n-Tuple relations and topologies on function spaces

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Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

ABSTRACT. In [7] some results concerning $S$-splitting, $S$-jointly continuous, $D$-splitting and $D$-jointly continuous topologies are considered, where $S$ and $D$ are the Sierpinski space and the double-point space, respectively. Here we generalize these results replacing the spaces $S$ and $D$ by any finite space.

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1. Introduction.

By $Y$ and $Z$ we denote two fixed topological spaces and by $t_Z$ the topology of $Z$. By $C(Y, Z)$ we denote the set of all continuous maps of $Y$ into $Z$. If $\tau$ is a topology on the set $C(Y, Z)$, then the corresponding topological space is denoted by $C_\tau(Y, Z)$.

Let $X$ be a space and $F : X \times Y \to Z$ be a continuous map. By $F_x$, where $x \in X$, we denote the continuous map of $Y$ into $Z$, for which $F_x(y) = F(x, y)$, for every $y \in Y$. By $\hat{F}$ we denote the map of $X$ into the set $C(Y, Z)$, for which $\hat{F}(x) = F_x$ for every $x \in X$.

Let $G$ be a map of the space $X$ into the set $C(Y, Z)$. By $\tilde{G}$ we denote the map of the space $X \times Y$ into the space $Z$, for which $\tilde{G}(x, y) = G(x)(y)$ for every $(x, y) \in X \times Y$.

A topology $t$ on $C(Y, Z)$ is called splitting if for every space $X$, the continuity of a map $F : X \times Y \to Z$ implies that of the map $\hat{F} : X \to C_t(Y, Z)$. A topology $t$ on $C(Y, Z)$ is called jointly continuous if for every space $X$, the continuity of a map $G : X \to C_t(Y, Z)$ implies that of the map $\tilde{G} : X \times Y \to Z$ (see [5], [1], [2] and [3]).

If in the above definitions it is assumed that the space $X$ belongs to a given family $A$ of spaces, then the topology $\tau$ is called $A$-splitting (respectively,
\(A-\textit{jointly continuous}\) (see [6]). In the present paper we shall considered only the case \(A = \{F\}\), where \(F\) is a space, and instead of \(A\)-splitting and \(A\)-jointly continuous we write \(F\)-splitting and \(F\)-jointly continuous.

Let \(X\) be a space with a topology \(\tau\). We denote (see, for example, [8]) by \(\leq\) (respectively, by \(\sim\)) a preorder (respectively, an equivalence relation) on \(X\) defined as follows: if \(x, y \in X\), then we write \(x \leq y\) (respectively, \(x \sim y\)) if and only if \(x \in \text{Cl}_X(\{y\})\) (respectively, \(x \in \text{Cl}_X(\{y\})\) and \(y \in \text{Cl}_X(\{x\})\)). (By \(\text{Cl}_X(Q)\) we denote the closure of a set \(Q\) in the space \(X\).

On the set \(C(Y, Z)\) we denote a preorder \(\leq\) (respectively, an equivalence relation \(\sim\)) as follows: if \(g, f \in C(Y, Z)\), then we write \(g \leq f\) (respectively, \(g \sim f\)) if \(g(y) \leq f(y)\) (respectively, \(g(y) \sim f(y)\)) for every \(y \in Y\) (see, for example, [7]).

By \(S\) we denote the Sierpinski space, that is, the set \(\{0, 1\}\) equipped with the topology \(\tau(S) \equiv \{\emptyset, \{0, 1\}, \{1\}\}\), and by \(D\) the set \(\{0, 1\}\) with the trivial topology. In [7] the notions of \(S\)-splitting and \(S\)-jointly continuous (respectively, \(D\)-splitting and \(D\)-jointly continuous) topologies are characterized by the above preceders (respectively, equivalence relations) on \(C(Y, Z)\). By the trivial topology on a set \(X\) we mean the topology \(\{\emptyset, X\}\).

