Topological groups with dense compactly generated subgroups

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ABSTRACT. A topological group $G$ is: (i) compactly generated if it contains a compact subset algebraically generating $G$, (ii) $\sigma$-compact if $G$ is a union of countably many compact subsets, (iii) $\aleph_0$-bounded if arbitrary neighborhood $U$ of the identity element of $G$ has countably many translates $xU$ that cover $G$, and (iv) finitely generated modulo open sets if for every non-empty open subset $U$ of $G$ there exists a finite set $F$ such that $F \cup U$ algebraically generates $G$. We prove that: (1) a topological group containing a dense compactly generated subgroup is both $\aleph_0$-bounded and finitely generated modulo open sets, (2) an almost metrizable topological group has a dense compactly generated subgroup if and only if it is both $\aleph_0$-bounded and finitely generated modulo open sets, and (3) an almost metrizable topological group is compactly generated if and only if it is $\sigma$-compact and finitely generated modulo open sets.

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1. Preliminaries

All topological groups in this article are assumed to be $T_1$ (and thus Tychonoff). For subsets $A$ and $B$ of a group $G$ let $AB = \{ab : a \in A \text{ and } b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$. For $a \in A$ and $b \in B$ we write $aB$ or $Ab$ rather than $\{a\}B$ or $A\{b\}$. If $A$ is a subset of a group $G$, then the smallest subgroup of $G$ that contains $A$ is denoted by $\langle A \rangle$.

Recall that a topological group $G$ is said to be:

(i) compactly generated if $G = \langle K \rangle$ for some compact subspace $K$ of $G$,

(ii) $\sigma$-compact provided that there exists a sequence $\{K_n : n \in \omega\}$ of compact subsets of $G$ such that $G = \bigcup \{K_n : n \in \omega\}$,
(iii) $\aleph_0$-bounded if for every neighborhood $U$ of the unit element there exists a countable set $S \subset G$ such that $US = G$ ([2]),

(iv) totally bounded if for every neighborhood $U$ of the unit element there exists a finite set $S \subset G$ such that $US = G$,

(v) finitely generated modulo open sets if for every non-empty open set $U \subseteq G$, there exists a finite set $F \subseteq G$ such that $\langle F \cup U \rangle = G$ ([1]).

Clearly, compactly generated groups are $\sigma$-compact. It is well-known that $\sigma$-compact groups, separable groups and their dense subgroups are $\aleph_0$-bounded ([2]).

2. The results

The main purpose of this note is to study the following question: When does a topological group contain a dense compactly generated subgroup? Our first result provides two necessary conditions:

**Theorem 2.1.** If a topological group $G$ contains a dense compactly generated subgroup, then $G$ is both $\aleph_0$-bounded and finitely generated modulo open sets.

**Proof.** Let $G$ be a topological group and $K$ be its compact subset such that $\langle K \rangle$ is dense in $G$. Then $G$ is $\aleph_0$-bounded ([2]), so it remains only to show that $G$ is finitely generated modulo open sets. Given a non-empty open set $U$, the group $G$ is divided into pairwise disjoint left-congruence classes modulo its subgroup $\langle U \rangle$. Let $X$ be a complete set of representatives of these congruence classes: $G = \bigcup_{x \in X} x\langle U \rangle$. Since each congruence class is an open set, finite number of those classes must cover the compact set $K$. Therefore there is a finite set $F \subset X$ such that $F \langle U \rangle \supseteq K$. Since $\langle K \rangle$ is dense in $G$, it follows that $G = U\langle K \rangle \subseteq U\langle F \langle U \rangle \rangle \subseteq \langle F \cup U \rangle \subseteq G$. □

In our future arguments we will make use of the following easy lemma:

**Lemma 2.2.** Let $X$ be a topological space. Let $K \subset X$ be a compact set with a neighborhood base $\{U_n\}_{n \in \omega}$. Suppose that we have compact sets $C_n \subset \bigcap_{k \leq n} U_k$ for all $n \in \omega$. Then the set $C = K \cup \bigcup_{n \in \omega} C_n$ is also compact.

A topological group $G$ is almost metrizable if there exist a non-empty compact set $K \subset G$ and a sequence $\{U_n\}_{n \in \omega}$ of open subsets of $G$ such that (1) $K \subset U_n$ for all $n \in \omega$ and (2) if $O$ is an open set containing $K$, then there is an $n \in \omega$ such that $K \subset U_n \subset O$. (Such a sequence $\{U_n\}_{n \in \omega}$ is called a neighborhood base of $K$ in $G$.) Both metric groups and locally compact groups are almost metrizable ([3]).

Our next theorem demonstrates that the necessary conditions for a topological group $G$ to have a dense compactly generated subgroup found in Theorem 2.1 are also sufficient in case $G$ is almost metrizable.

