On paracompact spaces and projectively inductively closed functors

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ABSTRACT. In this paper we introduce a notion of projectively inductively closed functor (p.i.c.-functor). We give sufficient conditions for a functor to be a p.i.c.-functor. In particular, any finitary normal functor is a p.i.c.-functor. We prove that every preserving weight p.i.c.-functor of a finite degree preserves the class of stratifiable spaces and the class of paracompact \(\sigma\)-spaces. The same is true (even if we omit a preservation of weight) for paracompact \(\Sigma\)-spaces and paracompact \(p\)-spaces.

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1. Introduction

By \(Tych\) we denote the category of all Tychonoff spaces and all their continuous functions. A Hausdorff compact space is called a compact space or just a compactum. By \(Comp\) we denote the full subcategory of \(Tych\), whose objects are compacta.

Recall that a covariant functor \(F: Comp \to Comp\) is said to be normal \([17]\) if it satisfies the following properties:

1. preserves the empty set and singletons, i.e., \(F(\emptyset) = \emptyset\) and \(F(\{1\}) = \{1\}\), where \(\{k\} (k \geq 0)\) denotes the set \(\{0, 1, \ldots, k-1\}\) of nonnegative integers smaller than \(k\). In this notation \(0 = \{\emptyset\}\).
2. is monomorphic, i.e., for any (topological) embedding \(f: A \to X\), the mapping \(F(f): F(A) \to F(X)\) is also an embedding.
3. is epimorphic, i.e., for any surjective mapping \(f: X \to Y\), \(F(f): F(X) \to F(Y)\) is also surjective.
4. continuous, i.e., for any inverse spectrum \(S = \{X_\alpha; \pi^\alpha_\beta: \alpha \in A\}\) of compact spaces, the limit \(f: F(\lim S) \to \lim F(S)\) of the mappings \(F(\pi_\alpha)\),
where $\pi_\alpha : \lim S \to X_\alpha$ are the limiting projections of the spectrum $S$, is a homeomorphism.

(5) preserves intersections, i.e., for any family $\{A_\alpha : \alpha \in A\}$ of closed subsets of a compact space $X$, the mapping $F(i) : \cap \{F(A_\alpha) : \alpha \in A\} \to F(X)$ defined by $F(i)(x) = F(i_\alpha)(x)$, where $i_\alpha : A_\alpha \to X$ is the identity embeddings for all $\alpha \in A$, is an embedding.

(6) preserves preimages, i.e., for any mapping $f : X \to Y$ and an arbitrary closed set $A \subset Y$, we have $F(f^{-1}(A)) = (F(f))^{-1}(F(A))$.

(7) preserves weight, i.e., $w(F(X)) = w(X)$ for any infinite compactum $X$.

In what follows we shall use bigger than normal classes of functors. But any of them shall preserve empty set, intersections and be monomorphic. By $\exp$ we denote the well-known hyperspace functor of non-empty closed subsets. This functor takes every (nonempty) compact space $X$ to the set of all its nonempty closed subsets endowed with the (finite) Vietoris topology (see [9]), and a continuous mapping $f : X \to Y$ to the mapping $\exp(f) : \exp(X) \to \exp(Y)$, defined by $F(f)(A) = A$.

For a functor $F$ and an element $a \in F(X)$, the support of $a$ is defined as intersection of all closed sets $A \subset X$ such that $a \in F(A)$ (recall that we consider only monomorphic functors preserving intersections). This support we denote by $\text{supp}_{F(X)}(a)$. When it is clear what functor and space are meant, we denote the support of $a$ merely by $\text{supp}(a)$.