Let \(\mathcal{U}\) be a quasi-uniformity on the space \(Z\) (see, for example, [4]). This quasi-uniformity defines on the set \(C(Y, Z)\) a quasi-uniformity \(\mathcal{Q}(\mathcal{U})\) as follows (see [11]): the set of all subsets of \(C(Y, Z)\) of the form

\[\{(f, g) \in C(Y, Z) \times C(Y, Z) : (f(y), g(y)) \in U, \text{ for every } y \in Y\},\]

where \(U \in \mathcal{U}\), is a basis for the quasi-uniformity \(\mathcal{Q}(\mathcal{U})\). We denote by \(\tau_{\mathcal{Q}(\mathcal{U})}\) (see [11]) the topology on \(C(Y, Z)\), which is defined by the quasi-uniformity \(\mathcal{Q}(\mathcal{U})\), that is: the subbasic neighborhoods of an arbitrary element \(f \in C(Y, Z)\) in \(\tau_{\mathcal{Q}(\mathcal{U})}\) are of the form: \((Y, U)[f] = \{g \in C(Y, Z) : (f, g) \in (Y, U)\},\) where \(U \in \mathcal{U}\). In this case we shall say also that \(\tau_{\mathcal{Q}(\mathcal{U})}\) is generated by the quasi-uniformity \(\mathcal{U}\).

Let \(\mathcal{O}(Y)\) be the family of all open sets of the space \(Y\). The \textit{Scott topology} on \(\mathcal{O}(Y)\) (see, for example, [8]) is defined as follows: a subset \(\mathcal{H}\) of \(\mathcal{O}(Y)\) is open if:

\((\alpha)\) the conditions \(U \in \mathcal{H}, V \in \mathcal{O}(Y),\) and \(U \subseteq V\) imply \(V \in \mathcal{H},\) and

\((\beta)\) for every collection of open sets of \(Y\), whose union belongs to \(\mathcal{H}\), there are finitely many elements of this collection whose union also belongs to \(\mathcal{H}\).

The \textit{Isbell topology} \(\tau_{\mathcal{I}}\) on \(C(Y, Z)\) (see [9] and [10]) is the topology for which the family of all sets of the form

\[(\mathcal{H}, U) = \{f \in C(Y, Z) : f^{-1}(U) \in \mathcal{H}\},\]

where \(\mathcal{H}\) is Scott open in \(\mathcal{O}(Y)\) and \(U \in \mathcal{O}(Z)\), is a subbasis.

The \textit{pointwise topology} (see, for example, [3]) \(\tau_{p}\) on \(C(Y, Z)\) is the topology for which the family of all sets of the form

\[(f, V) = \{y \in Y : f(y) \in V\},\]
Let \( j \in \mathbb{R} \) and denote the intersection of all open neighborhoods of \( j \) such that \( \tau = \text{splitting} \). Then, we have \( \tau = \text{splitting} \) if and only if the topology of \( C(Y, Z) \) is jointly continuous (see [11]).

Below, we recall some well known results:

1. The pointwise topology, the compact open topology and the Isbell topology on \( C(Y, Z) \) are always splitting (see, for example, [1], [2], [3], [5], [9] and [10]).
2. The compact open topology on \( C(Y, Z) \) is jointly continuous if \( Y \) is locally compact (see [5] and [2]).
3. The Isbell topology on \( C(Y, Z) \) is jointly continuous if \( Y \) is corecompact (see, for example, [9]).
4. The topology \( \tau_{\mathcal{U}(U)} \) is jointly continuous (see [11]).

\[ (\{y\}, U) = \{f \in C(Y, Z) : f(y) \in U\}, \]

where \( y \in Y \) and \( U \in \mathcal{O}(Z) \), is a subbasis.

The \textit{compact open} (see [5]) topology \( \tau_c \) on \( C(Y, Z) \) is the topology for which the family of all sets of the form

\[ (K, U) = \{f \in C(Y, Z) : f(K) \subseteq U\}, \]

where \( K \) is a compact subset of \( Y \) and \( U \in \mathcal{O}(Z) \), is a subbasis.

