefore, \( A \neq F_{P^+}^+ \)-lim \( A_\lambda \), which is a contradiction. Consequently, \( P - LsA_\lambda \subseteq A \).

The same reasoning proves the statement for \( Q \). \( \square \)

We characterize the Kuratowski-Painlevé convergence in terms of sequences of points.

**Proposition 4.6.** Let \((X, P, Q)\) be a quasi-pseudo-metrizable bitopological space and \( d \) a quasi-pseudo-metric on \( X \) compatible with the bitopological space. A sequence \( \{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X) \) is \( P \)-Kuratowski-Painlevé lower convergent to a set \( A \) if and only if each point \( a \in A \) is the limit of some \( T(d^{-1}) \)-convergent sequence \( \{a_n\}_{n \in \mathbb{N}} \) such that \( a_n \in A_n \) for all \( n \in \mathbb{N} \).

**Proof.** Let us suppose that \( \{A_n\}_{n \in \mathbb{N}} \) is \( F_\mathcal{P}^+ \)-convergent to \( A \in \mathcal{P}(X) \). Pick \( a \in A \) (if \( A = \emptyset \) the result is evident). Given \( k \in \mathbb{N} \), we have that \( A \in B_{d^{-1}}(a, 1/k) \) so there exists \( n_k \in \mathbb{N} \) such that \( A_n \in B_{d^{-1}}(a, 1/k) \) for all \( n \geq n_k \). We can suppose that \( n_1 < n_2 < \ldots < n_k < \ldots \). Therefore, we can find \( a_n \in A_n \cap B_{d^{-1}}(a, 1/k) \) for all \( n_k + 1 \geq n \geq n_k + 1 \). If we consider the sequence \( \{a_n\}_{n \in \mathbb{N}} \) where if \( n \in \{1, \ldots, n_1\} \) we consider a fixed point \( a_n \in A_n \), we have that this sequence is \( T(d^{-1}) \)-convergent to \( a \).

Conversely, let \( \{A_n\}_{n \in \mathbb{N}} \) be a sequence and \( A \subseteq X \) satisfying our assumption. If \( a \in A \cap G \) where \( G \) is a \( T(d^{-1}) \)-open set, there exists a sequence \( \{a_n\}_{n \in \mathbb{N}} \) \( T(d^{-1}) \)-convergent to \( a \) verifying that \( a_n \in A_n \) for all \( n \in \mathbb{N} \). On the other hand, we can find \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) such that \( B_{d^{-1}}(a, \varepsilon) \subseteq G \) and \( d^{-1}(a, a_n) < \varepsilon \) for all \( n \geq n_0 \). Therefore, \( A_n \in G \) for all \( n \geq n_0 \). \( \square \)

We can also obtain a characterization of the Kuratowski-Painlevé upper convergence in terms of sequences.

**Proposition 4.7.** Let \((X, P, Q)\) be a quasi-pseudo-metrizable bitopological space and \( d \) a quasi-pseudo-metric on \( X \) compatible with the bitopological space. A sequence \( \{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X) \) is \( P \)-Kuratowski-Painlevé upper convergent to a set \( A \) if and only if whenever there exist positive integers \( n_1 < n_2 < \ldots \) and \( a_k \in A_{n_k} \) for all \( k \in \mathbb{N} \) such that \( \{a_k\}_{k \in \mathbb{N}} \) is \( T(d^{-1}) \)-convergent to a then \( a \in A \).
Proof. Let us suppose that \( \{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X) \) is \( P \)-Kuratowski-Painlevé convergent to \( A \). If \( \{a_n\}_{n \in \mathbb{N}} \) is a sequence as in the statement, it is evident that \( a \in P\text{-}LsA_n \subseteq A \).

