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n-Tuple relations and topologies on function spaces

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Dedicated to Professor S. Naimpally on the occasion of his 70^{th} birthday.

Abstract. In [7] some results concerning **S**-splitting, **S**-jointly continuous, **D**-splitting and **D**-jointly continuous topologies are considered, where **S** and **D** are the Sierpinski space and the double-point space, respectively. Here we generalize these results replacing the spaces **S** and **D** by any finite space.

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1. Introduction.

By Y and Z we denote two fixed topological spaces and by t_Z the topology of Z. By $C(Y,Z)$ we denote the set of all continuous maps of Y into Z. If τ is a topology on the set $C(Y,Z)$, then the corresponding topological space is denoted by $C_\tau(Y,Z)$.

Let X be a space and $F : X \times Y \to Z$ be a continuous map. By F_x, where $x \in X$, we denote the continuous map of Y into Z, for which $F_x(y) = F(x,y)$, for every $y \in Y$. By $\widehat{F}$ we denote the map of X into the set $C(Y,Z)$, for which $\widehat{F}(x) = F_x$ for every $x \in X$.

Let G be a map of the space X into the set $C(Y,Z)$. By $\widetilde{G}$ we denote the map of the space $X \times Y$ into the space Z, for which $\widetilde{G}(x,y) = G(x)(y)$ for every $(x,y) \in X \times Y$.

A topology t on $C(Y,Z)$ is called *splitting* if for every space X, the continuity of a map $F : X \times Y \to Z$ implies that of the map $\widehat{F} : X \to C_t(Y,Z)$. A topology t on $C(Y,Z)$ is called *jointly continuous* if for every space X, the continuity of a map $G : X \to C_t(Y,Z)$ implies that of the map $\widetilde{G} : X \times Y \to Z$ (see [5], [1], [2] and [3]).

If in the above definitions it is assumed that the space X belongs to a given family $\mathcal{A}$ of spaces, then the topology τ is called $\mathcal{A}-$*splitting* (respectively,

$\mathcal{A}-$*jointly continuous*) (see [6]). In the present paper we shall considered only the case $\mathcal{A} = \{\mathbf{F}\}$, where $\mathbf{F}$ is a space, and instead of $\mathcal{A}$-splitting and $\mathcal{A}$-jointly continuous we write $\mathbf{F}$-splitting and $\mathbf{F}$-jointly continuous.

Let X be a space with a topology τ. We denote (see, for example, [8]) by $\overset{\tau}{\leq}$ (respectively, by $\overset{\tau}{\sim}$) a preorder (respectively, an equivalence relation) on X defined as follows: if $x, y \in X$, then we write $x \overset{\tau}{\leq} y$ (respectively, $x \overset{\tau}{\sim} y$) if and only if $x \in \mathrm{Cl}_X(\{y\})$ (respectively, $x \in \mathrm{Cl}_X(\{y\})$ and $y \in \mathrm{Cl}_X(\{x\})$). (By $\mathrm{Cl}_X(Q)$ we denote the closure of a set Q in the space X).

On the set $C(Y,Z)$ we denote a preorder $\leq$ (respectively, an equivalence relation $\sim$) as follows: if $g, f \in C(Y,Z)$, then we write $g \leq f$ (respectively, $g \sim f$) if $g(y) \overset{\tau_Z}{\leq} f(y)$ (respectively, $g(y) \overset{\tau_Z}{\sim} f(y)$) for every $y \in Y$ (see, for example, [7]).

By $\mathbf{S}$ we denote the Sierpinski space, that is, the set $\{0,1\}$ equipped with the topology $\tau(\mathbf{S}) \equiv \{\varnothing, \{0,1\}, \{1\}\}$, and by $\mathbf{D}$ the set $\{0,1\}$ with the trivial topology. In [7] the notions of $\mathbf{S}$-splitting and $\mathbf{S}$-jointly continuous (respectively, $\mathbf{D}$-splitting and $\mathbf{D}$-jointly continuous) topologies are characterized by the above preorders (respectively, equivalence relations) on $C(Y,Z)$. By the trivial topology on a set X we mean the topology $\{\varnothing, X\}$.

