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Near metrizability via a new approach

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Abstract

The present article deals with near metrizability, initiated in an earlier paper [7], with a new orientation and approach. The notions of nearly regular and uniform pseudo-bases are introduced and analogues of some results concerning metrizability and paracompactness are obtained for near metrizability and near paracompactness respectively via the proposed approach, suitably formulated.

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1. Introduction

The idea of near paracompactness, a well known weaker form of paracompactness, was initiated by Singal and Arya [9], followed by an extensive study of the concept by many topologists from different perspectives and with different applications (for instance see [3], [4], [5], [6], [8]). Now, in [7] we introduced a neighbouring form of metrizability, termed near metrizability, which plays the same role with regard to near paracompactness as is done by metrizability vis-a-vis paracompactness. It was shown in [7] that there exist nearly metrizable, non-metrizable spaces that are not paracompact, moreover some other facts were established in [7].

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The intent of the present article is to do a further study of nearly mertizable spaces from an altogether new approach. The notion of pseudo-base was introduced and studied in [7], and here, we define regular and uniform pseudo-bases, and ultimately achieve analogues of two well known results on metrizability in our setting.

At the outset we recall a few definitions which may be found in [1, 2]. A base $\mathcal{B}$ for a topological space X is called regular if for each $x \in X$ and any neighbourhood U of x, there exists a neighbourhood O of x such that the set of all members of $\mathcal{B}$ that meet both O and $X \setminus U$, is finite; and a base $\mathcal{B}$ is called a uniform base if for each $x \in X$ and every neighbourhood U of x, the set of all members of $\mathcal{B}$ that contain x and meet $X \setminus U$, is finite. It is clear that every regular base is a uniform base. The next two metrization theorems are known (see [1, 2]), which have been formulated in terms of the above special base.

Theorem 1.1.

(a). *A T_3-paracompact space X with a uniform base $\mathcal{B}$ is metrizable.*
(b). *Every T_1-space X with a regular base $\mathcal{B}$ is metrizable.*

As already proposed, our principal aim in this paper is to achieve analogous versions of the results in Theorem 1.1 for near metrizability with accessories formulated suitably.

In what follows, by a space X we shall mean a topological space X endowed with a topology τ(say). The notations 'clA', '$intA$' and '$|A|$' will respectively stand for the closure, interior and cardinality of a set A of a space X. A set $A(\subseteq X)$ is called regular open if $A = intclA$, and the complement of a regular open set is called regular closed. The set of all regular open (resp. closed) sets of a space X will be denoted by $RO(X)$(resp. $RC(X)$). We shall sometimes write A^* for $intclA$ for a subset A of X and $\mathcal{C}^\# = \{A^* : A \in \mathcal{C}\}$, for any open cover $\mathcal{C}$ of a space X.

Singal and Arya formulated the following definitions which are quite well known by now.

Definition 1.2 ([10])**.** A topological space X is called nearly paracompact if every regular open cover of X has a locally finite open refinement.

Definition 1.3 ([9])**.** A topological space X is said to be almost regular, if for any regular closed set A and any $x \in X \setminus A$, there exist disjoint open sets U and V in X such that $x \in U$ and $A \subseteq V$.

2. Main results

We start by recalling a few definitions from [7] as follows:

Definition 2.1. If X and Y are two topological spaces, then a continuous, injective map $f : X \to Y$ is called a pseudo-embedding of X into Y, if for any $A \in RO(X)$, $f(A)$ is open.

If there is a pseudo-embedding f of X into Y, then we say that X is pseudo-embeddable in Y. If a pseudo-embedding $f : X \to Y$ is surjective, we say that f is a pseudo-embedding of X onto Y.

It is known [7] that every embedding is a pseudo-embedding; but the converse is false.

Definition 2.2 ([7])**.** A space X is called nearly metrizable if it is pseudo-embeddable in a metric space Y.

Definition 2.3 ([7])**.** Suppose $\mathcal{B}$ is a family of open subsets of X. We say that $\mathcal{B}$ is a pseudo-base in X if for any $A \in RO(X)$, there is a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $A = \bigcup\{B : B \in \mathcal{B}_0\}$.

We now define a family $\mathcal{B}$ of open subsets of X to be a pseudo-base at a point $x \in X$ if for each $U \in RO(X)$ containing x, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Clearly, a family $\mathcal{B}$ of open subsets of a space X is pseudo-base for X if and only if it is so at each $x \in X$.

