On topological sequence entropy of circle maps

JOSE S. CÁNOVAS*

ABSTRACT. We classify completely continuous circle maps from the point of view of topological sequence entropy. This improves a result of Roman Hric.

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1. Introduction

Let $(X,d)$ be a compact metric space and let $f : X \to X$ be a continuous map. Denote by $C(X,X)$ the set of continuous maps $f : X \to X$. $(X,f)$ is called a discrete dynamical system. The map $f$ is said chaotic in the sense of Li–Yorke (or simply chaotic) if there is an uncountable set $S \subset X$ such that for any $x, y \in S$, $x \neq y$, it holds that

\begin{align}
\liminf_{n \to \infty} d(f^n(x), f^n(y)) &= 0, \\
\limsup_{n \to \infty} d(f^n(x), f^n(y)) &> 0.
\end{align}

$S$ is said a scrambled set of $f$ (see [10]). When $f$ is chaotic we say that $(X,f)$ is chaotic.

The notion of chaos plays an special role in the setting of discrete dynamical systems. So, some topological invariants have been porposed to give a characterization of chaos. Maybe, the most important topological invariant in this setting is the topological entropy (see [1]). When $X = [a,b], a,b \in \mathbb{R}$, it is well-known that positive topological entropy implies that $f$ is chaotic, while the converse result is false (see [12]).

So, in order to characterize chaotic interval maps we need an extension of topological entropy called topological sequence entropy (see [7]). Given an

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increasing sequence of positive integers $A = (a_i)_{i=0}^\infty$, a number $h_A(f)$ can be associated to each $f \in C(X, X)$. This number is also a topological invariant. Then, defining $h_\infty(f) = \sup_A h_A(f)$, chaotic interval maps can be characterized by the following result.

**Theorem 1.1.** Let $f \in C([a, b], [a, b])$. Then

(a) $f$ is non-chaotic iff $h_\infty(f) = 0$.
(b) $f$ is chaotic with zero topological entropy iff $h_\infty(f) = \log 2$.
(c) $f$ is chaotic with positive topological entropy iff $h_\infty(f) = \infty$.

Theorem 1.1 establishes a complete classification of maps from the point of view of topological sequence entropy. The part (a) was proved by Franzová and Smital in [6]. (a) provides that any chaotic map holds $h_\infty(f) > 0$. In [3] was proved (b) and (c) in case of piecewise monotonic maps. This result was extended to the general case in [4].

Following [7], a map $f \in C(X, X)$ is said null if $h_\infty(f) = 0$. $f$ is said bounded if $h_\infty(f) < \infty$ and unbounded if $h_\infty(f) = \infty$. In the general case, it is unknown when a continuous map is null, bounded or unbounded. It is easy to see that when $f$ is stable in the Lyapunov sense ($f$ has equicontinuous powers) the map is null (see [7]). Theorem 1.1 establishes a characterization of null, bounded and unbounded continuous interval maps.

The aim of this paper is to prove Theorem 1.1 in the setting of continuous circle maps. This will provide a classification of unbounded, bounded and null continuous circle maps. Before starting with this classification, let us point out that for any $f \in C(S^1, S^1)$, Hric proved in [8] that it is non-chaotic iff $h_\infty(f) = 0$, which classifies chaotic circle maps from the point of view of topological sequence entropy.

2. Preliminaries

Let $(X, d)$ be a compact metric space and let $f : X \to X$ be a continuous map. Denote by $C(X, X)$ the set of continuous maps $f : X \to X$. Let $f^0$ be the identity on $X$, $f^1 := f$ and $f^n = f \circ f^{n-1}$ for all $n \geq 1$. Consider an increasing sequence of positive integers $A = (a_i)_{i=1}^\infty$ and let $Y \subseteq X$ and $\varepsilon > 0$. We say that a subset $E \subseteq Y$ is $(A, \varepsilon, n, Y, f)$-separated if for any $x, y \in E, x \neq y$, there is an $i \in \{1, 2, ..., n\}$ such that $d(f^{a_i}(x), f^{a_i}(y)) > \varepsilon$. Denote by $s_n(A, \varepsilon, Y, f)$ the cardinality of any maximal $(A, \varepsilon, n, Y, f)$-separated set. Define

$$s(A, \varepsilon, Y, f) := \limsup_{n \to \infty} \frac{1}{n} \log s_n(A, \varepsilon, Y, f).$$