Let $\mathcal{U}$ be a quasi-uniformity on the space Z (see, for example, [4]). This quasi-uniformity defines on the set $C(Y,Z)$ a quasi-uniformity $\mathcal{Q}(\mathcal{U})$ as follows (see [11]): the set of all subsets of $C(Y,Z)$ of the form

$$(Y,U) = \{(f,g) \in C(Y,Z) \times C(Y,Z) : (f(y), g(y)) \in U, \text{ for every } y \in Y\},$$

where $U \in \mathcal{U}$, is a basis for the quasi-uniformity $\mathcal{Q}(\mathcal{U})$. We denote by $\tau_{\mathcal{Q}(\mathcal{U})}$ (see [11]) the topology on $C(Y,Z)$, which is defined by the quasi-uniformity $\mathcal{Q}(\mathcal{U})$, that is: the subbasic neighborhoods of an arbitrary element $f \in C(Y,Z)$ in $\tau_{\mathcal{Q}(\mathcal{U})}$ are of the form: $(Y,U)[f] = \{g \in C(Y,Z) : (f,g) \in (Y,U)\}$, where $U \in \mathcal{U}$. In this case we shall say also that $\tau_{\mathcal{Q}(\mathcal{U})}$ is *generated* by the quasi-uniformity $\mathcal{U}$.

Let $\mathcal{O}(Y)$ be the family of all open sets of the space Y. The *Scott topology* on $\mathcal{O}(Y)$ (see, for example, [8]) is defined as follows: a subset $I\!H$ of $\mathcal{O}(Y)$ is open if:

(α) the conditions $U \in I\!H$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in I\!H$, and

(β) for every collection of open sets of Y, whose union belongs to $I\!H$, there are finitely many elements of this collection whose union also belongs to $I\!H$.

The *Isbell topology* τ_{is} on $C(Y,Z)$ (see [9] and [10]) is the topology for which the family of all sets of the form

$$(I\!H, U) = \{f \in C(Y,Z) : f^{-1}(U) \in I\!H\},$$

where $I\!H$ is Scott open in $\mathcal{O}(Y)$ and $U \in \mathcal{O}(Z)$, is a subbasis.

The *pointwise topology* (see, for example, [3]) τ_p on $C(Y,Z)$ is the topology for which the family of all sets of the form

$$(\{y\}, U) = \{f \in C(Y,Z) : f(y) \in U\},$$

where $y \in Y$ and $U \in \mathcal{O}(Z)$, is a subbasis.

The *compact open* (see [5]) topology τ_c on $C(Y,Z)$ is the topology for which the family of all sets of the form

$$(K, U) = \{f \in C(Y,Z) : f(K) \subseteq U\},$$

where K is a compact subset of Y and $U \in \mathcal{O}(Z)$, is a subbasis.

Below, we recall some well known results:

(1) The pointwise topology, the compact open topology and the Isbell topology on $C(Y,Z)$ are always splitting (see, for example, [1], [2], [3], [5], [9] and [10]).

(2) The compact open topology on $C(Y,Z)$ is jointly continuous if Y is locally compact (see [5] and [2]).

(3) The Isbell topology on $C(Y,Z)$ is jointly continuous if Y is corecompact (see, for example, [9]).

(4) The topology $\tau_{\mathcal{Q}(\mathcal{U})}$ is jointly continuous (see [11]).

2. F-splitting and F-jointly continuous topologies.

In the paper we denote by $\mathbf{F}$ a non-discrete space which is the set $\{0, 1, ..., n\}$, $n > 0$, equipped with an arbitrary fixed topology. By $\mathbf{U}_j$, $j = 0, 1, ..., n$, we denote the intersection of all open neighborhoods of j in $\mathbf{F}$.

It is clear that if $\mathbf{F}$ is the discrete space, then every topology τ on $C(Y,Z)$ is $\mathbf{F}$-splitting and $\mathbf{F}$-jointly continuous.

Theorem 2.1. *The trivial topology and, hence, every topology on the set $C(Y,Z)$ is $\mathbf{F}$-jointly continuous if and only if the topology of Z is trivial.*

Proof. Suppose that the topology of Z is trivial. Then for any topology τ on $C(Y,Z)$ and any continuous map $G : \mathbf{F} \to C_\tau(Y,Z)$, the map $\widetilde{G} : \mathbf{F} \times Y \to Z$ is trivially continuous, that is τ is $\mathbf{F}$-jointly continuous.