We shall call a pseudo-base $\mathcal{B}$ σ-locally finite if $\mathcal{B}$ can be expressed as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where $\mathcal{B}_n$ is locally finite, for each $n \in \mathbb{N}$.

We now define another type of bases as follows:

Definition 2.4. Let (X, τ) be a topological space.

(a) A family $\mathcal{B}$ of subsets of X is called nearly regular if for each $U \in \mathcal{B}$ and any point $x \in U$, there exists a regular open set O_x containing x such that the set $\{V \in \mathcal{B} : V \bigcap O_x \neq \phi$ and $V \bigcap (X \setminus U) \neq \phi\}$ is finite.

(b) A pseudo-base $\mathcal{B}$ for X is called nearly regular if for each $x \in X$ and any regular open set O_x containing x, there exists a regular open set G_x containing x such that the set $\{U \in \mathcal{B} : U \bigcap G_x \neq \phi$ and $U \bigcap (X \setminus O_x) \neq \phi\}$ is finite.

Remark 2.5. It is clear from the above definition that a subfamily of a nearly regular family is a nearly regular family.

Proposition 2.6. *If $\mathcal{B}$ is a nearly regular pseudo-base for a space X, then so is $\mathcal{B}^{\#} = \{B^* : B \in \mathcal{B}\}$.*

Proof. First let $x \in X$ and U a regular open set in X such that $x \in U$. As $\mathcal{B}$ is a pseudo-base for X, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $x \in B^* \subseteq U^* = U$, and hence $\mathcal{B}^{\#}$ is a pseudo-base for X.
Next, let $x \in X$ and O_x be any regular open set in X containing x. As $\mathcal{B}$ is a nearly regular pseudo-base, there exists a regular open set G_x containing x such that the set $\{B \in \mathcal{B} : B \bigcap G_x \neq \phi \neq B \bigcap (X \setminus O_x)\}$ is finite.
It suffices to show that $\{B^* \in \mathcal{B}^{\#} : B^* \bigcap G_x \neq \phi \neq B^* \bigcap (X \setminus O_x)\}$ is finite, for which we need only to show that $\{B^* \in \mathcal{B}^{\#} : B^* \bigcap G_x \neq \phi \neq B^* \bigcap (X \setminus O_x)\} \subseteq \{B \in \mathcal{B} : B \bigcap G_x \neq \phi \neq B \bigcap (X \setminus O_x)\}$. In fact, $B \bigcap G_x = \phi \Leftrightarrow intclB \bigcap intclG_x = \phi \Leftrightarrow B^* \bigcap G_x = \phi$, and $B \bigcap (X \setminus O_x) = \phi \Rightarrow B \subseteq O_x \Rightarrow B^* \subseteq intclO_x = O_x \Rightarrow B^* \bigcap (X \setminus O_x) = \phi$. $\square$

We shall call a space X to be an almost T_3-space if it is almost regular and Hausdorff.

Theorem 2.7. *A T_2-space X, possessing a nearly regular pseudo-base $\mathcal{B}$ is an almost T_3-space.*