A. Ch. Chigogidze [7] extended an arbitrary intersection-preserving monomorphic functor $F : \text{Comp} \to \text{Comp}$ to the category $\text{Tych}$ by setting

$$F_{\beta}(X) = \{a \in F(\beta X) : \text{supp}(a) \subset X\}$$

for any Tychonoff space $X$. If $f : X \to Y$ is a continuous mapping of Tychonoff spaces and $\beta f : \beta X \to \beta Y$ is the (unique) extension of $f$ over their Stone-Čech compactifications, then

$$F(\beta f)(F(\beta X)) \subset F_{\beta}(X).$$

The last inclusion is a corollary of a trivial fact

$$f(\text{supp}(a)) \supset \text{supp}(F(f)(a)).$$

Therefore, we can define the mapping

$$F_{\beta}(f) = F(\beta f)|X,$$

which makes $F_{\beta}$ into a functor.

A. Ch. Chigogidze proved [7] that if a functor $F$ has certain normality property, then $F_{\beta}$ has the same property (modified when necessary). In what follows by a covariant functor $F : \text{Tych} \to \text{Tych}$ we shall mean a functor of type $F_{\beta}$. For such a functor $F$ and any compact space $X$ the space $F(X)$ is a compact space.

For a set $A$ by $|A|$ we denote the cardinality of $A$. For a subset $A$ of a space $X$ by $\overline{A}^X$ we denote the closure of $A$ in $X$. 
In this paper we introduce a notion of *projectively inductively closed* functor (p.i.c.-functor). We give sufficient conditions for a functor to be a p.i.c.-functor (Theorem 3.5). In particular, any finitary normal functor is a p.i.c.-functor (Corollary 3.6). We prove that every preserving weight p.i.c.-functor of a finite degree preserves the class of stratifiable spaces and the class of paracompact $\sigma$-spaces (Theorem 3.7). The same is true (even if we omit a preservation of weight) for paracompact $\Sigma$-spaces and paracompact $p$-spaces (Theorem 3.8).

All spaces are assumed to be Tychonoff, and all mappings, continuous. Any additional information on general topology and covariant functors one can find, for example, in ([8], [9], [17]).

2. Preliminaries

In this section we recall some definitions and facts, which will be useful in establishing our main results (see Section 3).

**Definition 2.1** ([2]). A *network* for a space $X$ is a collection $\mathcal{N}$ of subsets of $X$ such that whenever $x \in U$ with $U$ open, there exists $F \in \mathcal{N}$ with $x \in F \subset U$.

An elementary corollary of this definition is that every base of a space $X$ is a network of $X$.

A family $\mathcal{A}$ of subsets of $X$ is said to be *$\sigma$-locally finite* if it is a union of countably many families $\mathcal{A}_n$ which are locally finite in $X$.

**Definition 2.2** ([16]). A topological space $X$ is called a *$\sigma$-space*, if it has a $\sigma$-locally finite network.

**Remark 2.3.** A rather simple observation of the definition 2.2 shows us that every closed subset of a $\sigma$-space is a $\sigma$-space.

**Proposition 2.4** ([11]). *Every closed image of a $\sigma$-space is a $\sigma$-space.*

A well-known theorem of E. Michael [13] states that every closed image of a paracompact space is a paracompact space. So, from Proposition 2.4 we get

**Theorem 2.5** ([11]). *Every closed image of a paracompact $\sigma$-space is a paracompact $\sigma$-space.*

**Theorem 2.6** ([11]). *A countable product of paracompact $\sigma$-spaces is a paracompact $\sigma$-space.*

In 1969 K. Nagami [15] introduced more general class than class of $\sigma$-spaces.

**Definition 2.7.** A space $X$ is a *$\Sigma$-space* if there exists a $\sigma$-discrete collection $\mathcal{N}$, and a cover $c$ of $X$ by closed countably compact sets such that, whenever $C \in c$ and $C \subset U$ with open $U$, then $C \subset F \subset U$ for some $F \in \mathcal{N}$.

Clearly, from Definitions 2.1 and 2.7 we have.

**Proposition 2.8.** *Every perfect preimage of a $\sigma$-space is a $\Sigma$-space. In particular, every $\sigma$-space is a $\Sigma$-space.*

**Proposition 2.9.** *Every closed subspace $Y$ of a $\Sigma$-space $X$ is a $\Sigma$-space.*
Indeed, evidently, that the families \( \mathcal{N} \mid Y \) and \( c \mid Y \), where \( \mathcal{N} \) and \( c \) are from Definition 2.7, satisfy Definition 2.7 for \( Y \).