It is clear from the definition that if $Y_1 \subseteq Y_2 \subseteq X$, then

$$s(A, \varepsilon, Y_1, f) \leq s(A, \varepsilon, Y_2, f).$$

Let

$$h_A(f, Y) := \lim_{\varepsilon \to 0} s(A, \varepsilon, Y, f).$$

The topological sequence entropy of $f$ respect to the sequence $A$ is defined by

$$h_A(f) := h_A(f, X).$$
When $A = (i)_{i=0}^{\infty}$, we receive the classical definition of topological entropy (see [1]).

Finally, let

$$h_{\infty}(f, Y) := \sup_{A} h_{A}(f, Y)$$

and

$$h_{\infty}(f) := \sup_{A} h_{A}(f).$$

An $x \in X$ is periodic if there is an $n \in \mathbb{N}$ such that $f^{n}(x) = x$. The smallest positive integer holding this condition is called the period of $x$. The set of periods of $f$, $P(f)$, is defined by

$$P(f) := \{n \in \mathbb{N} : \exists x \in X \text{ periodic point of period } n\}.$$ 

3. Results on topological sequence entropy

In this section we prove some useful results concerning topological sequence entropy of continuous maps defined on arbitrary compact metric spaces.

**Proposition 3.1.** Let $f \in C(X, X)$. For all $n \in \mathbb{N}$ it holds that $h_{\infty}(f^{n}) = h_{\infty}(f)$.

**Proof.** First, we prove that $h_{\infty}(f^{n}) \leq h_{\infty}(f)$. In order to see this, let $A = (a_{i})_{i=1}^{\infty}$ be an increasing sequence of positive integers and define $nA = (na_{i})_{i=1}^{\infty}$. Then it is straightforward to see that $h_{A}(f^{n}) = h_{nA}(f)$ and hence

$$h_{\infty}(f^{n}) = \sup_{A} h_{A}(f^{n}) = \sup_{A} h_{nA}(f) \leq \sup_{B} h_{B}(f) = h_{\infty}(f).$$

Now, we prove the converse inequality. Let $A$ be an increasing sequence of positive integers. By [8], there is another sequence $B = B(A)$ such that $h_{A}(f) \leq h_{B}(f^{n})$. Then

$$h_{\infty}(f) = \sup_{A} h_{A}(f) \leq \sup_{A} h_{B(A)}(f^{n}) \leq \sup_{A} h_{A}(f^{n}) = h_{\infty}(f^{n}),$$

which ends the proof. \qed

**Corollary 3.2.** Under the conditions of Proposition 3.1, the following statements hold:

(a) $f$ is null iff $f^{n}$ is null for all $n \in \mathbb{N}$.
(b) $f$ is bounded iff $f^{n}$ is bounded for all $n \in \mathbb{N}$.
(c) $f$ is unbounded iff $f^{n}$ is unbounded for all $n \in \mathbb{N}$.

**Proposition 3.3.** Let $f \in C(X, X)$ have positive topological entropy. Then $h_{\infty}(f) = \infty$. 
Proof. Since \( h(f) > 0 \) it follows by [7] that for any increasing sequence of positive integers \( A = (a_i)_{i=1}^{\infty}, \) \( h_A(f) = K(A)h(f), \) where

\[
K(A) = \lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n} \text{Card}\{a_i, a_i + 1, \ldots, a_i + k : 1 \leq i \leq n\}.
\]

Taking \( A = (2^i)_{i=1}^{\infty} \) it holds that \( K(A) = \infty \) and hence \( h_A(f) = \infty. \)

Proposition 3.4. Let \( (X,d) \) and \( (Y,e) \) be compact metric spaces and let \( f : X \to X \) and \( g : Y \to Y \) be continuous maps. Let \( \pi : X \to Y \) be continuous and surjective such that \( \pi \circ f = g \circ \pi. \) Let \( A \) be an increasing sequence of positive integers \( A \) and let \( Y_1 \subseteq Y. \) Then, for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
s(A, \delta, \pi^{-1}(Y_1), f) \geq s(A, \varepsilon, Y_1, g).
\]