Proof. Let F be a regular closed set and $x \in X \setminus F$. Then there exists a regular open set O_x containing x such that $O_x \bigcap F = \phi$, i.e., $F \subseteq X \setminus O_x$.
By hypothesis, there exists a regular open set G_x containing x such that the family $\mathcal{U} = \{U \in \mathcal{B} : U \bigcap G_x \neq \phi$ and $U \bigcap (X \setminus O_x) \neq \phi\}$ is finite. Put $O = O_x \bigcap G_x$. Then O is a regular open set containing x such that $O \bigcap F = \phi$.
Consider the family $\mathcal{C} = \{U \in \mathcal{B} : U \bigcap O \neq \phi$ and $U \bigcap F \neq \phi\}$. Since $F \subseteq X \setminus O_x$, $\mathcal{C}$ is finite.
Now for each $U \in \mathcal{C}$, $|U| \geq 2$ as $O \bigcap F = \phi$.
Let $\mathcal{B}' = \mathcal{B} \setminus \mathcal{C}$. We show that $\mathcal{B}'$ is a pseudo-base for X. In fact, let $p \in X$ and W a regular open set containing p. Let us enumerate $\mathcal{C}$ as $\{W_1, W_2, ..., W_n\}$ and let x_1, x_2,..., x_n be points from W_1, W_2,..., W_n respectively different from p.
Since X is T_2, each $\{x_i\}$ is regular closed and so $X \setminus \{x_1, x_2, ..., x_n\}$ is a regular open set containing p and hence there exists a $B_1 \in \mathcal{B}$ such that $p \in B_1 \subseteq X \setminus \{x_1, x_2, ..., x_n\}$. Again there exists $B_2 \in \mathcal{B}$ such that $p \in B_2 \subseteq W$. Thus there exists $B_3 \in \mathcal{B}$ such that $p \in B_3 \subseteq B_1 \bigcap B_2 \subseteq W$ i.e., $p \in B_3 \subseteq W$ where $B_3 \notin \mathcal{C}$. This shows that $\mathcal{B}'$ is a pseudo-base for X.
Put $\mathcal{G} = \{U \in \mathcal{B}' : U \bigcap F \neq \phi\}$ and $G = \bigcup\{U : U \in \mathcal{G}\}$. Then $F \subseteq G$ and $G \bigcap O = \phi$ with $x \in O$ (since for $U \in \mathcal{G}$, if $U \bigcap O \neq \phi$ then $U \in \mathcal{C}$, a contradiction).
This shows that F and x are strongly separated. Thus X is almost regular and consequently X is an almost T_3-space. □

Definition 2.8 ([2]). Let X be a topological space and $\mathcal{B}$ a family of subsets of X. An element U of $\mathcal{B}$ is called a maximal element of $\mathcal{B}$ if it is not contained in any element of $\mathcal{B}$ other than U. We denote by $m(\mathcal{B})$, the set of all maximal elements of $\mathcal{B}$ and call $m(\mathcal{B})$ the surface of $\mathcal{B}$.

Theorem 2.9. *Let $\mathcal{B}$ be a nearly regular family which is a cover of X. Then the surface $m(\mathcal{B})$ of $\mathcal{B}$ is a cover of X and is locally finite.*

Proof. Let $x \in X$ be taken arbitrarily and kept fixed, and let $U \in \mathcal{B}$ such that $x \in U$. If $U \notin m(\mathcal{B})$, then the family $\lambda_U = \{V \in \mathcal{B} : V \supsetneqq U\}$ is finite. In fact, by definition of $\mathcal{B}$, there exists a regular open set O_x containing x such that the collection $\mathcal{D} = \{V \in \mathcal{B} : V \bigcap O_x \neq \phi$ and $V \bigcap (X \setminus U) \neq \phi\}$ is finite. Clearly, $\lambda_U \subseteq \mathcal{D}$ and therefore λ_U is finite (note that $x \in V \bigcap O_x$). Consequently λ_U has a maximal element $V'(say)$. Again $x \in V'$ and $V' \in m(\mathcal{B})$. Hence $m(\mathcal{B})$ is a cover of X.
We now show that $m(\mathcal{B})$ is locally finite. As $m(\mathcal{B}) \subseteq \mathcal{B}$ and $\mathcal{B}$ is nearly regular, $m(\mathcal{B})$ is nearly regular. Again every element of $m(\mathcal{B})$ is maximal in $m(\mathcal{B})$ (because it is maximal in $\mathcal{B}$ and $m(\mathcal{B}) \subseteq \mathcal{B}$). Let $x \in X$. Then there exists a $U \in m(\mathcal{B})$ such that $x \in U$. Since $m(\mathcal{B})$ is nearly regular, there exists a regular open set O_x containing x such that the family $\mathcal{B}' = \{V \in m(\mathcal{B}) : V \bigcap O_x \neq \phi$ and $V \bigcap (X \setminus U) \neq \phi\}$ is finite. But $V \setminus U \neq \phi$ for all $V \in m(\mathcal{B})$ with $V \neq U$

(because every element V in $m(\mathcal{B})$ is maximal, there is no set $L \in m(\mathcal{B})$ which properly contains V).
Thus $\{V \in m(\mathcal{B}) : V \bigcap O_x \neq \phi\} = \mathcal{B}' \bigcup \{U\}$ is a finite set and hence $m(\mathcal{B})$ is locally finite. ☐

Theorem 2.10. *A space possessing a nearly regular pseudo-base $\mathcal{B}$ is nearly paracompact.*