Now, let \( a \in P\text{-}LsA_n \) and \( \{B_{d^{-1}}(a,1/n) : n \in \mathbb{N}\} \) a countable \( T(d^{-1}) \)-neighborhood base of \( a \). Choose \( n_1 \in \mathbb{N} \) such that \( B_{d^{-1}}(a,1) \cap A_{n_1} \neq \emptyset \). Since \( \{n \in \mathbb{N} : B_{d^{-1}}(a,1/2) \cap A_n\} \) is infinite, we can find \( n_2 > n_1 \) verifying \( B_{d^{-1}}(a,1/2) \cap A_{n_2} \neq \emptyset \). Following this procedure, we can construct a strictly increasing sequence of positive integers \( \{n_k\}_{k \in \mathbb{N}} \) such that \( B_{d^{-1}}(a,1/k) \cap A_{n_k} \neq \emptyset \) for all \( k \in \mathbb{N} \). If \( a_k \in B_{d^{-1}}(a,1/k) \cap A_{n_k} \) for all \( k \in \mathbb{N} \), it is evident that this sequence is \( T(d^{-1}) \)-convergent to \( a \), so by assumption \( a \in A \). \( \square \)

Now, we can extend an interesting result due to Mrowka (see [14]).

**Theorem 4.8 (Mrowka).** Let \( (X, P, Q) \) be a bitopological space and let \( \{A_{\lambda}\}_{\lambda \in \Lambda} \) be a net in \( \mathcal{P}(X) \). Then \( \{A_{\lambda}\}_{\lambda \in \Lambda} \) has a \( P \)-Kuratowski-Painlevé convergent subnet and a \( Q \)-Kuratowski-Painlevé convergent subnet.

**Proof.** Let \( B \) be a base for the topology \( Q \). Let us consider the space \( \{0, 1\} \) with the discrete topology. For each \( \lambda \in \Lambda \), we define \( f_{\lambda} : B \to \{0, 1\} \) as follows:

\[
f_{\lambda}(V) = \begin{cases} 
1 & \text{if } A_{\lambda} \cap V \neq \emptyset \\
0 & \text{if } A_{\lambda} \cap V = \emptyset
\end{cases}
\]

By the Tychonoff’s theorem, \( \{f_{\lambda}\}_{\lambda \in \Lambda} \) has a convergent subnet \( \{f_{\lambda'}\}_{\lambda' \in \Lambda'} \). For each \( V \in B \), we obtain that \( f_{\lambda'}(V) = 1 \) eventually if and only if \( f_{\lambda'}(V) = 1 \) frequently. Therefore, if \( A_{\lambda} \cap V \neq \emptyset \) frequently then \( A_{\lambda} \cap V \neq \emptyset \) eventually, so \( \{A_{\lambda'}\}_{\lambda' \in \Lambda'} \) is \( P \)-Kuratowski-Painlevé convergent.

The reasoning for \( P \) is similar. \( \square \)

**Theorem 4.9.** Let \( (X, d) \) be a quasi-pseudo-metric space. Let \( P = T(d) \) and \( Q = T(d^{-1}) \). Let us consider \( \{A_{\lambda}\}_{\lambda \in \Lambda} \) a net in \( CL_Q^0(X) \) and \( \{B_{\gamma}\}_{\gamma \in \Gamma} \) a net in \( CL_P^0(X) \).

i) If \( A = T^+(W_d)\text{-}\lim A_{\lambda} \) and \( B = T^+(W_{d^{-1}})\text{-}\lim B_{\gamma} \) then \( A = K_P^+\text{-}\lim A_{\lambda} \) and \( B = K_Q^+\text{-}\lim B_{\gamma} \).

ii) If \( A = K_P^+\text{-}\lim A_{\lambda} \) and \( B = K_Q^+\text{-}\lim B_{\gamma} \), implies \( A = T^+(W_Q)\text{-}\lim A_{\lambda} \) and \( B = T^+(W_{d^{-1}})\text{-}\lim B_{\gamma} \) if and only if \( (X, d) \) has nice closed balls.