**Theorem 2.3.** An almost metrizable topological group $G$ contains a dense compactly generated subgroup if and only if it is $\aleph_0$-bounded and finitely generated modulo open sets.
Proof. The “only if” part of our theorem follows from Theorem 2.1, so it remains only to prove the “if” part. Let $K$ be a compact subset of $G$ with a neighborhood base $\{U_n\}_{n \in \omega}$. Since $G$ is $\aleph_0$-bounded, for each $n \in \omega$ there is a countable set $S_n \subseteq G$ such that $G = S_n U_n$. The set $S = \bigcup_{n \in \omega} S_n$ is countable, so we can fix its enumeration $S = \{s_n\}_{n \in \omega}$. Let $g \in G$. Let $V$ be any neighborhood of the unit element of $G$. Then $KV^{-1}$ is an open set containing $K$, and so there is an $n \in \omega$ such that $U_n \subseteq KV^{-1}$. Since $S_n U_n = G$, there is an $s \in S_n$ such that $g \in s U_n \subseteq s KV^{-1}$. Let $g = skv^{-1}$ with $k \in K$ and $v \in V$. Then $gv = sk \in gV \cap SK \neq \emptyset$. Since $V$ and $g$ are arbitrary, it follows that $SK$ is dense in $G$. Since $G$ is finitely generated modulo open sets, there are finite sets $F_n$ such that $G = \langle F_n \cup U_n \rangle$ for each $n \in \omega$. Set $E_0 = F_0 \cup \{s_0\}$. It follows that $G = \langle E_0 \cup U_0 \rangle$. So there is a finite set $E_1 \subseteq U_0$ such that $F_1 \cup \{s_1\} \subseteq \langle E_0 \cup E_1 \rangle$. From this it follows that $\langle E_0 \cup E_1 \cup U_1 \rangle = G$. So there is a finite set $E_2 \subseteq U_1$ such that $F_2 \cup \{s_2\} \subseteq \langle E_0 \cup E_1 \cup E_2 \rangle$. In this way we obtain finite sets $E_{n+1} \subseteq U_n$ (for $n \in \omega$) such that $F_{n+1} \cup \{s_{n+1}\} \subseteq \langle E_0 \cup E_1 \cup \cdots \cup E_{n+1} \rangle$. By Lemma 2.2, the set $C = K \cup \bigcup_{n \in \omega} E_n$ is compact. The subgroup $\langle C \rangle$ is dense, since it contains $SK$. Thus $G$ contains a compactly generated dense subgroup. \hfill \square

Since every metrizable group is almost metrizable ([3]), and $\aleph_0$-boundedness is equivalent to separability for metrizable groups, from Theorem 2.3 we obtain:

Corollary 2.4. A metrizable group contains a dense compactly generated subgroup if and only if it is separable and finitely generated modulo open sets.

Our next result generalizes Theorem 4 from [1].

Lemma 2.5. If a $\sigma$-compact almost metrizable group $G$ contains a dense compactly generated subgroup, then $G$ itself is compactly generated.

Proof. Suppose $G = \bigcup_{n \in \omega} L_n$, with $L_n$ compact. Suppose also that $H = \langle L_0 \rangle$ is dense in $G$. Let $K \subseteq G$ be a compact set with a neighborhood base $\{U_n\}_{n \in \omega}$. By regularity of the topology of $G$ and compactness of $K$, we may assume without loss of generality that each $U_n$ contains the closure of $U_{n+1}$. By compactness of $L_n$ and denseness of $H$, there is a finite subset $F_n$ of $H$ such that $L_n \subseteq U_{n+1} F_n$. Let $C_n = L_n F_n^{-1} \cap U_{n+1}$. Then $C_n$ is compact, because it is a closed subset of the union of finitely many copies of $L_n$. We also have $C_n \subseteq U_n$ and $L_n \subseteq C_n F_n \subseteq \langle C_n \cup L_0 \rangle$. Therefore, setting $C = L_0 \cup K \cup \bigcup_{n \in \omega} C_n$, we obtain $\langle C \rangle = G$. It follows from Lemma 2.2 that $C$ is compact. \hfill \square

Combining Theorem 2.3 and Lemma 2.5, we obtain our next theorem which extends the main result of [1]:

Theorem 2.6. An almost metrizable topological group is compactly generated if and only if it is $\sigma$-compact and finitely generated modulo open sets.

Theorems 2.3 and 2.6 become especially simple for locally compact groups:

Theorem 2.7. For a locally compact group $G$ the following conditions are equivalent:
(i) $G$ has a dense compactly generated subgroup,
(ii) $G$ is compactly generated,
(iii) $G$ is finitely generated modulo open sets.

Proof. Let $U$ be an open neighbourhood of the identity element having compact closure $\overline{U}$.

(i)→(ii). Let $K$ be a compact subset of $G$ such that $\langle K \rangle$ is dense in $G$. Then $\overline{U} \cup K$ is also compact and $\langle \overline{U} \cup K \rangle \supseteq U \langle K \rangle = G$ because $\langle K \rangle$ is dense in $G$ and $U$ is an open neighbourhood of the identity.

(ii)→(iii) follows from Theorem 2.1.

(iii)→(i). Assume that $G$ is finitely generated modulo open sets. Then there exists a finite set $F \subseteq G$ with $\langle F \cup U \rangle = G$. Now note that $G = \langle F \cup U \rangle \subseteq \langle F \cup \overline{U} \rangle \subseteq G$. Since $\langle F \cup \overline{U} \rangle$ is compact, $G$ is compactly generated. \hfill \Box

Since for every non-empty open subset $U$ of a topological group $G$ the set $\langle U \rangle$ is an open subgroup of $G$, it follows that a topological group without proper open subgroups is finitely generated modulo open sets ([1]). Therefore, from Theorem 2.6 we obtain

Corollary 2.8. An almost metrizable, $\sigma$-compact group without proper open subgroups is compactly generated.

Corollary 2.9. A metric $\sigma$-compact group without proper open subgroups is compactly generated.

Totally bounded groups are finitely generated modulo open sets, and so we get

Corollary 2.10. Every $\sigma$-compact totally bounded almost metrizable group is compactly generated.

References


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