Proof. Let $\mathcal{G}$ be any regular open cover of X and let $\mathcal{G}_{\mathcal{B}} = \{U \in \mathcal{B} : \exists G \in \mathcal{G}$ with $U \subseteq G\}$.
We check that $\mathcal{G}_{\mathcal{B}}$ is a pseudo-base for X. In fact, let $x \in X$ and G be any regular open set containing x. Now $\mathcal{G}$ being a cover, there exists $G_1 \in \mathcal{G}$ such that $x \in G_1$. Thus $G \bigcap G_1$ is a regular open set containing x. Since $\mathcal{B}$ is a pseudo-base for X, there exists $U \in \mathcal{B}$ such that $x \in U \subseteq G \bigcap G_1 \subseteq G_1 \in \mathcal{G} \Rightarrow U \in \mathcal{G}_{\mathcal{B}}$ with $x \in U \subseteq G \Rightarrow \mathcal{G}_{\mathcal{B}}$ is a pseudo-base for X.
Since $\mathcal{B}$ is nearly regular and $\mathcal{G}_{\mathcal{B}} \subseteq \mathcal{B}$, $\mathcal{G}_{\mathcal{B}}$ is nearly regular. Thus by Theorem 2.9, $m(\mathcal{G}_{\mathcal{B}})$ is an open cover of X and locally finite. Also clearly $m(\mathcal{G}_{\mathcal{B}})$ is an open refinement of $\mathcal{G}$. Hence X is nearly paracompact. ☐

Analogous to the concept of uniform base, we now define a special type of base as follows:

Definition 2.11. A pseudo-base $\mathcal{B}$ for a space X is called a uniform pseudo-base if for each $x \in X$ and each regular open set O_x containing x, $\mathcal{U}_{O_x} = \{U \in \mathcal{B} : x \in U$ and $U \bigcap (X \setminus O_x) \neq \phi\}$ is finite.

Lemma 2.12. *Let $\mathcal{B}$ be a family of open sets of a space X such that $\mathcal{B}^{\#}$ is a uniform pseudo-base for X. Then the surface $m(\mathcal{B}^{\#})$ is a point finite regular open cover of X.*

Proof. Let $x \in X$. Then there exists $U^* \in \mathcal{B}^{\#}$ (where $U \in \mathcal{B}$) such that $x \in U^*$. If $U^* \notin m(\mathcal{B}^{\#})$ then the set $\lambda_{U^*} = \{V \in \mathcal{B}^{\#} : V \supseteq U^*\}$ is finite. In fact, U^* is a regular open set containing x and hence the family $\mathcal{V} = \{V \in \mathcal{B}^{\#} : x \in V$ and $V \bigcap (X \setminus U^{\#}) \neq \phi\}$ is finite and $\lambda_{U^*} \subseteq \mathcal{V} \bigcup \{U^*\}$. Then λ_{U^*} has a maximal element $m(\lambda_{U^{\#}})$ which is also a maximal element of $\mathcal{B}^{\#}$ and which also contains x. Hence $m(\mathcal{B}^{\#})$ is a regular open cover of X.
We now show that $m(\mathcal{B}^{\#})$ is point finite. If possible let $x \in X$ be such that x belongs to an infinite collection $\mathcal{D}$ of members of $m(\mathcal{B}^{\#})$. Then we claim that $\mathcal{D}$ is a pseudo-base for X at x.
If $\mathcal{D}$ is not a pseudo-base for X at x, there exists a regular open set W containing x such that $x \in D \subseteq W$ holds for no $D \in \mathcal{D}$, i.e., for all $D \in \mathcal{D}$, $D \bigcap (X \setminus W) \neq \phi$. But $\{B \in \mathcal{D} : B \bigcap (X \setminus W) \neq \phi\}$ is finite as $\mathcal{B}^{\#}$ is a uniform pseudo-base. Hence $\mathcal{D}$ is a pseudo-base for X at x.
Next let, U and V be two distinct (and hence non comparable) elements of $\mathcal{D}$. Since $x \in U \bigcap V$ and $U \bigcap V$ is a regular open set, there exists a $W \in \mathcal{D}$ such that $x \in W \subsetneqq U \bigcap V$ (note that $U \bigcap V \notin \mathcal{D}$, since otherwise $U \bigcap V \subsetneqq U$ would contradict the maximality of $U \bigcap V$), i.e., $x \in W \subsetneqq U$ and hence W is not a

maximal element of $\mathcal{D}$ although $\mathcal{D} \subseteq m(\mathcal{B}^{\#})$, a contradiction. Hence $m(\mathcal{B}^{\#})$ is a point finite regular open cover of X. □