K. Nagami [15] has shown that the class of \( \Sigma \)-spaces is strictly larger than the class of perfect preimages of \( \sigma \)-spaces. On the other hand, the class of perfect preimages of \( \sigma \)-spaces is much larger than the class of \( \sigma \)-spaces. For example, every compact \( \sigma \)-space is metrizable.

Paracompact \( \Sigma \)-spaces behave nicely with respect to countable products and perfect images.

**Proposition 2.10** ([15]). The countable product of paracompact \( \Sigma \)-spaces is a paracompact \( \Sigma \)-space.

**Proposition 2.11** ([15]). Every perfect image of a paracompact \( \Sigma \)-space is a paracompact \( \Sigma \)-space.

The class of paracompact \( p \)-spaces in sense of A. V. Arhangel’-skii is a proper subclass of paracompact \( \Sigma \)-spaces.

**Definition 2.12** ([3]). A space \( X \) is called a \( p \)-space if there exists a countable family \( u_n \) such that:

1) \( u_n \) consists of open subsets of \( \beta X \);
2) \( X \subset \cup u_n \) for each \( n \);
3) \( \cap_n st(x, u_n) \subset X \) for every \( x \in X \).

Here for a family \( v \) of subsets of a space \( Y \) by \( st(y, v) \) we denote the set \( \cup \{ V \in v : y \in V \} \).

**Theorem 2.13** ([3]). The class of paracompact \( p \)-spaces coincides with the class of perfect preimages of metrizable spaces.

**Corollary 2.14.** Every paracompact \( p \)-space is a perfect preimage of a paracompact \( \sigma \)-space and, consequently, is a paracompact \( \Sigma \)-space.

Theorem 2.13 also yields

**Corollary 2.15** ([3]). Every countable product of paracompact \( p \)-spaces is a paracompact \( p \)-space.

**Proposition 2.16** ([3]). Every closed subspace of a paracompact \( p \)-space is a paracompact \( p \)-space.

**Theorem 2.17** ([10]). Every perfect image of a paracompact \( p \)-space is a paracompact \( p \)-space.

Let us recall some more notions and facts.

**Definition 2.18** ([6]). A space \( X \) is stratifiable if there is a function \( G \) which assigns to each \( n \in \omega \) and closed set \( H \subset X \), an open set \( G(n, H) \) containing \( H \) such that

1) if \( H \subset K \), then \( G(n, H) \subset G(n, K) \);
2) \( H = \cap_n G(n, H) \).
The class of stratifiable spaces was defined in 1961 by J.Ceder [6]. But he called these spaces by $M_3$-spaces. The latter form was proposed by C.R. Borges [5] in 1966.

In the definition of a stratifiable space we can also use the following additional condition:

(3) $G(n + 1, H) \subset G(n, H)$.

Indeed, we define new stratification $G'$ by

$$G'(n, H) = \cap_{i \leq n} G(i, H).$$

The following dual characterization of stratifiable spaces is sometimes useful:

$X$ is stratifiable if and only if for each open $U \subset X$ and $n \in \omega$ one can assign an open set $U_n$ such that $U_n \subset U$, $U = \cup_n U_n$ and $U \subset V$ implies $U_n \subset V_n$.

To get this characterization from a function $G$ satisfying definition 2.18, let $U_n = X \backslash G(n, X \backslash U)$. On the other hand, to get $G$ from $U_n$’s let $G(n, X) = X \backslash \left( \bigcup_n (X \backslash H) \right)_n$.