Conversely, suppose that the trivial topology τ on $C(Y,Z)$ is $\mathbf{F}$-jointly continuous. We prove that the topology of Z is trivial. Indeed, in the opposite case, there exist two distinct elements z_1, z_2 of Z and an open subset U of Z such that $z_1 \in U$ and $z_2 \notin U$. We consider the maps $f, g \in C(Y,Z)$ such that $f(Y) = \{z_1\}$ and $g(Y) = \{z_2\}$. Denote by i, the element of $\mathbf{F}$ such that $\mathbf{U}_i \neq \{i\}$. Let $G : \mathbf{F} \to C_\tau(Y,Z)$ be a map such that $G(i) = f$ and $G(j) = g$, for every $j \in \mathbf{F} \setminus \{i\}$. Since τ is trivial, the map G is continuous. Since τ is $\mathbf{F}$-jointly continuous, the map $\widetilde{G} : \mathbf{F} \times Y \to Z$ is also continuous. By the definition of $\widetilde{G}$, $\widetilde{G}(i,y) = G(i)(y) = f(y) = z_1 \in U$, $y \in Y$. Therefore for a fixed $y \in Y$ there exists an open neighborhood V_y such that $\widetilde{G}(\mathbf{U}_i \times V_y) \subseteq U$. Let $j \in \mathbf{U}_i \setminus \{i\}$. Then, we have $\widetilde{G}(j,y) = G(j)(y) = g(y) = z_2 \notin U$ which is a contradiction. Thus the topology of Z is trivial. □

Theorem 2.2. *If the discrete topology, and hence, every topology on $C(Y,Z)$ is $\mathbf{F}$-splitting, then Z is a T_0 space.*

Proof. Suppose that the discrete topology τ on $C(Y,Z)$ is **F**-splitting and Z is not T_0 space. We shall construct a continuous map $F : \mathbf{F} \times Y \to Z$ such that $\widehat{F}$ is not continuous, which will be a contradiction.

There exist two distinct elements z_1, z_2 of Z such that either $z_1, z_2 \in V$ or $z_1, z_2 \notin V$ for every open subset V of Z. Let i be an element of **F** such that $\mathbf{U}_i \neq \{i\}$. We consider the map $F : \mathbf{F} \times Y \to Z$ such that $F(i,y) = z_1$ for every $y \in Y$, and $F(j,y) = z_2$ for every $j \in \mathbf{F} \setminus \{i\}$ and $y \in Y$. Let V be an open subset of Z. Then, either $F^{-1}(V) = \mathbf{F} \times Y$ or $F^{-1}(V) = \varnothing$, which means that F is continuous.

By the definition of $\widehat{F} : \mathbf{F} \to C_\tau(Y,Z)$ we have $\widehat{F}(i)(Y) = \{z_1\}$, and $\widehat{F}(j)(Y) = \{z_2\}$ for every $j \in \mathbf{F} \setminus \{i\}$. Let $j \in \mathbf{U}_i \setminus \{i\}$. Then $\widehat{F}(j) \notin \{\widehat{F}(i)\}$, that is, $\widehat{F}(\mathbf{U}_i) \not\subseteq \{\widehat{F}(i)\}$, which means that $\widehat{F}$ is not continuous. □

Theorem 2.3. *Let Z be a T_1 space. Then, the discrete topology, and hence, every topology on $C(Y,Z)$ is* **F***-splitting.*

Proof. Let τ be the discrete topology on $C(Y,Z)$ and $F : \mathbf{F} \times Y \to Z$ a continuous map. We prove that the map $\widehat{F} : \mathbf{F} \to C_\tau(Y,Z)$ is continuous.

Let $i \in \mathbf{F}$ and $\widehat{F}(i) = f$. Then $f \in \{f\} \in \tau$. It is suffices to prove that $\widehat{F}(\mathbf{U}_i) \subseteq \{f\}$, that is, $\widehat{F}(j) = f$ for every $j \in \mathbf{U}_i$. Let $j \in \mathbf{U}_i$ and y be an arbitrary point of Y. We need to prove that $\widehat{F}(j)(y) = f(y)$. Let U be an arbitrary open neighborhood of $f(y) = \widehat{F}(i)(y) = F(i,y)$ in Z. Since the map F is continuous there exists an open neighborhood V_y of y in Y such that $F(\mathbf{U}_i \times V_y) \subseteq U$. Therefore, $F(j,y) = \widehat{F}(j)(y) \in U$, which means that $f(y) \in \mathrm{Cl}_Z(\{\widehat{F}(j)(y)\})$. Since Z is a T_1 space, $f(y) = \widehat{F}(j)(y)$. Hence, $\widehat{F}(j) = f$. Thus, the map $\widehat{F} : \mathbf{F} \to C_\tau(Y,Z)$ is continuous and therefore the topology τ on $C(Y,Z)$ is **F**-splitting. □