Lemma 2.13. *Let $\mathcal{B}$ be a family of open sets of a T_2-space X such that $\mathcal{B}^{\#}$ is a uniform pseudo-base. Then there exists a countable family of point finite regular open covers which taken together is a pseudo-base for X.*

Proof. Let $\mathcal{B}_1^{\#} = \mathcal{B}^{\#}$ and $\mathcal{B}_2^{\#} = \mathcal{B}_1^{\#} \setminus m^*(\mathcal{B}_1^{\#})$, where $m^*(\mathcal{B}_1^{\#})$ is the collection of all maximal elements of $\mathcal{B}_1^{\#}$ each of which contains at least two points. We first show that $\mathcal{B}_2^{\#}$ is a pseudo-base for X. In fact, let $x \in X$. Then by Lemma 2.12, x belongs to only finitely many members U_1, U_2,..., U_n *(say)* of $m^*(\mathcal{B}_1^{\#})$. Let $x_i \in U_i$ with $x \neq x_i$ for $i = 1, 2, ..., n$. Since X is T_2, $X \setminus \{x_1, x_2, ..., x_n\}$ is a regular open set containing x and so there exists B in $\mathcal{B}^{\#}$ such that $x \in B \subseteq X \setminus \{x_1, x_2, ..., x_n\}$. Let W be any regular open set containing x. Then there exists a $B' \in \mathcal{B}^{\#}$ such that $x \in B' \subseteq W$. Again there exists $B_1 \in \mathcal{B}^{\#}$ such that $x \in B_1 \subseteq B \bigcap B' \Rightarrow x \in B_1 \subseteq W$ and $B_1 \notin m^*(\mathcal{B}_1^{\#})$ $[B_1 \in m^*(\mathcal{B}_1^{\#}) \Rightarrow B_1 = U_i$ for some $i = 1, 2, ..., n \Rightarrow x_i \in B_1$ but $(x_i \notin B) \Rightarrow B_1 \not\subseteq B$, a contradiction]. Therefore, $x \in B_1 \subseteq W$ and $B_1 \in \mathcal{B}_2^{\#}$. Again $\mathcal{B}_2^{\#} \subseteq \mathcal{B}_1^{\#}$ and $\mathcal{B}_1^{\#}$ is a uniform pseudo-base $\Rightarrow \mathcal{B}_2^{\#}$ is a uniform pseudo-base.

Now proceed by induction, if $\mathcal{B}_k^{\#}$ is already defined then put $\mathcal{B}_{k+1}^{\#} = \mathcal{B}_k^{\#} \setminus m^*(\mathcal{B}_k^{\#})$ and as above, $\mathcal{B}_{k+1}^{\#}$ is a uniform pseudo-base for X. Then for each $n \in \mathbb{N}$, $\mathcal{B}_n^{\#}$ is a uniform pseudo-base for X and so $m(\mathcal{B}_n^{\#})$ is a point finite regular open cover of X (by Lemma 2.12).

Consider an arbitrary $x \in X$. For each $n \in \mathbb{N}$, choose $U_n \in m(\mathcal{B}_n^{\#})$ such that $x \in U_n$.

If there is $n \in \mathbb{N}$ satisfying $|U_n| = 1$ then $\{U_n : n \in \mathbb{N}\}$ is a pseudo-base at x.

If $|U_n| \geq 2$ for all $n \in \mathbb{N}$ then by definition of $\mathcal{B}_n^{\#}$, $U_n \neq U_m$ for $n \neq m$. Hence $\mathcal{L} = \{U_n : n \in \mathbb{N}\}$ is an infinite set of elements of the uniform pseudo-base $\mathcal{B}_n^{\#}$, each containing x. We claim that $\mathcal{L}$ is a pseudo-base for X at x. If not, then for some regular open set D containing x, there does not exist any $C \in \mathcal{L}$ such that $x \in C \subseteq D$ holds, i.e., for all $C \in \mathcal{L}$, $C \bigcap (X \setminus D) \neq \phi$. But since $\mathcal{L} \subseteq \mathcal{B}^{\#}$, $\{V \in \mathcal{B}^{\#} : x \in U \text{ and } U \bigcap (X \setminus D) \neq \phi\}$ is finite, a contradiction. Consequently, $\mathcal{L}$ is a pseudo-base for X at x. Hence $\{m(\mathcal{B}_n^{\#}) : n \in \mathbb{N}\}$ is the required family. □