From this characterization of stratifiable spaces and Michael’s theorem [14] characterizing a paracompact space by $\sigma$-cushioned coverings, we get

**Theorem 2.19** ([6]). Stratifiable spaces are paracompact.

**Corollary 2.20** ([5]). Stratifiable spaces are perfectly normal.

Indeed, every stratifiable space is normal in view of Theorem 2.19. On the other hand, each closed subset of $X$ is a $G_\delta$-set by Definition 2.18.

**Theorem 2.21** ([12]). Every stratifiable space is a $\sigma$-space.

As a corollary of Theorems 2.19 and 2.21 we get

**Theorem 2.22.** Every stratifiable space is a paracompact $\sigma$-space.

**Theorem 2.23** ([5]). Every subspace of a stratifiable space is stratifiable.

**Theorem 2.24** ([5]). A countable product of stratifiable spaces is stratifiable.

**Theorem 2.25** ([5]). Stratifiable spaces are preserved by closed mappings.

From Theorem 2.25 we get

**Corollary 2.26.** An image of a metrizable space under closed mapping is stratifiable. In particular, every metrizable space is stratifiable.

Going back to functors $F: \text{Comp} \to \text{Comp}$, we, evidently, have

(2.1) $a \in F(\text{supp}(a))$.

If a functor $F$ preserves preimages, then $F$ preserves supports [17], i.e.

(2.2) $f(\text{supp}(a)) = \text{supp}(F(f)(a))$.

The property (2.2) can be conversed.

**Proposition 2.27** ([17]). Any monomorphic preserving intersections functor preserves supports if and only if it preserves preimages.
Definition of the functor $\mathcal{F}$ and property (2.2) imply that
\begin{equation}
(2.3) \quad f(\text{supp}_{\mathcal{F}}(X))(a) = \text{supp}_{\mathcal{F}^Y}(Y) \mathcal{F}_Y(f)(a)
\end{equation}
for any preimage preserving functor $\mathcal{F}: \text{Comp} \to \text{Comp}$, continuous mapping $f: X \to Y$, and $a \in \mathcal{F}_Y(X)$.

Now we recall one construction given by V. N. Basmanov [4]. Let $\mathcal{F}: \text{Comp} \to \text{Comp}$ be a functor. By $C(X,Y)$ we denote the space of all continuous mappings from $X$ to $Y$ with compact-open topology.

In particular, $C(\{k\},Y)$ is naturally homeomorphic to the $k$-th power $Y^k$ of the space $Y$; the homeomorphism takes each mapping $\xi: \{k\} \to Y$ to the point $(\xi(0), \ldots, \xi(k-1)) \in Y^k$.

For a functor $\mathcal{F}$, compact space $X$, and a positive integer $k$, V. N. Basmanov [4] defined the mapping
\[
\pi_{\mathcal{F},X,k}: C(\{k\},X) \times \mathcal{F}(\{k\}) \to \mathcal{F}(X)
\]
by
\[
\pi_{\mathcal{F},X,k}(\xi,a) = \mathcal{F}(\xi)(a)
\]
for any $\xi \in C(\{k\},X)$ and $a \in \mathcal{F}(\{k\})$.

When it is clear what functor $\mathcal{F}$ and what space $X$ are meant, we omit the subscripts $\mathcal{F}$ and $X$ and write $\pi_{X,k}$ or $\pi_k$ instead of $\pi_{\mathcal{F},X,k}$.

According to Shcepin’s theorem ([17], Theorem 3.1), the mapping
\[
\mathcal{F}: C(Z,Y) \to \mathcal{F}(\mathcal{F}(Z),\mathcal{F}(Y))
\]
is continuous for any continuous functor $\mathcal{F}$ and compact spaces $Z$ and $Y$. This implies the following assertion.

**Proposition 2.28 ([4]).** If $\mathcal{F}$ is a continuous functor, $X$ is a compact space, and $k$ is a positive integer, then the mapping $\pi_{\mathcal{F},X,k}$ is continuous.