**Proof.** i) Let us suppose that \( A = T^+(W_d)\text{-}\lim A_{\lambda} \) and \( B = T^+(W_{d^{-1}})\text{-}\lim B_{\gamma} \). Let \( a \in P\text{-}LsA_{\lambda} \) and suppose that \( a \notin A \). Thus \( d(A, a) > 0 \). Therefore, given \( 0 < \delta < d(A, a) \) since \( a \in P\text{-}LsA_{\lambda} \) we obtain that \( d(A_{\lambda}, a) < \delta \) frequently so

\[d(A, a) - d(A_{\lambda}, a) > d(A, a) - \delta > 0\]

frequently which contradicts that \( d(A, a) = T(\ell)\text{-}\lim d(A_{\lambda}, a) \). The same reasoning shows that \( B \subseteq Q\text{-}LsB_{\gamma} \).

ii) This statement is similar to the proof of Theorem 2.7. \( \square \)

We recall the following definitions (see [15]).
Definition 4.10. Let \((X, P, Q)\) be a quasi-uniformizable bitopological space and \(U\) a quasi-uniformity compatible with the bitopological space. The double upper \(U\)-proximal topological space is defined as the double topological space \(((\text{CL}_0(X), T^+(\delta_U)), (\text{CL}_0(X), T^+(\delta_{U^{-1}})))\) where \(T^+(\delta_U)\) is the topology generated by all sets of the form \(G^{++} = \{A \in \text{CL}_0(X) : \text{there exists } U \in U \text{ such that } U(A) \subseteq G\} \subseteq \text{CL}_0(X)\) and \(G\) is a \(P\)-open set. The topology \(T^+(\delta_{U^{-1}})\) is defined in a similar way by writing \(P\) instead of \(Q\), \(Q\) instead of \(P\) and \(U^{-1}\) instead of \(U\). The pair \((T^+(\delta_U), T^+(\delta_{U^{-1}}))\) is called the double upper \(U\)-proximal topology.

The double lower \(U\)-proximal topological space is defined as the double topological space \(((\text{CL}_0(X), T^-(\delta_U)), (\text{CL}_0(X), T^-(\delta_{U^{-1}})))\) where this double topological space coincides with the double lower Fell topological space. The pair \((T^-(\delta_U), T^-(\delta_{U^{-1}}))\) is called the double lower \(U\)-proximal topology.

The double \(U\)-proximal topological space is defined as the double topological space \(((\text{CL}_0(X), T^+(\delta_U)), (\text{CL}_0(X), T^+(\delta_{U^{-1}})))\) where \(T^+(\delta_U) = T^+(\delta_U) \cap T^-(\delta_U)\) and \(T^+(\delta_{U^{-1}}) = T^+(\delta_{U^{-1}}) \cap T^-(\delta_{U^{-1}})\). The pair \((T^+(\delta_U), T^+(\delta_{U^{-1}}))\) is called the double \(U\)-proximal topology.

Definition 4.11. Let \((X, P, Q)\) be a quasi-uniformizable bitopological space and \(U\) a quasi-uniformity compatible with \((X, P, Q)\). We say that \((X, P, Q)\) has the property pairwise star if

i) given \(A \in \text{CL}_0(X)\) and \(B \in \text{CL}_0(X)\) with \(A_{\delta_U}B\) there exist \(x_1, \ldots, x_n \subseteq X\) and \(U_1, \ldots, U_n \in U\) such that \(A \cap (\cup_{i=1}^n U_{i^{-1}}(x_i)^P) = \emptyset\) and \(B \subseteq \cup_{i=1}^n U_{i^{-1}}(x_i)\).

ii) given \(A \in \text{CL}_0(X)\) and \(B \in \text{CL}_0(X)\) with \(A_{\delta_{U^{-1}}}B\) there exist \(x_1, \ldots, x_n \subseteq X\) and \(U_1, \ldots, U_n \in U\) such that \(A \cap (\cup_{i=1}^n U_i(x_i)^Q) = \emptyset\) and \(B \subseteq \cup_{i=1}^n U_i(x_i)\).

Definition 4.12. Let \((X, U)\) be a quasi-uniform space. We say that it has nice closed balls if every proper set of the form \(U(x)^T(U^{-1})\) or \(U^{-1}(x)^T(U)\) is \(T(U')\)-compact, where \(U \in U\) and \(x \in X\).