Theorem 2.4. *The pointwise topology τ_p, the compact-open topology τ_c, and the Isbell topology τ_{is} on $C(Y,Z)$ are* **F***-splitting and* **F***-jointly continuous.*

Proof. First, we prove that τ_p is **F**-jointly continuous. Let $G : \mathbf{F} \to C_{\tau_p}(Y,Z)$ be a continuous map. We need to prove that the map $\widetilde{G} : \mathbf{F} \times Y \to Z$ is continuous.

Let $(i,y) \in \mathbf{F} \times Y$ and U be an arbitrary open neighborhood of $\widetilde{G}(i,y) = G(i)(y)$ in Z. Then $G(i) \in (\{y\},U)$. Since G is continuous, $G(\mathbf{U_i}) \subseteq (\{y\},U)$. Also, since the map $G(j)$, $j \in \mathbf{U}_i$, is continuous and $G(j)(y) \in U$ there exists an open neighborhood V_y^j of y in Y such that $G(j)(V_y^j) \subseteq U$. Let $V_y = \cap\{V_y^j : j \in \mathbf{U}_i\}$. Then, $\widetilde{G}(\mathbf{U}_i \times V_y) \subseteq U$. Thus, the map $\widetilde{G}$ is continuous and the therefore the topology τ_p is **F**-jointly continuous.

Since $\tau_p \subseteq \tau_c$ and $\tau_p \subseteq \tau_{is}$ (see [10]) the topologies τ_c and τ_{is} are also **F**-jointly continuous.

Finally, since the topologies τ_p, τ_c and τ_{is} are splitting, they are also **F**-splitting. □

Theorem 2.5. *The topology $\tau_{\mathcal{Q}(\mathcal{U})}$ on the set $C(Y,Z)$ generated by a quasi-uniformity $\mathcal{U}$ on the space Z is $\mathbf{F}$-splitting and $\mathbf{F}$-jointly continuous.*

Proof. Let $\mathcal{U}$ be a quasi-uniformity on the space Z. Since $\tau_{\mathcal{Q}(\mathcal{U})}$ is jointly continuous (see [11]), this topology is also $\mathbf{F}$-jointly continuous.

We prove that $\tau_{\mathcal{Q}(\mathcal{U})}$ is $\mathbf{F}$-splitting. Let $F : \mathbf{F} \times Y \to Z$ be a continuous map. We need to prove that $\widehat{F} : \mathbf{F} \to C_{\tau_{\mathcal{Q}(\mathcal{U})}}(Y,Z)$ is continuous. Let $i \in \mathbf{F}$ and $\widehat{F}(i) = f_i$. The set $(Y,U)[f_i] = \{h \in C(Y,Z) : (f_i, h) \in (Y,U)\}$, where U is an element of $\mathcal{U}$, is an open neighborhood of f_i in $C_{\tau_{\mathcal{Q}(\mathcal{U})}}(Y,Z)$. We prove that $\widehat{F}(\mathbf{U_i}) \subseteq (Y,U)[f_i]$. Let $j \in \mathbf{U}_i$. It is suffices to prove that $\widehat{F}(j) = f_j \in (Y,U)[f_i]$, that is $f_j \in (Y,U)[f_i]$ or $(f_i(y), f_j(y)) \in U$ for every $y \in Y$. Let $y \in Y$ and $U[f_i(y)] = \{z \in Z : (f_i(y), z) \in U\}$. Since F is continuous there exists an open neighborhood V_y of y in Y such that $F(\mathbf{U}_i \times V_y) \subseteq U[f_i(y)]$. So, for the element (j,y) of $\mathbf{U}_i \times V_y$ we have $F(j,y) = f_j(y) \in U[f_i(y)]$ or $(f_i(y), f_j(y)) \in U$. Thus, the map $\widehat{F}$ is continuous and therefore the topology $\tau_{\mathcal{Q}(\mathcal{U})}$ is $\mathbf{F}$-splitting. □

Definition 2.6. For every space X with a topology t we define an $(n+1)$-tuple relation denoted by R^t in X as follows: an $(n+1)$-tuple $(x_0, x_1, ..., x_n)$ of elements of X belongs to R^t if for every $i, j \in \mathbf{F}$, $x_i \in \mathrm{Cl}_X(\{x_j\})$ provided that $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$.