Below, we recall some well known results:

1. The pointwise topology, the compact open topology and the Isbell topology on \( C(Y, Z) \) are always splitting (see, for example, [1], [2], [3], [5], [9] and [10]).
2. The compact open topology on \( C(Y, Z) \) is jointly continuous if \( Y \) is locally compact (see [5] and [2]).
3. The Isbell topology on \( C(Y, Z) \) is jointly continuous if \( Y \) is corecompact (see, for example, [9]).
4. The topology \( \tau_{\mathcal{U}(U)} \) is jointly continuous (see [11]).

2. \textbf{\textit{F}}-\textit{Splitting and F-Jointly Continuous Topologies.}

In the paper we denote by \( \mathbf{F} \) a non-discrete space which is the set \( \{0, 1, \ldots, n\} \), \( n > 0 \), equipped with an arbitrary fixed topology. By \( U_j, j = 0, 1, \ldots, n \), we denote the intersection of all open neighborhoods of \( j \) in \( \mathbf{F} \).

It is clear that if \( \mathbf{F} \) is the discrete space, then every topology \( \tau \) on \( C(Y, Z) \) is \( \mathbf{F} \)-splitting and \( \mathbf{F} \)-jointly continuous.

\textbf{Theorem 2.1.} The trivial topology and, hence, every topology on the set \( C(Y, Z) \) is \( \mathbf{F} \)-jointly continuous if and only if the topology of \( Z \) is trivial.

\textbf{Proof.} Suppose that the topology of \( Z \) is trivial. Then for any topology \( \tau \) on \( C(Y, Z) \) and any continuous map \( G : \mathbf{F} \to C_\tau(Y, Z) \), the map \( \tilde{G} : \mathbf{F} \times Y \to Z \) is trivially continuous, that is \( \tau \) is \( \mathbf{F} \)-jointly continuous.

Conversely, suppose that the trivial topology \( \tau \) on \( C(Y, Z) \) is \( \mathbf{F} \)-jointly continuous. We prove that the topology of \( Z \) is trivial. Indeed, in the opposite case, there exist two distinct elements \( z_1, z_2 \) of \( Z \) and an open subset \( U \) of \( Z \) such that \( z_1 \in U \) and \( z_2 \notin U \). We consider the maps \( f, g \in C(Y, Z) \) such that \( f(Y) = \{z_1\} \) and \( g(Y) = \{z_2\} \). Denote by \( i \), the element of \( \mathbf{F} \) such that \( U_i \neq \{i\} \). Let \( G : \mathbf{F} \to C_\tau(Y, Z) \) be a map such that \( G(i) = f \) and \( G(j) = g \), for every \( j \in \mathbf{F} \setminus \{i\} \). Since \( \tau \) is trivial, the map \( G \) is continuous. Since \( \tau \) is \( \mathbf{F} \)-jointly continuous, the map \( \tilde{G} : \mathbf{F} \times Y \to Z \) is also continuous.

By the definition of \( \tilde{G} \), \( \tilde{G}(i, y) = G(i)(y) = f(y) = z_1 \in U, y \in Y \). Therefore for a fixed \( y \in Y \) there exists an open neighborhood \( V_y \) such that \( \tilde{G}(U_i \times V_y) \subseteq U \).

Let \( j \in U_i \setminus \{i\} \). Then, we have \( \tilde{G}(j, y) = G(j)(y) = g(y) = z_2 \notin U \) which is a contradiction. Thus the topology of \( Z \) is trivial. \( \Box \)

\textbf{Theorem 2.2.} If the discrete topology, and hence, every topology on \( C(Y, Z) \) is \( \mathbf{F} \)-splitting, then \( Z \) is a \( T_0 \) space.
Proof. Suppose that the discrete topology $\tau$ on $C(Y,Z)$ is $F$-splitting and $Z$ is not $T_0$ space. We shall construct a continuous map $F : F \times Y \to Z$ such that $\hat{F}$ is not continuous, which will be a contradiction.