In particular, \( h_{\infty}(f) \geq h_{\infty}(g). \)

Proof. Let \( E \subseteq Y_1 \) be a maximal subset \( (A, n, \varepsilon, Y_1, g) \)-separated. Let \( F \subseteq \pi^{-1}(Y_1) \) be a set containing exactly one element from \( \pi^{-1}(y) \) for all \( y \in E. \)

We claim that \( F \) is an \( (A, n, \delta, \pi^{-1}(Y_1), f) \)-separated subset for some \( \delta > 0. \)

Assume the contrary. Since \( \pi \) is uniformly continuous, there is a \( \delta = \delta(\varepsilon) > 0 \) such that \( d(x_1, x_2) < \delta, x_1, x_2 \in X, \) implies \( e(\pi(x_1), \pi(x_2)) < \varepsilon. \) Now let \( x_1, x_2 \in F \) be such that

\[
d(f^{\delta_i}(x_1), f^{\delta_i}(x_2)) < \delta
\]

for all \( i \in \{1, 2, \ldots, n\}. \) Let \( y_1, y_2 \in E \) be such that \( \pi(x_j) = y_j \) for \( j = 1, 2. \) Then, for all \( i \in \{1, 2, \ldots, n\} \) we have that

\[
e(g^{\delta_i}(y_1), g^{\delta_i}(y_2)) = e(g^{\delta_i}(\pi(x_1)), g^{\delta_i}(\pi(y_2)))
\]

\[
= e(\pi \circ f^{\delta_i}(x_1), \pi \circ f^{\delta_i}(y_2)) \leq \varepsilon,
\]

which leads us to a contradiction. Then \( s_n(A, \delta, \pi^{-1}(Y_1), f) \geq s_n(A, \varepsilon, Y_1, f) \)

and hence

\[
s(A, \delta, \pi^{-1}(Y_1), f) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(A, \delta, \pi^{-1}(Y_1), f)
\]

\[
\geq \limsup_{n \to \infty} \frac{1}{n} \log s_n(A, \varepsilon, Y_1, g)
\]

\[
= s(A, \varepsilon, Y_1, g),
\]

which ends the proof. \( \Box \)

Under the conditions of Proposition 3.4, if \( \pi \) is an homemorphism, then \( f \) and \( g \) are said to be conjugate. Then

Corollary 3.5. Under the conditions of Proposition 3.4, if \( f \) and \( g \) are conjugate, then \( h_{\infty}(f) = h_{\infty}(g). \)
4. Main results

In the sequel we will discuss the space of continuous circle maps denoted by $C(S^1, S^1)$. Let $f \in C(S^1, S^1)$ and let $l : \mathbb{R} \to S^1$ be defined by $l(x) = \exp(2\pi i x)$ for all $x \in \mathbb{R}$. Then, there are a countable number of continuous maps $F : \mathbb{R} \to \mathbb{R}$ such that $l \circ F = f \circ l$. An $F$ holding this condition is called a lifting of $f$. If $F$ is another lifting of $f$, then

$$F - F = k \in \mathbb{N}.$$ 

By $|J|$ we denote the length of an interval $J \subseteq \mathbb{R}$.

**Theorem 4.1.** Let $f \in C(S^1, S^1)$. Then

(a) $f$ is non-chaotic iff $h_\infty(f) = 0$.

(b) $f$ is chaotic with zero topological entropy iff $h_\infty(f) = \log 2$.

(c) $f$ is chaotic with positive topological entropy iff $h_\infty(f) = \infty$.