Theorem 4.13. Let \((X, P, Q)\) be a quasi-uniformizable bitopological space and \(U\) a quasi-uniformity compatible with the bitopological space. Let us consider \(\{A_\lambda\}_{\lambda \in \Lambda}\) a net in \(\text{CL}_0(X)\) and \(\{B_\gamma\}_{\gamma \in \Gamma}\) a net in \(\text{CL}_0(X)\).

i) If \(A = T^+(\delta_U)\)-lim \(A_\lambda\) and \(B = T^+(\delta_{U^{-1}})\)-lim \(B_\gamma\) then \(A = K_{P^+}\)-lim \(A_\lambda\) and \(B = K_{Q^+}\)-lim \(B_\gamma\).

ii) If \(A = K_{P^+}\)-lim \(A_\lambda\) and \(B = K_{Q^+}\)-lim \(B_\gamma\) implies \(A = T^+(\delta_U)\)-lim \(A_\lambda\) and \(B = T^+(\delta_{U^{-1}})\)-lim \(B_\gamma\) if and only if \((X, P, Q)\) has the property pairwise star and \((X, U)\) has nice closed balls.

Proof. i) Suppose that \(A = T^+(\delta_U)\)-lim \(A_\lambda\) and \(B = T^+(\delta_{U^{-1}})\)-lim \(B_\gamma\). If there exists \(a \in P\cdot LsA_\lambda\), we can find \(U \in U\) such that \(U^{-1}(a) \cap A = \emptyset\). It is easy to prove that \(A \in (\text{int}_PV(A))^{++}\), where \(V \in U\) and \(V^2 \subseteq U\). Therefore, \(A_\lambda \subseteq (\text{int}_PV(A))^{++}\) for all \(\lambda\) in a residual subset of \(\Lambda\). Furthermore,
\[ V(A) \cap V^{-1}(a) = \emptyset. \] On the other hand, since \( a \in P-LsA_\lambda \) we obtain that \( V^{-1}(a) \cap A_\lambda \neq \emptyset \) for a cofinal subset of \( \Lambda \) which is not possible. Therefore, \( LsA_\lambda \subseteq A \). \( B = T^+(\delta_{U,-1})\lim B_\gamma \) implies \( B = K_Q^+\lim B_\gamma \) can be proved in a similar way.

ii) Let us suppose that there exist \( U \in \mathcal{U} \) and \( x \in X \) such that \( \overline{U^{-1}(x)}^P \) is a proper set and is not \( P \vee Q \)-compact. Then, we can find \( y_0 \in X \setminus \overline{U^{-1}(x)}^P \) and a net \( \{x_\lambda\}_{\lambda \in \Lambda} \subseteq \overline{U^{-1}(x)}^P \) such that it does not admit a \( P \vee Q \)-cluster point. We can easily deduce that the net \( \{A_\lambda\}_{\lambda \in \Lambda} \) is \( P \)-Kuratowski-Painlevé upper convergent to \( \overline{\{y_0\}^Q} \), where \( A_\lambda = \overline{\{x_\lambda\}^Q} \cup \overline{\{y_0\}^Q} \) for all \( \lambda \in \Lambda \). On the other hand, if we consider the \( T^+(\delta_U)-\)open set \( (\interior_P V(\overline{\{y_0\}^Q}))+^+ \) where \( V, U_0 \in \mathcal{U} \), \( V^2 \subseteq U_0 \) and \( U_0(\{y_0\}) \cap \overline{U^{-1}(x)}^P = \emptyset \), we have that \( x_\lambda \not\in \interior_P V(\overline{\{y_0\}^Q}) \) for all \( \lambda \in \Lambda \) since if there exists \( z \in \overline{\{y_0\}^Q} \) verifying \( (z, x_\lambda) \in V \), for some \( \lambda \in \Lambda \), we obtain that \( (y_0, x_\lambda) \in U_0 \) which is not possible. Consequently, \( \{A_\lambda\}_{\lambda \in \Lambda} \) is not \( T^+(\delta_U)-\)convergent to \( \overline{\{y_0\}^Q} \). A contradiction. In a similar way, it can be proved that the proper sets of the form \( \overline{U(x)}^Q \) are \( P \vee Q \)-compact. Consequently, \( (X, \mathcal{U}) \) has nice closed balls.