There exist two distinct elements $z_1, z_2$ of $Z$ such that either $z_1, z_2 \in V$ or $z_1, z_2 \not\in V$ for every open subset $V$ of $Z$. Let $i$ be an element of $F$ such that $U_i \neq \{i\}$. We consider the map $F : F \times Y \to Z$ such that $F(i, y) = z_1$ for every $y \in Y$, and $F(j, y) = z_2$ for every $j \in F \setminus \{i\}$ and $y \in Y$. Let $V$ be an open subset of $Z$. Then, either $F^{-1}(V) = F \times Y$ or $F^{-1}(V) = \emptyset$, which means that $F$ is continuous.

By the definition of $\hat{F} : F \to C_{\tau}(Y,Z)$ we have $\hat{F}(i)(Y) = \{z_1\}$, and $\hat{F}(j)(Y) = \{z_2\}$ for every $j \in F \setminus \{i\}$. Let $j \in U_i \setminus \{i\}$. Then $\hat{F}(j) \not\in \{\hat{F}(i)\}$, that is, $\hat{F}(U_i) \not\subseteq \{\hat{F}(i)\}$, which means that $\hat{F}$ is not continuous. $\square$

**Theorem 2.3.** Let $Z$ be a $T_1$ space. Then, the discrete topology, and hence, every topology on $C(Y,Z)$ is $F$-splitting.

Proof. Let $\tau$ be the discrete topology on $C(Y,Z)$ and $F : F \times Y \to Z$ a continuous map. We prove that the map $\hat{F} : F \to C_{\tau}(Y,Z)$ is continuous.

Let $i \in F$ and $\hat{F}(i) = f$. Then $f \in \{f\} \in \tau$. It is suffices to prove that $\hat{F}(U_i) \subseteq \{f\}$, that is, $\hat{F}(j) = f$ for every $j \in U_i$. Let $j \in U_i$ and $y$ be an arbitrary point of $Y$. We need to prove that $\hat{F}(j)(y) = f(y)$. Let $U$ be an arbitrary open neighborhood of $f(y) = \hat{F}(i)(y) = F(i, y)$ in $Z$. Since the map $F$ is continuous there exists an open neighborhood $V_y$ of $y$ in $Y$ such that $F(U_i \times V_y) \subseteq U$. Therefore, $F(j, y) = \hat{F}(j)(y) \in U$, which means that $f(y) \in Cl_Z(\{\hat{F}(j)(y)\})$. Since $Z$ is a $T_1$ space, $f(y) = \hat{F}(j)(y)$. Hence, $\hat{F}(j) = f$. Thus, the map $\hat{F} : F \to C_{\tau}(Y,Z)$ is continuous and therefore the topology $\tau$ on $C(Y,Z)$ is $F$-splitting. $\square$

**Theorem 2.4.** The pointwise topology $\tau_p$, the compact-open topology $\tau_c$, and the Isbell topology $\tau_{is}$ on $C(Y,Z)$ are $F$-splitting and $F$-jointly continuous.

Proof. First, we prove that $\tau_p$ is $F$-jointly continuous. Let $G : F \to C_{\tau_p}(Y,Z)$ be a continuous map. We need to prove that the map $\tilde{G} : F \times Y \to Z$ is continuous.

Let $(i, y) \in F \times Y$ and $U$ be an arbitrary open neighborhood of $\tilde{G}(i, y) = G(i)(y)$ in $Z$. Then $G(i) \in (\{y\}, U)$. Since $G$ is continuous, $G(U_i) \subseteq (\{y\}, U)$. Also, since the map $G(j, \cdot) \in U_i \times Y$ is continuous and $G(j)(y) \in U$ there exists an open neighborhood $V^j_y$ of $y$ in $Y$ such that $G(j)(V^j_y) \subseteq U$. Let $V_y = \cap\{V^j_y : j \in U_i\}$. Then, $\tilde{G}(U_i \times V_y) \subseteq U$. Thus, the map $\tilde{G}$ is continuous and the therefore the topology $\tau_p$ is $F$-jointly continuous.