Definition 2.14 ([11])**.** Let $\mathcal{A}$ be a family of subsets of a space X. The star of a point $x \in X$ in $\mathcal{A}$, denoted by $St(x, \mathcal{A})$, is defined by the union of all members of $\mathcal{A}$ which contain x. A family $\mathcal{A}$ of subsets of a space X is said to be a star refinement of another family $\mathcal{B}$ of subsets of X if the family of all stars of points of X in $\mathcal{A}$ forms a covering of X which refines $\mathcal{B}$.

Theorem 2.15 ([10])**.** *An almost regular space X is nearly paracompact if and only if every regular open covering of X has a regular open star refinement.*

Definition 2.16. Let X be a topological space and Γ a family of covers of X. We call Γ refined if for any point $x \in X$ and any regular open set O_x containing

x, there exists $\mathcal{B} \in \Gamma$ such that $St(x, \mathcal{B}) \subseteq O_x$.
If all the members of Γ are regular open covers, then we say that Γ is a refined family of regular open covers.

Theorem 2.17. *Let $\mathcal{B}$ be a family of open sets of an almost T_3 nearly paracompact space X such that $\mathcal{B}^\#$ is a uniform pseudo-base for X. Then X has a countable refined family of regular open covers.*

Proof. By Lemma 2.13, there exists a countable family of point finite regular open covers $\mathcal{B}_n$, which taken together is a pseudo-base for X. Since X is almost regular and nearly paracompact, by Theorem 2.15, each $\mathcal{B}_n$ has a regular open star refinement $\mathcal{U}_n$.
Now fix $x \in X$, and for each $n \in \mathbb{N}$, choose $B_n \in \mathcal{B}_n$ so that $St(x, \mathcal{U}_n) \subseteq B_n$. Then $\{B_n : n = 1, 2, ...\}$ is a pseudo-base for X at x. Let U be a regular open set containing x. Then there exists $B_k(say)$ such that $x \in B_k \subseteq U$ and then $x \in St(x, \mathcal{U}_k) \subseteq B_k \subseteq U$. Thus $\{\mathcal{U}_n : n = 1, 2, ...\}$ is a countable refined family of regular open covers. □

Theorem 2.18 ([7])**.** *A space X is nearly metrizable if and only if it is almost T_3 and possesses a σ-locally finite pseudo-base.*

Theorem 2.19. *Let X be an almost T_3 nearly paracompact space such that X has a countable refined family $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of regular open covers. Then X is nearly metrizable.*

Proof. Since X is nearly paracompact, each $\mathcal{U}_i$ has a locally finite open refinement $\mathcal{B}_i$. Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$. We show that $\mathcal{B}$ is a pseudo-base for X. In fact, let $x \in X$ and U be any regular open set containing x. Then since $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a refined family of covers there exists $k \in \mathbb{N}$ such that $x \in St(x, \mathcal{U}_k) \subseteq U$. But $\mathcal{B}_k$ being a cover of X, there exists $B_k \in \mathcal{B}_k$ such that $x \in B_k$ and B_k is contained in some member of $\mathcal{U}_k$ containing x and hence is contained in $St(x, \mathcal{U}_k)$. Thus $x \in B_k \subseteq U$. Hence $\mathcal{B}$ is a σ-locally finite pseudo-base for X and hence by Theorem 2.18, X is nearly metrizable. □

Theorem 2.20. *Let $\mathcal{B}$ be a family of open sets of an almost T_3 nearly paracompact space X such that $\mathcal{B}^\#$ is a uniform pseudo-base for X. Then X is nearly metrizable.*

Proof. Follows from Theorems 2.17 and 2.19. □

Theorem 2.21. *Every almost T_3-space X with a nearly regular pseudo-base $\mathcal{B}$ is nearly mertizable.*

Proof. By Theorem 2.10, X is nearly paracompact. Again by Proposition 2.6, $\mathcal{B}^\#$ is a nearly regular pseudo-base. Since every nearly regular pseudo-base is a uniform pseudo-base, $\mathcal{B}^\#$ is a uniform pseudo-base for X, and then by Theorem 2.20, it follows that X is nearly metrizable. □

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