Let $\mathcal{F}_k$ be a subfunctor of a functor $\mathcal{F}$ defined as follows. For a compact space $X$, $\mathcal{F}_k(X)$ is the image of the mapping $\pi_{\mathcal{F},X,k}$ and for a mapping $f: X \to Y$, $\mathcal{F}_k(f)$ is the restriction of $\mathcal{F}(f)$ to $\mathcal{F}_k(X)$. Denote by $\mathcal{T}: C(\{k\},X) \to (C\{k\},Y)$ the mapping which takes $\xi$ to composition $f \circ \xi$. It is easy to see that
\begin{equation}
(2.4) \quad \pi_{Y,k} \circ \mathcal{T} \times \text{id}_{\mathcal{F}(\{k\})} = \mathcal{F}(f) \circ \pi_{X,k}.
\end{equation}

Therefore, $\mathcal{F}(f)(\mathcal{F}_k(X)) \subset \mathcal{F}_k(Y)$. Hence, $\mathcal{F}_k$ is a functor.

A functor $\mathcal{F}$ is called a **functor of degree** $n$, if $\mathcal{F}_n(X) = \mathcal{F}(X)$ for any compact space $X$, but $\mathcal{F}_{n-1}(X) \neq \mathcal{F}(X)$ for some $X$. The next assertion (Proposition 2.29) is Shcepin’s definition of the functor $\mathcal{F}_k$. But using Basmanov’s definition we should prove it. One can find the proof in [28].

**Proposition 2.29.** For any continuous functor $\mathcal{F}$ and a compact space $X$, we have
\[
\mathcal{F}_k(X) = \{a \in \mathcal{F}(X) : |\text{supp}(a)| \leq k\}.
\]

**Corollary 2.30.** For any compact space $X$, we have
\[
\text{exp}_k(X) = \{a \in \exp(X) : |a| \leq k\}.
\]
The definition of a support and the property (2.1) imply.

**Proposition 2.31.** For a functor $\mathcal{F}$, a compact space $X$, and a closed subset $A$ of $X$,

\[ \mathcal{F}(A) = \{ a \in \mathcal{F}(X) : \text{supp}(a) \subseteq A \}. \]

For a Tychonoff space $X$, a functor $\mathcal{F} : \text{Comp} \to \text{Comp}$, and a positive integer $k$, we put

\[ \mathcal{F}_k(X) = \pi_{\mathcal{F}, \beta X, k}(C(\{k\}), X) \times \mathcal{F}(\{k\}) \]

and denote the restriction of $\pi_{\mathcal{F}, \beta X, k}$ to $C(\{k\}) \times \mathcal{F}(\{k\})$ by $\pi_{\mathcal{F}, X, k}$. If $f : X \to Y$ is a continuous mapping, then

\[ \mathcal{F}(\beta f)(\mathcal{F}_k(X)) \subseteq \mathcal{F}_k(Y), \]

in view of the equality (2.4) for the mapping $\beta f$. Therefore, setting

\[ \mathcal{F}_k(f) = \mathcal{F}_k(\beta f)|\mathcal{F}(X), \]

we obtained a mapping

\[ \mathcal{F}_k(f) : \mathcal{F}_k(X) \to \mathcal{F}_k(Y). \]

Thus, we have defined the covariant functor

\[ \mathcal{F}_k : \text{Tych} \to \text{Tych}, \]

that extends the functor $\mathcal{F}_k : \text{Comp} \to \text{Comp}$ to the category $\text{Tych}$. Proposition 2.29 implies the following assertion.

**Proposition 2.32 ([18]).** If $\mathcal{F} : \text{Comp} \to \text{Comp}$ is a continuous functor, then $\mathcal{F}_k : \text{Tych} \to \text{Tych}$ is a subfunctor of the functor $\mathcal{F}_\beta$, and

\[ \mathcal{F}_k(X) = \mathcal{F}_\beta(X) \cap \mathcal{F}_k(\beta X). \]

**Proposition 2.33 ([18]).** For a compact space $X$, a continuous functor $\mathcal{F}$ and a positive integer $k$, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}(X)$.