We observe that if t_1, t_2 are two topologies on a set X such that $t_1 \subseteq t_2$, then $R^{t_2} \subseteq R^{t_1}$.

Definition 2.7. On the set $C(Y,Z)$ we define an $(n+1)$-tuple relation denoted by R as follows: an $(n+1)$-tuple $(f_0, f_1, ..., f_n)$ of elements of $C(Y,Z)$ belongs to R if $(f_0(y), f_1(y), ..., f_n(y)) \in R^{t_Z}$ for every $y \in Y$.

Below we give necessary and sufficient conditions for an arbitrary topology τ on $C(Y,Z)$ to be $\mathbf{F}$-splitting or $\mathbf{F}$-jointly continuous.

Theorem 2.8. *A topology τ on $C(Y,Z)$ is $\mathbf{F}$-splitting if and only if $R \subseteq R^\tau$.*

Proof. Let τ be an $\mathbf{F}$-splitting topology on $C(Y,Z)$. Suppose that $(f_0, f_1, ..., f_n) \in R$. We need to prove that $(f_0, f_1, ..., f_n) \in R^\tau$.

Let $F : \mathbf{F} \times Y \to Z$ be a map for which $F(i,y) = f_i(y)$, for every $i \in \mathbf{F}$ and $y \in Y$. This map is continuous. Indeed, let U be an open neighborhood of $f_i(y)$ in Z. Since f_i is continuous, the set $f_i^{-1}(U)$ is open neighborhood of y in Y. Therefore it is sufficient to prove that:

$$F(\mathbf{U}_i \times f_i^{-1}(U)) \subseteq U.$$

Let $(j, y') \in \mathbf{U}_i \times f_i^{-1}(U)$. By the definition of F, $F(j,y') = f_j(y')$. Since $j \in \mathbf{U}_i$ we have $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$. Also, by the definition of the $(n+1)$-tuple relation R we have $f_i(y') \in \mathrm{Cl}_Z(f_j(y'))$. Since $f_i(y') \in U$ we have $f_j(y') \in U$. Thus, $F(\mathbf{U}_i \times f_i^{-1}(U)) \subseteq U$, that is F is continuous.

Furthermore, since τ is $\mathbf{F}$-splitting, the map $\widehat{F} : \mathbf{F} \to C_\tau(Y,Z)$ is continuous.

Now, we prove that $(f_0, f_1, ..., f_n) \in R^{\tau}$. Let $i, j \in \mathbf{F}$ such that $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$. We need to prove that $f_i \in \mathrm{Cl}_{C_\tau(Y,Z)}(\{f_j\})$. Let W be an open neighborhood of f_i in $C_\tau(Y, Z)$. Then, $\widehat{F}^{-1}(W)$ is an open neighborhood of i in $\mathbf{F}$ and therefore $j \in \widehat{F}^{-1}(W)$. This means that $\widehat{F}(j) = f_j \in W$ and therefore $f_i \in \mathrm{Cl}_{C_\tau(Y,Z)}(\{f_j\})$. Hence, $(f_0, f_1, ..., f_n) \in R^{\tau}$.

Conversely, let τ be a topology on $C(Y, Z)$ such that the condition $(f_0, f_1, .., f_n) \in R$ implies $(f_0, f_1, ..., f_n) \in R^{\tau}$. We prove that τ is $\mathbf{F}$-splitting.

Let $F : \mathbf{F}\times Y \to Z$ be a continuous map. Consider the map $\widehat{F} : \mathbf{F}\to C_\tau(Y, Z)$ and let $\widehat{F}(i) = f_i$, $i \in \mathbf{F}$. First, we prove that $(f_0, f_1, ..., f_n) \in R$. Indeed, let $y \in Y$. Consider the $(n+1)$-tuple $(f_0(y), f_1(y), ..., f_n(y))$ and suppose that $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$. Let U be an open neighborhood of $f_i(y)$ in Z. Since $F(i, y) = f_i(y)$ and F is continuous, the set $F^{-1}(U)$ is an open neighborhood of (i, y) in $\mathbf{F}\times Y$. Therefore there exist open sets V and W of $\mathbf{F}$ and Y, respectively, such that $(i, y) \in V \times W \subseteq F^{-1}(U)$. This means that $j \in V$ and $F(j, y) = f_j(y) \in U$ and therefore $f_i(y) \in \mathrm{Cl}_Z(f_j(y))$, that is $(f_0(y), f_1(y), ..., f_n(y)) \in R^{tz}$. Hence $(f_0, f_1, ..., f_n) \in R$. By the assumption, $(f_0, f_1, ..., f_n) \in R^{\tau}$.