Since $\tau_p \subseteq \tau_c$ and $\tau_p \subseteq \tau_{is}$ (see [10]) the topologies $\tau_c$ and $\tau_{is}$ are also $F$-jointly continuous.

Finally, since the topologies $\tau_p$, $\tau_c$ and $\tau_{is}$ are splitting, they are also $F$-splitting. $\square$
Theorem 2.5. The topology $\tau_{Q(\mathcal{U})}$ on the set $C(Y, Z)$ generated by a quasi-uniformity $\mathcal{U}$ on the space $Z$ is $\mathbf{F}$-splitting and $\mathbf{F}$-jointly continuous.

Proof. Let $\mathcal{U}$ be a quasi-uniformity on the space $Z$. Since $\tau_{Q(\mathcal{U})}$ is jointly continuous (see [11]), this topology is also $\mathbf{F}$-jointly continuous.

We prove that $\tau_{Q(\mathcal{U})}$ is $\mathbf{F}$-splitting. Let $F : \mathbf{F} \times Y \to Z$ be a continuous map. We need to prove that $\hat{F} : \mathbf{F} \to C_{\tau_{Q(\mathcal{U})}}(Y, Z)$ is continuous. Let $i \in \mathbf{F}$ and $\hat{F}(i) = f_i$. The set $(Y, U)[f_i] = \{h \in C(Y, Z) : (f_i, h) \in (Y, U)\}$, where $U$ is an element of $\mathcal{U}$, is an open neighborhood of $f_i$ in $C_{\tau_{Q(\mathcal{U})}}(Y, Z)$. We prove that $\hat{F}(U_i) \subseteq (Y, U)[f_i]$. Let $j \in U_i$. It is sufficient to prove that $\hat{F}(j) = f_j \in (Y, U)[f_i]$, that is, $f_j \in (Y, U)[f_i]$ or $(f_i(y), f_j(y)) \in U$ for every $y \in Y$. Let $y \in Y$ and $U[f_i(y)] = \{z \in Z : (f_i(y), z) \in U\}$. Since $F$ is continuous there exists an open neighborhood $V_y$ of $y$ in $Y$ such that $F(U_i \times V_y) \subseteq U[f_i(y)]$. So, for the element $(j, y)$ of $U_i \times V_y$ we have $F(j, y) = f_j(y) \in U[f_i(y)]$ or $(f_i(y), f_j(y)) \in U$. Thus, the map $\hat{F}$ is continuous and therefore the topology $\tau_{Q(\mathcal{U})}$ is $\mathbf{F}$-splitting.

Definition 2.6. For every space $X$ with a topology $t$ we define an $(n+1)$-tuple relation denoted by $R_t^1$ in $X$ as follows: an $(n+1)$-tuple $(x_0, x_1, ..., x_n)$ of elements of $X$ belongs to $R_t^1$ if for every $i, j \in \mathbf{F}$, $x_i \in \text{Cl}_X(\{x_j\})$ provided that $i \in \text{Cl}_F(\{j\})$.

We observe that if $t_1, t_2$ are two topologies on a set $X$ such that $t_1 \subseteq t_2$, then $R_{t_2}^1 \subseteq R_{t_1}^1$.

Definition 2.7. On the set $C(Y, Z)$ we define an $(n+1)$-tuple relation denoted by $R$ as follows: an $(n+1)$-tuple $(f_0, f_1, ..., f_n)$ of elements of $C(Y, Z)$ belongs to $R$ if $(f_0(y), f_1(y), ..., f_n(y)) \in R_{t_j}^1$ for every $y \in Y$.

Below we give necessary and sufficient conditions for an arbitrary topology $\tau$ on $C(Y, Z)$ to be $\mathbf{F}$-splitting or $\mathbf{F}$-jointly continuous.

Theorem 2.8. A topology $\tau$ on $C(Y, Z)$ is $\mathbf{F}$-splitting if and only if $R \subseteq R_\tau$.