Propositions 2.32 and 2.33 imply

**Proposition 2.34 ([18]).** For a Tychonoff space $X$, a continuous functor $\mathcal{F}$, and a positive integer $k$, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_\beta(X)$.

**Corollary 2.35.** For a Tychonoff space $X$, a continuous functor $\mathcal{F}$, and a positive integer $k$, the set $\mathcal{F}_k(X)$ is closed in $\mathcal{F}_{k+1}(X)$.

### 3. Projectively inductively closed functors

We start recalling that a functor $\mathcal{F}$ is said to be **finitely open** [18], if the set $\mathcal{F}_k(\{k+1\})$ is open in $\mathcal{F}(\{k+1\})$ for any positive integer $k$. The dual for this definition states that $\mathcal{F}(\{k+1\}) \setminus \mathcal{F}_k(\{k+1\})$ is closed in $\mathcal{F}(\{k+1\})$.

**Remark 3.1.** As an example of a finitely open functor one can take any finitary functor $\mathcal{F}$, i.e., a functor $\mathcal{F}$ such that $\mathcal{F}(\{k\})$ is finite for any positive integer $k$. In particular, the hyperspace functor exp is a finitary and, consequently, a finitely open functor.
Lemma 3.2. For any continuous, preserving preimages functor $\mathcal{F}_\beta$, the mapping $\pi_{\mathcal{F}_\beta,X,1}$ is a homeomorphism.

Proof. At first we show that $\pi_{\mathcal{F}_\beta,X,1}$ is a bijective mapping. In view of (2.3) for any $\xi \in C(\{1\}, X)$ and $a \in \mathcal{F}(\{1\})$ we have $\mathcal{F}(\xi)(a) = \xi(0)$, since we consider functors preserving empty set. Since the set $\{1\}$ consists of one point 0, every mapping $\xi: \{1\} \to X$ is a monomorphism. But we consider only monomorphic functors. Hence, the mapping $\mathcal{F}(\xi)$ is a monomorphism. On the other hand, $\pi_{\mathcal{F}_\beta,X,1}(\xi,a) = \xi(0)$. Consequently, $\pi_1$ is an injective mapping. Furthere, the mapping $\pi_{\mathcal{F}_\beta,X,k}: C(\{k\}, X) \times \mathcal{F}(\{k\}) \to \mathcal{F}_k(X)$ is epimorphic for any positive integer $k$, in particular, for $k = 1$ by definition of $\mathcal{F}(X)$. Thus, $\pi_1$ is a bijective mapping.

Hence, $\pi_1$ is a homeomorphism for a compact space $X$ ($\pi_1$ is continuous in view of Proposition 2.28). If $X$ is a Tychonoff space, then by definition, the mapping $\pi_{\mathcal{F}_\beta,X,k}$ is a restriction of $\pi_{\mathcal{F}_\beta,X,k}$ to $C(\{k\}, X) \times \mathcal{F}(\{k\})$. Therefore, the mapping $\pi_{\mathcal{F}_\beta,X,1}$ is a homeomorphism as a restriction of the homeomorphism $\pi_{\mathcal{F}_\beta,X,1}$ to a subset. The proof is complete. □

Definition 3.3. An epimorphism $f: X \to Y$ is called inductively closed if there exists a closed subset $A$ of $X$ such that $f(A) = Y$ and $f|A$ is a closed mapping.

Definition 3.4. A functor $\mathcal{F}_\beta$ is said to be projectively inductively closed (p.i.c.) if the mapping $\pi_{\mathcal{F}_\beta,X,k}$ is inductively closed for any Tychonoff space $X$ and positive integer $k$.

The next theorem gives us sufficient conditions for a functor $\mathcal{F}_\beta$ to be projectively inductively closed (a p.i.c.-functor).

Theorem 3.5. Every continuous, monomorphic, finitely open functor $\mathcal{F}_\beta: \text{Tych} \to \text{Tych}$, that preserves empty set, intersections, and preimages is a p.i.c.-functor.