**Proof.** According to Chapter 3 from [2], $C(S^1, S^1)$ can be decomposed into the following classes:

\begin{align*}
C_1 &= \{f \in C(S^1, S^1) : f \text{ has no periodic points}\}; \\
C_2 &= \{f \in C(S^1, S^1) : P(f^n) = \{1\} \text{ or } P(f^n) = \{1, 2, 2^2, \ldots\} \text{ for some } n \in \mathbb{N}\}; \\
C_3 &= \{f \in C(S^1, S^1) : P(f^n) = \mathbb{N} \text{ for some } n \in \mathbb{N}\}.
\end{align*}

According to [8], any $f \in C_1$ is non-chaotic and holds that $h_\infty(f) = 0$. Let $f \in C_2$. Again by [8], it holds that $f$ is chaotic and $h(f) > 0$. Then, by Proposition 3.3 we have that $h_\infty(f) = \infty$. So, we must consider only maps from $C_2$.

Let $f \in C_2$ and let $n \in \mathbb{N}$ be such that $P(f^n) = \{1\} \text{ or } P(f^n) = \{1, 2, 2^2, \ldots\}$. It is well-known that $f$ is chaotic iff $f^n$ is chaotic. So, applying Proposition 3.1, it is not restrictive to assume that $n = 1$. Since $f$ has a fixed point, by Lemma 2.5 from [9], there is a lifting $F : \mathbb{R} \to \mathbb{R}$ and there is a compact interval $J$, with $|J| > 1$, such that $F(J) = J$. For the rest of the proof call $l = l|_J$. First assume that $f$ is non-chaotic. Then by [8] it holds that $h_\infty(f) = 0$. Secondly, assume that $f$ is chaotic. Hence $F$ is also chaotic (see [8]) and has zero topological entropy (see [12]). By Proposition 3.4, for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

\begin{align*}
s(A, \varepsilon, S^1, f) \leq s(A, \delta, l^{-1}(S^1), F) &= s(A, \delta, J, F).
\end{align*}

On the other hand, by [4], there is a compact interval $J_i \subseteq J$, holding that $F^2(i) = J_i \text{ such that}$

\begin{align*}
s(A, \delta, J, F) \leq s(A, \delta/6, \bigcup_{j=1}^{2^i} F^j(J_i), F) \leq \log 2.
\end{align*}

By (4.16) and (4.17) we conclude that

\begin{align*}
s(A, \varepsilon, S^1, f) \leq \log 2.
\end{align*}

Since $\varepsilon$ was arbitrary chosen, we obtain

\begin{align*}
h_A(f) \leq \log 2,
\end{align*}
and

\[(4.20) \quad h_\infty(f) = \sup_A h_A(f) \leq \log 2.\]

Now, we prove the converse inequality. By [12], there is a compact interval \(J_i\), with \(|J_i| < 1\) and \(F^{2^i}(J_i) = J_i\) such that \(F^{2^i}|_{J_i}\) is chaotic. By [6], there is an increasing sequence of positive integers \(B\) such that \(s(B, \varepsilon, J_i, F^{2^i}) \geq \log 2\) for a suitable \(\varepsilon > 0\). Since \(l|_{J_i} : J_i \to I(J_i)\) is an homeomorphism, we can apply Proposition 3.4 to \(l|_{J_i} = l\) to obtain a \(\delta > 0\) such that

\[(4.21) \quad s(A, \delta, l(J_i), f^{2^i}) \geq s(A, \varepsilon, J_i, F^{2^i}) \geq \log 2.\]

Hence

\[(4.22) \quad h_\infty(f) \geq h_{2^i,A}(f) = h_A(f^{2^i}) \geq s(A, \delta, l(J_i), f^{2^i}) \geq \log 2,\]

which concludes the proof. \(\square\)

**Remark 4.2.** When two-dimensional maps are concerned, Theorems 1.1 and 4.1 are false in general. More precisely, in [11] and [5] a chaotic map \(F \in C([0,1]^2, [0,1]^2)\) with \(h_\infty(F) = 0\) and a non chaotic map \(G \in C([0,1]^2, [0,1]^2)\) holding \(h_\infty(G) > 0\) have been constructed. It seems that the dimension of the space \(X\) plays a special role in Theorems 1.1 and 4.1. We conjecture that Theorem 1.1 remains true for continuous maps defined on finite graphs, that is, in the special setting of one-dimensional dynamics.

**References**


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José S. Cánovas
Dep. de Matemática Aplicada y Estadística
Universidad Politécnica de Cartagena
30203 Cartagena
Spain
E-mail address: Jose.Canovas@upct.es