Therefore, \( (X, P, Q) \) is a locally bicompact space. Applying Theorem 4.5, we deduce that the double Fell topology agrees with the double proximal topology which implies (see [15]) that the bitopological space has the property pairwise star.

Conversely, if \( (X, P, Q) \) has the property pairwise star and \( (X, \mathcal{U}) \) has nice closed balls it can be proved (see [15]) that the double upper Fell topology agrees with the double upper proximal topology, and the statement follows directly. \( \square \)

At last, we establish the relationship of the Kuratowski-Painlevé convergence and the convergence in the Vietoris hypertopology.

**Theorem 4.14.** Let \((X, P, Q)\) be a quasi-uniformizable bitopological space. Let us consider \( \{A_\lambda\}_{\lambda \in \Lambda} \) a net in \( CL_Q^0(X) \) and \( \{B_\gamma\}_{\gamma \in \Gamma} \) a net in \( CL_Q^0(X) \). Then

i) If \( A = V_P^+\lim A_\lambda \) and \( B = V_Q^+\lim B_\gamma \) then \( A = K_P^+\lim A_\lambda \) and \( B = K_Q^+\lim B_\gamma \).

ii) \( A = K_P^+\lim A_\lambda \) and \( B = K_Q^+\lim B_\gamma \) implies \( A = V_P^+\lim A_\lambda \) and \( B = V_Q^+\lim B_\gamma \) if and only if \((X, P \vee Q)\) is a compact space.

**Proof.**

i) The proof is similar to the part i) of the above theorem.

ii) Let us suppose that \((X, P \vee Q)\) is not a compact space. Therefore, there exists a net \( \{x_\lambda\}_{\lambda \in \Lambda} \) that does not admit a \( P \vee Q \)-cluster point. If we fix \( y_0 \in X \) it is easy to prove that \( \{\overline{\{x_\lambda\}^P} \cup \overline{\{y_0\}^Q}\}_{\lambda \in \Lambda} \) is \( P \)-Kuratowski-Painlevé upper convergent to \( \overline{\{y_0\}^Q} \). We can also prove that the net \( \{\overline{\{x_\lambda\}^P} \cup \overline{\{y_0\}^P}\}_{\lambda \in \Lambda} \) is \( Q \)-Kuratowski-Painlevé upper convergent to \( \overline{\{y_0\}^P} \).

On the other hand, since \( y_0 \) is not a \( P \vee Q \)-cluster point of \( \{x_\lambda\}_{\lambda \in \Lambda} \), we can find
$U \in \mathcal{U}$ such that $x_\lambda \not\in U^s(y_0)$ for all $\lambda$ in whatever cofinal subset $\Lambda_0$ of $\Lambda$. We can choose $\Lambda_0$ in such a way that we only have to distinguish two possibilities:

i) $x_\lambda \not\in U(y_0)$ for all $\lambda \in \Lambda_0$. Therefore, it is evident that if we consider the $P$-open set $\text{int}_P V((y_0)^Q)$ where $V \in \mathcal{U}$ and $V^2 \subseteq U$, we have that $x_\lambda \not\in V((y_0)^Q)$ for all $\lambda \in \Lambda_0$ so $\{(x_\lambda)^Q \cup (y_0)^Q\}_{\lambda \in \Lambda}$ is not $V_P^+$-convergent to $(y_0)^Q$. A contradiction.

ii) $x_\lambda \not\in U^{-1}(y_0)$ for all $\lambda \in \Lambda_0$. Reasoning as above we obtain that $\{(x_\lambda)^P \cup (y_0)^P\}_{\lambda \in \Lambda}$ is not $V_Q^+$-convergent to $(y_0)^P$. A contradiction.

Conversely, since $(X, P \vee Q)$ is a compact space then $F_P^+ = V_P^+$ on $\text{CL}_Q^Q(X)$ and $F_Q^+ = V_Q^+$ on $\text{CL}_P^P(X)$ (see [15]), so the proof is evident. \qed

We can also define the Kuratowski-Painlevé convergence in a bitopological sense in a different way.