Proof. It is necessary to check, that for any Tychonoff space $X$ and positive integer $k$, the mapping

$$\pi_{\mathcal{F}_\beta,X,k}: X^k \times \mathcal{F}(\{k\}) \to \mathcal{F}_k(X)$$

is inductively closed. We shall prove it by induction on $k$. If $k = 1$, the mapping $\pi_{\mathcal{F},X,1}$ is inductively closed, since it is a homeomorphism by Lemma 3.2.

Assume that our assertion is proved for all integers $k \leq l$. Let us prove it for $k = l + 1$. Fix some point $x_0 \in X^l$. Consider the embedding $i: X^l \to X^{l+1}$ defined as

$$i(x_1,\ldots,x_l) = (x_1,\ldots,x_l,x_0).$$

Define a mapping $j: \mathcal{F}(\{l\}) \to \mathcal{F}(\{l+1\})$ by the equality $j(a) = \mathcal{F}(h)(a)$, where $h: \{l\} \to \{l+1\}$ is an identical embedding, i.e., $h(m) = m$ for any
$m \leq l - 1$. Since $F$ is a monomorphic functor, the mapping $j$ is an embedding. Hence, we defined the embedding

$$e = i \times j : X^l \times F(\{l\}) \to X^{l+1} \times F(\{l+1\}).$$

It follows from definitions that

$$\pi F_{\beta,X,l+1} \circ e = \pi F_{\beta,X,l}.$$  

From property (3.1) we get, that on the set $e(X^l \times F(\{l\}))$ the next equality holds:

$$\pi F_{\beta,X,l} = \pi F_{\beta,X,l+1} \circ e^{-1}.$$  

Since the mapping $\pi F_{\beta,X,l}$ is inductively closed by an inductive assumption, there exists a closed subset $A$ of $X^l \times F(\{l\})$ such that $\pi F_{\beta,X,l}(A) = F_l(X)$, and the mapping $\pi F_{\beta,X,l}|A$ is closed. Since the mapping $e^{-1}$ is a homeomorphism on the set $e(A)$, equality (3.2) and Corollary 2.35 imply that

$$\pi F_{\beta,X,l+1}|A$$ is a closed mapping.

Moreover, it is clear, that

$$\pi F_{\beta,X,l+1}(A) = F_l(X).$$

Now we put

$$\Phi = F(\{l+1\}) \setminus F(\{l+1\}).$$

The set $\Phi$ is compact, because the functor $F$ is finitely open. Now we define the sets

$$Z_0 = X^{l+1} \times \Phi$$ and

$$Z_1 = (\beta X)^{l+1} \times \Phi.$$  

By $f_i$, $i < 2$, we denote restrictions of the mapping $\pi F_{\beta,X,l+1}$ to the sets $Z_i$. Let us show that

$$Z_0 = f_1^{-1}(f_1(Z_0)).$$

To verify this equality, we remark that the functor $F_{\beta}$ preserves monomorphisms, intersections, and preimages. Hence, it preserves supports (look at (2.2)). Therefore,

$$\text{supp}(\pi_{l+1}(\xi, a)) = \xi(\text{supp}(a))$$ for any $\xi \in C(\{l+1\}, \beta X)$ and $a \in F(\{l+1\})$. But if $(\xi, a) \in Z_1$, then $\text{supp}(a) = \{l+1\}$. Consequently,

$$\text{supp}(f_1(\xi, a)) = \xi(\{l+1\}).$$

Hence,

$$Z_0 = \{(\xi, a) \in Z_1 : \xi(\{l+1\}) \subset X\}.$$  

Thus, if $f_1(\xi_0, a_0) = f_1(\xi_1, a_1)$ and $(\xi_0, a_0) \in Z_0$, then $(\xi_1, a_1) \in Z_0$. Hence, equality (3.7) is verified.
Compactness of the set $Z_1$ and the equality (3.7) imply that the mapping $f_0: Z_0 \rightarrow f_0(Z_0)$ is closed.