Now, we prove that $\widehat{F}$ is continuous. Let $\widehat{F}(i) = f_i$ and H be an open neighborhood of f_i in $C_\tau(Y, Z)$. It suffices to prove that

$$\widehat{F}(\mathbf{U}_i) \subseteq H.$$

Let $j \in \mathbf{U}_i$. Then $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$. Since $(f_0, f_1, ..., f_n) \in R$ we have $(f_0(y), f_1(y), ..., f_n(y)) \in R^{tz}$ for every $y \in Y$. Therefore $f_i(y) \in \mathrm{Cl}_Z(\{f_j(y)\})$ for every $y \in Y$, that is $f_i \in \mathrm{Cl}_{C_\tau(Y,Z)}(\{f_j\})$ which means that $\widehat{F}(j) = f_j \in H$.

Hence the map $\widehat{F}$ is continuous and the topology τ is $\mathbf{F}$-splitting. □

The next corollary follows by the fact that for $\mathbf{F}=\mathbf{S}$ (respectively, for $\mathbf{F}=\mathbf{D}$) then the 2-tuple relations R and R^{τ} on $C(Y, Z)$ coincide with the relations $\leq$ and $\overset{\tau}{\leq}$ (respectively, with the relations $\sim$ and $\overset{\tau}{\sim}$).

Corollary 2.9. *The following (see* [7]*) are true:*

(1) A topology τ on $C(Y, Z)$ is $\mathbf{S}$-splitting if and only if the condition $f \leq g$ implies $f \overset{\tau}{\leq} g$.

(2) A topology τ on $C(Y, Z)$ is $\mathbf{D}$-splitting if and only if the condition $f \sim g$ implies $f \overset{\tau}{\sim} g$.

Theorem 2.10. *A topology τ on $C(Y, Z)$ is $\mathbf{F}$-jointly continuous if and only if $R^{\tau} \subseteq R$.*

Proof. Let τ be an $\mathbf{F}$-jointly continuous topology on $C(Y, Z)$. Suppose that $(f_0, f_1, ..., f_n) \in R^{\tau}$. We need to prove that $(f_0, f_1, ..., f_n) \in R$.

Let $G : \mathbf{F}\to C_\tau(Y, Z)$ be a map for which $G(i) = f_i$ for every $i \in \mathbf{F}$. We prove that G is continuous. Let H be an open subset of $C_\tau(Y, Z)$ such that $f_i \in H$. It is suffices to prove that $G(\mathbf{U}_i) \subseteq H$. Let $j \in \mathbf{U}_i$. Since, $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$. and $(f_0, f_1, ..., f_n) \in R^{\tau}$ we have $f_i \in \mathrm{Cl}_{C_\tau(Y,Z)}(\{f_j\})$. Therefore $G(j) = f_j \in H$, that is the map G is continuous.

Moreover, since τ is $\mathbf{F}$-jointly continuous, the map $\widetilde{G} : \mathbf{F}\times Y \to Z$ is also continuous.

Now, we prove that $(f_0, f_1, .., f_n) \in R$. Let $y \in Y$. Consider the $(n+1)$-tuple $(f_0(y), f_1(y), ..., f_n(y))$ and let $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$. It is suffices to prove prove that $f_i(y) \in \mathrm{Cl}_Z(\{f_j(y)\})$. Let U be an open neighborhood of $f_i(y)$ in Z. Since $\widetilde{G}(i,y) = f_i(y)$ we have $\widetilde{G}^{-1}(U)$ is an open subset of $\mathbf{F}\times Y$ containing the point (i,y). There exist an open neighborhood V of i in $\mathbf{F}$ and an open neighborhood W of y in Y such that $V \times W \subseteq \widetilde{G}^{-1}(U)$. Since $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$ we have that $j \in V$ and therefore $(j,y) \in \widetilde{G}^{-1}(U)$, which means that $\widetilde{G}(j,y) = f_j(y) \in U$. Thus, $f_i(y) \in \mathrm{Cl}_Z(\{f_j(y)\})$. Hence, $(f_0, f_1, ..., f_n) \in R$.