**Definition 4.15.** Let $(X, P, Q)$ be a bitopological space and let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a net of subsets of $X$ and $A \in P(X)$.

We say that $\{A_\lambda\}_{\lambda \in \Lambda}$ is $P$-mixed Kuratowski-Painlevé convergent (resp. $Q$-mixed Kuratowski-Painlevé convergent) to $A$ if $A = K_Q^{-}\text{-lim} A_\lambda$ and $A = K_P^{-}\text{-lim} A_\lambda$ (resp. $A = K_P^{+}\text{-lim} A_\lambda$ and $A = K_Q^{+}\text{-lim} A_\lambda$). We write $A = MKP^{-}\text{-lim} A_\lambda$ (resp. $A = MKQ^{-}\text{-lim} A_\lambda$).

With this definition, we can also wonder under which conditions we can topologize this convergence. For the other definition, the condition was to make the space locally bicompact. We observe that we use this condition only to reconcile the Kuratowski-Painlevé upper convergence with the convergence in the upper Fell topology. So we have that this is an appropriate concept to work with the double Fell topology.

**Proposition 4.16.** Let $(X, P, Q)$ be a locally bicompact bitopological space. Then the mixed Kuratowski-Painlevé convergence agrees with the convergence in the double Fell topology.

Consequently, the concept of Kuratowski-Painlevé convergence is suitable to obtain relationships with the double mixed Fell topology and the topologization of the mixed Kuratowski-Painlevé convergence is the double Fell topology.

It is natural to wonder if we can obtain conditions for the bitopological space $(X, P, Q)$ in order to obtain the coincidence of the Kuratowski-Painlevé convergence and the other Fell topologies defined. We give a positive answer to this question. It is natural to look for other definitions of local compactness in bitopological spaces. The next definition is due to Stoltenberg.

**Definition 4.17 ([22]).** Let $(X, P, Q)$ be a bitopological space. We say that $P$ is locally compact with respect to $Q$ if for all $x \in X$ there exists a $P$-neighborhood $G$ of $x$ such that the $Q$-closure of $G$ is $Q$-compact.

We say that $(X, P, Q)$ is pairwise locally compact if $P$ is locally compact with respect to $Q$ and $Q$ is locally compact with respect to $P$. 

This definition is not suitable here. If we want that our techniques work with this definition, we have to define another upper Fell topology for $P$ considering that this topology is generated by the sets of the form $G^+$ where $G$ is $P$-open and $X \setminus G$ is $P$-compact. The upper Fell topology for $Q$ would be defined in a similar way. But this topology does not give good results.

Taking into account this, we propose the following definition.

**Definition 4.18.** Let $(X, P, Q)$ be a bitopological space. We say that $(X, P, Q)$ is bilocally compact if $(X, P)$ and $(X, Q)$ are locally compact spaces.

It is clear that if $(X, P, Q)$ is locally bicompact then it is bilocally compact. With this definition we have the following obvious result.

**Proposition 4.19.** Let $(X, P, Q)$ be a bitopological space and suppose that $A \in CL^Q_0(X)$ (resp. $A \in CL^P_0(X)$) and that $\{A_\lambda\}_{\lambda \in \Lambda}$ is a net in $CL^Q_0(X)$ (resp. $CL^P_0(X)$). Then

i) If $A = K^+_P\lim A_\lambda$ (resp. $A = K^+_Q\lim A_\lambda$) then $A = FF^+_P\lim A_\lambda$ (resp. $A = FF^+_Q\lim A_\lambda$).

ii) If $(X, P, Q)$ is a bilocally compact pairwise Hausdorff bitopological space and $A = FF^+_P\lim A_\lambda$ (resp. $A = FF^+_Q\lim A_\lambda$) then $A = K^+_P\lim A_\lambda$ (resp. $A = K^+_Q\lim A_\lambda$).

**Proof.** The proof is similar to Theorem 4.5. □

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**References**


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