Proof. Let $\tau$ be an $\mathbf{F}$-splitting topology on $C(Y, Z)$. Suppose that $(f_0, f_1, ..., f_n) \in R$. We need to prove that $(f_0, f_1, ..., f_n) \in R_\tau$.

Consider $F : \mathbf{F} \times Y \to Z$ a map for which $F(i, y) = f_i(y)$, for every $i \in \mathbf{F}$ and $y \in Y$. This map is continuous. Indeed, let $U$ be an open neighborhood of $f_i(y)$ in $Z$. Since $f_i$ is continuous, the set $f_i^{-1}(U)$ is open neighborhood of $y$ in $Y$. Therefore it is sufficient to prove that:

\[F(U_i \times f_i^{-1}(U)) \subseteq U.\]

Let $(j, y') \in U_i \times f_i^{-1}(U)$. By the definition of $F$, $F(j, y') = f_j(y')$. Since $j \in U_i$ we have $i \in \text{Cl}_F(\{j\})$. Also, by the definition of the $(n+1)$-tuple relation $R$ we have $f_i(y') \in \text{Cl}_Z(f_i(y'))$. Since $f_i(y') \in U$ we have $f_j(y') \in U$. Thus, $F(U_i \times f_i^{-1}(U)) \subseteq U$, that is $F$ is continuous.

Furthermore, since $\tau$ is $\mathbf{F}$-splitting, the map $\hat{F} : \mathbf{F} \to C_\tau(Y, Z)$ is continuous.
Now, we prove that \((f_0, f_1, \ldots, f_n) \in R^r\). Let \(i, j \in F\) such that \(i \in \text{Cl}_F\{j\}\). We need to prove that \(f_i \in \text{Cl}_{C_r(Y, Z)}(\{f_j\})\). Let \(W\) be an open neighborhood of \(f_i\) in \(C_r(Y, Z)\). Then, \(\widetilde{F}^{-1}(W)\) is an open neighborhood of \(i\) in \(F\) and therefore \(j \in \widetilde{F}^{-1}(W)\). This means that \(\widetilde{F}(j) = f_j \in W\) and therefore \(f_i \in \text{Cl}_{C_r(Y, Z)}(\{f_j\})\). Hence, \((f_0, f_1, \ldots, f_n) \in R^r\).

Conversely, let \(\tau\) be a topology on \(C(Y, Z)\) such that the condition
\((f_0, f_1, \ldots, f_n) \in R\) implies \((f_0, f_1, \ldots, f_n) \in R^r\). We prove that \(\tau\) is \(F\)-splitting.

Let \(F : F \times Y \to Z\) be a continuous map. Consider the map \(\widehat{F} : F \to C_r(Y, Z)\) and let \(\widehat{F}(i) = f_i, i \in F\). First, we prove that \((f_0, f_1, \ldots, f_n) \in R\). Indeed, let \(y \in Y\). Consider the \((n + 1)\)-tuple \((f_0(y), f_1(y), \ldots, f_n(y))\) and suppose that \(i \in \text{Cl}_F\{j\}\). Let \(U\) be an open neighborhood of \(f_i(y)\) in \(Z\). Since \(F(y, y) = f_i(y)\) and \(F\) is continuous, the set \(F^{-1}(U)\) is an open neighborhood of \((i, y)\) in \(F \times Y\). Therefore there exist open sets \(V\) and \(W\) of \(F\) and \(Y\), respectively, such that \((i, y) \in V \times W \subseteq F^{-1}(U)\). This means that \(j \in V\) and \(F(j, y) = f_j(y) \in U\) and therefore \(f_j(y) \in \text{Cl}_Z(f_j(y))\), that is \((f_0(y), f_1(y), \ldots, f_n(y)) \in R^r\). Hence \((f_0, f_1, \ldots, f_n) \in R\). By the assumption, \((f_0, f_1, \ldots, f_n) \in R^r\).