Now we shall verify that

(3.11) \[ f_0(Z_0) = f_1(Z_1) \cap F_{l+1}(X). \]

It is sufficient to check the inclusion $\supset$. Let $f_1(\xi, a) \in f_1(Z_1) \cap F_{l+1}(X)$. Then $X \supset \text{supp}
(f_1(\xi, a)) = \xi(l + 1)$ by (3.9). Consequently, $(\xi, a) \in Z_0$ according to (3.10). Thus, the equality (3.11) is checked.

This equality and compactness of $f_1(Z_1)$ imply that $f_0(Z_0)$ is closed in $F_{l+1}(X)$. Hence, the mapping $f_0: Z_0 \rightarrow F_{l+1}(X)$ is closed. Let $B = A_0 \cup Z_0$. Then the mapping

$\pi_{F_\beta, X, l+1}|B: B \rightarrow F_{l+1}(X)$

is closed as a union of closed mappings on two closed subsets $A$ and $Z_0$.

To complete the proof, it suffices to check that

(3.12) \[ f_0(Z_0) \supset F_{l+1}(X) \backslash F_l(X). \]

But this inclusion is a corollary of (3.11) and an evident inclusion

$f_1(Z_1) \supset F_{l+1}(X) \backslash F_l(X).$

The proof is complete. \(\square\)

**Corollary 3.6.** Every finitary normal functor, in particular the functor $\exp_n$, is a p.i.c.-functor.

**Theorem 3.7.** Let $F_\beta$ be a weight preserving p.i.c.-functor of a finite degree $m$. Then $F_\beta$ preserves the class of stratifiable spaces and the class of paracompact $\sigma$-spaces.

**Proof.** First we consider the case of stratifiable spaces. By Theorem 2.24 the space $X^m \times F_\beta\{m\}$ is stratifiable as a finite product of stratifiable spaces ($F_\beta\{m\}$ is stratifiable being a metrizable compact space, because $F_\beta$ is a weight preserving functor). Since $F_\beta$ is a p.i.c.-functor, there exists a closed subset $A$ of $X^m \times F_\beta\{m\}$ such that $\pi_{F_\beta, X, m}(A) = F_\beta(X)$ and the mapping $\pi_{F_\beta, X, m}|A \rightarrow F_\beta(X)$ is closed. But every subspace of a stratifiable space is stratifiable in view of Theorem 2.23. Hence, $A$ is a stratifiable space. Then the space $F_\beta(X)$ is stratifiable like an image of a stratifiable space under closed mapping (look at Theorem 2.25). \(\square\)

The proof of the assertion for paracompact $\sigma$-spaces repeats the previous proof. The necessary changings are the following: instead of Theorems 2.24, 2.23, and 2.25, we use Theorem 2.6, Remark 2.3, and Theorem 2.5 respectively.

By the same procedure we get

**Theorem 3.8.** Let $F_\beta$ be a p.i.c.-functor of a finite degree. Then $F_\beta$ preserves the class of paracompact $\Sigma$-spaces and the class of paracompact $p$-spaces.
Proof of this theorem repeats the proof of Theorem 3.7 for stratifiable spaces. The necessary changings are: 1) we don’t need that $\mathcal{F}_\beta(m)$ is metrizable; 2) instead of Theorems 2.24, 2.23, and 2.25, we use Propositions 2.10, 2.9, and 2.11 in the case of paracompact $\Sigma$-spaces; 3) in the case of paracompact $p$-spaces we use respectively Corollary 2.15, Proposition 2.16, and Theorem 2.17.

Corollary 3.6, Theorems 3.7, and 3.8 yield

**Corollary 3.9.** Every normal finitary functor of a finite degree, in particular the functor $\exp_m$, preserves the class of stratifiable spaces, the class of paracompact $\sigma$-spaces, and the class of paracompact $\Sigma$-spaces.

**Remark 3.10.** As for paracompact $p$-spaces, they are preserved by any normal functor $\mathcal{F}_\beta$ (look at [1]).

**References**


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