Conversely, let τ be a topology on $C(Y,Z)$ such that the condition $(f_0, f_1, ..., f_n) \in R^\tau$ implies $(f_0, f_1, ..., f_n) \in R$. We prove that τ is $\mathbf{F}$-jointly continuous.

Let $G : \mathbf{F} \to C_\tau(Y,Z)$ be a continuous map such that $G(i) = f_i$ for every $i \in \mathbf{F}$. Then the $(n+1)$-tuple $(f_0, f_1, ..., f_n)$ belongs to R^τ. Indeed, let $i \in \mathrm{Cl}_{\mathbf{F}}(\{j\})$ and H be an open neighborhood of f_i in $C_\tau(Y,Z)$. Since G is continuous, the set $G^{-1}(H)$ is an open subset of $\mathbf{F}$ containing the point i. Hence, $j \in G^{-1}(H)$ and, therefore, $G(j) = f_j \in H$, which means that $f_i \in \mathrm{Cl}_{C_\tau(Y,Z)}(\{f_j\})$. Thus, $(f_0, f_1, ..., f_n) \in R^\tau$.

Now, we consider the map $\widetilde{G} : \mathbf{F}\times Y \to Z$ and prove that this map is continuous. Let $(i,y) \in \mathbf{F}\times Y$. Suppose that U is an open subset of Z such that $\widetilde{G}(i,y) = G(i)(y) = f_i(y) \in U$. Since the map f_i is continuous and $f_i(y) \in U$, there exists an open neighborhood W of y in Y such that $f_i(W) \subseteq U$. To prove that $\widetilde{G}$ is continuous it is suffices to prove that

$$\widetilde{G}(\mathbf{U}_i \times W) \subseteq U.$$

Indeed, let $(j,y') \in \mathbf{U}_i \times W$. Then $j \in \mathbf{U}_i$, that is $i \in Cl_{\mathbf{F}}(\{j\})$. By the above $f_i \in \mathrm{Cl}_{C_\tau(Y,Z)}(\{f_j\})$. Since $(f_0, f_1, ..., f_n) \in R^\tau$, by assumption we have $(f_0, f_1, ..., f_n) \in R$. Thus $f_i(y) \in \mathrm{Cl}_Z(\{f_j(y)\})$ for every $y \in Y$ and therefore $f_i(y') \in \mathrm{Cl}_Z(\{f_j(y')\})$. Hence $\widetilde{G}(j,y') = G(j)(y') = f_j(y') \in U$.

Thus, $\widetilde{G}$ is continuous and therefore τ is an $\mathbf{F}$-jointly continuous topology. □

Corollary 2.11. *The following (see* [7]*) are true:*

(1) A topology τ on $C(Y,Z)$ is **S***-jointly continuous if and only if the condition $g \overset{\tau}{\leq} f$ implies $g \leq f$.*

(2) A topology τ on $C(Y,Z)$ is **D***-jointly continuous if and only if the condition $f \overset{\tau}{\sim} g$ implies $f \sim g$.*

Remark 2.12. The first five Theorems of this paper can be obtained by the last two Theorems provided that:

(1) For the trivial topology, and hence, for every topology τ on the set $C(Y,Z)$ we have $R^\tau \subseteq R$ if and only if the topology of Z is trivial.

(2) If for the discrete topology, and hence, for every topology τ on $C(Y,Z)$ we have $R \subseteq R^\tau$, then Z is T_0 space.

(3) Let Z be a T_1 space. Then, for the discrete topology, and hence, for every topology τ on $C(Y,Z)$ we have $R \subseteq R^{\tau}$.

(4) For the pointwise topology, for the compact open topology, and for the Isbell topology τ on $C(Y,Z)$ we have $R^{\tau} = R$.

(5) For the topology $\tau_{\mathcal{Q}(\mathcal{U})}$ on the set $C(Y,Z)$ which generated by a quasi-uniformity $\mathcal{U}$ we have $R = R^{\tau_{\mathcal{Q}(\mathcal{U})}}$.

The above statements can be easily proved.

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