Now, we prove that \(\widehat{F}\) is continuous. Let \(\widehat{F}(i) = f_i\) and \(H\) be an open neighborhood of \(f_i\) in \(C_r(Y, Z)\). It suffices to prove that
\[\widehat{F}(U_i) \subseteq H.\]

Let \(j \in U_i\). Then \(i \in \text{Cl}_F\{j\}\). Since \((f_0, f_1, \ldots, f_n) \in R\) we have \((f_0(y), f_1(y), \ldots, f_n(y)) \in R^r\) for every \(y \in Y\). Therefore \(f_i(y) \in \text{Cl}_Z(f_j(y))\) for every \(y \in Y\), that is \(f_i \in \text{Cl}_C_r(Y, Z)(f_j)\) which means that \(\widehat{F}(j) = f_j \in H\).

Hence the map \(\widehat{F}\) is continuous and the topology \(\tau\) is \(F\)-splitting.

The next corollary follows by the fact that for \(F = S\) (respectively, for \(F = D\)) then the 2-tuple relations \(R\) and \(R^r\) on \(C(Y, Z)\) coincide with the relations \(\leq\) and \(\leq\) (respectively, with the relations \(\sim\) and \(\sim\)).

**Corollary 2.9.** The following (see [7]) are true:

1. A topology \(\tau\) on \(C(Y, Z)\) is \(S\)-splitting if and only if the condition \(f \leq g\) implies \(f \leq g\).
2. A topology \(\tau\) on \(C(Y, Z)\) is \(D\)-splitting if and only if the condition \(f \sim g\) implies \(f \sim g\).

**Theorem 2.10.** A topology \(\tau\) on \(C(Y, Z)\) is \(F\)-jointly continuous if and only if \(R^r \subseteq R\).

**Proof.** Let \(\tau\) be an \(F\)-jointly continuous topology on \(C(Y, Z)\). Suppose that \((f_0, f_1, \ldots, f_n) \in R^r\). We need to prove that \((f_0, f_1, \ldots, f_n) \in R\).

Let \(G : F \to C_r(Y, Z)\) be a map for which \(G(i) = f_i\) for every \(i \in F\). We prove that \(G\) is continuous. Let \(H\) be an open subset of \(C_r(Y, Z)\) such that \(f_i \in H\). It is sufficient to prove that \(G(U_i) \subseteq H\). Let \(j \in U_i\). Since, \(i \in \text{Cl}_F\{j\}\) and \((f_0, f_1, \ldots, f_n) \in R^r\) we have \(f_i \in \text{Cl}_{C_r(Y, Z)}(\{f_j\})\). Therefore \(G(j) = f_j \in H\), that is the map \(G\) is continuous.
Moreover, since $\tau$ is $\mathbf{F}$-jointly continuous, the map $\tilde{G} : \mathbf{F} \times Y \to Z$ is also continuous.

Now, we prove that $(f_0, f_1, \ldots, f_n) \in R$. Let $y \in Y$. Consider the $(n+1)$-tuple $(f_0(y), f_1(y), \ldots, f_n(y))$ and let $i \in \operatorname{Cl}_\mathbf{F}(\{j\})$. It is suffices to prove that $f_i(y) \in \operatorname{Cl}_2(\{f_j(y)\})$. Let $U$ be an open neighborhood of $f_i(y)$ in $Z$. Since $\tilde{G}(i, y) = f_i(y)$ we have $\tilde{G}^{-1}(U)$ is an open subset of $\mathbf{F} \times Y$ containing the point $(i, y)$. There exist an open neighborhood $V$ of $i$ in $\mathbf{F}$ and an open neighborhood $W$ of $y$ in $Y$ such that $V \times W \subseteq \tilde{G}^{-1}(U)$. Since $i \in \operatorname{Cl}_\mathbf{F}(\{j\})$ we have that $j \in V$ and therefore $(j, y) \in \tilde{G}^{-1}(U)$, which means that $\tilde{G}(j, y) = f_j(y) \in U$. Thus, $f_i(y) \in \operatorname{Cl}_2(\{f_j(y)\})$. Hence, $(f_0, f_1, \ldots, f_n) \in R$.

Conversely, let $\tau$ be a topology on $C(Y, Z)$ such that the condition $(f_0, f_1, \ldots, f_n) \in R^\tau$ implies $(f_0, f_1, \ldots, f_n) \in R$. We prove that $\tau$ is $\mathbf{F}$-jointly continuous.

Let $G : \mathbf{F} \to C_\tau(Y, Z)$ be a continuous map such that $G(i) = f_i$ for every $i \in \mathbf{F}$. Then the $(n+1)$-tuple $(f_0, f_1, \ldots, f_n)$ belongs to $R^\tau$. Indeed, let $i \in \operatorname{Cl}_\mathbf{F}(\{j\})$ and $H$ be an open neighborhood of $f_i$ in $C_\tau(Y, Z)$. Since $G$ is continuous, the set $G^{-1}(H)$ is an open subset of $\mathbf{F}$ containing the point $i$. Hence, $j \in G^{-1}(H)$ and, therefore, $G(j) = f_j \in H$, which means that $f_i \in \operatorname{Cl}_{C_\tau(Y, Z)}(\{f_j\})$. Thus, $(f_0, f_1, \ldots, f_n) \in R^\tau$.

Now, we consider the map $\tilde{G} : \mathbf{F} \times Y \to Z$ and prove that this map is continuous. Let $(i, y) \in \mathbf{F} \times Y$. Suppose that $U$ is an open subset of $Z$ such that $\tilde{G}(i, y) = G(i)(y) = f_i(y) \in U$. Since the map $f_i$ is continuous and $f_i(y) \in U$, there exists an open neighborhood $W$ of $y$ in $Y$ such that $f_i(W) \subseteq U$. To prove that $\tilde{G}$ is continuous it is suffices to prove that

$$\tilde{G}(U_i \times W) \subseteq U.$$ 

Indeed, let $(j, y') \in U_i \times W$. Then $j \in U_i$, that is $i \in \operatorname{Cl}_\mathbf{F}(\{j\})$. By the above $f_i \in \operatorname{Cl}_{C_\tau(Y, Z)}(\{f_j\})$. Since $(f_0, f_1, \ldots, f_n) \in R^\tau$, by assumption we have $(f_0, f_1, \ldots, f_n) \in R$. Thus $f_i(y) \in \operatorname{Cl}_2(\{f_j(y)\})$ for every $y \in Y$ and therefore $f_i(y') \in \operatorname{Cl}_2(\{f_j(y')\})$. Hence $G(j, y') = G(j)(y') = f_j(y') \in U$.

Thus, $\tilde{G}$ is continuous and therefore $\tau$ is an $\mathbf{F}$-jointly continuous topology. \hfill $\square$

**Corollary 2.11.** The following (see [7]) are true:

1. A topology $\tau$ on $C(Y, Z)$ is $\mathbf{S}$-jointly continuous if and only if the condition $g \lesssim f$ implies $g \leq f$.

2. A topology $\tau$ on $C(Y, Z)$ is $\mathbf{D}$-jointly continuous if and only if the condition $f \sim g$ implies $f \sim g$.

**Remark 2.12.** The first five Theorems of this paper can be obtained by the last two Theorems provided that:

1. For the trivial topology, and hence, for every topology $\tau$ on the set $C(Y, Z)$ we have $R^\tau \subseteq R$ if and only if the topology of $Z$ is trivial.

2. If for the discrete topology, and hence, for every topology $\tau$ on $C(Y, Z)$ we have $R \subseteq R^\tau$, then $Z$ is $T_0$ space.
Let \( Z \) be a \( T_1 \) space. Then, for the discrete topology, and hence, for every topology \( \tau \) on \( C(Y,Z) \) we have \( R \subseteq R^\tau \).

For the pointwise topology, for the compact open topology, and for the Isbell topology \( \tau \) on \( C(Y,Z) \) we have \( R^\tau = R \).

For the topology \( \tau_{Q(U)} \) on the set \( C(Y,Z) \) which generated by a quasi-uniformity \( U \) we have \( R = R^{\tau_{Q(U)}} \).

The above statements can be easily proved.

References


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