Quasi-pseudometric properties of the Nikodym-Saks space

Jesús Ferrer*

Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

ABSTRACT. For a non-negative finite countably additive measure \( \mu \) defined on the \( \sigma \)-field \( \Sigma \) of subsets of \( \Omega \), it is well known that a certain quotient of \( \Sigma \) can be turned into a complete metric space \( \Sigma(\Omega) \), known as the Nikodym-Saks space, which yields such important results in Measure Theory and Functional Analysis as Vitali-Hahn-Saks and Nikodym's theorems. Here we study some topological properties of \( \Sigma(\Omega) \) regarded as a quasi-pseudometric space.

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1. Introduction.

All throughout this paper we shall assume that the measure space \((\Omega, \Sigma, \mu)\) corresponds to a non-negative finite countably additive measure \( \mu \) defined in the \( \sigma \)-algebra \( \Sigma \) of subsets of \( \Omega \), \( 0 < \mu(\Omega) < \infty \).

Following the terminology of [5, p.156], by \( \Sigma(\Omega) \) we denote the quotient space obtained after identifying the measurable sets \( A, B \) such that their symmetric difference \( A \bigtriangleup B \) has zero measure. For the sake of convenience, we shall not use any special symbol to distinguish between the elements of \( \Sigma \) and the equivalence classes in \( \Sigma(\Omega) \).

It is shown in [5, p.156], and also in [3, p.86] and [7, p.208], that the function \( \mu(A \bigtriangleup B) \) defines a metric in \( \Sigma(\Omega) \) such that it becomes a complete metric space. This property allows one to apply Baire category arguments to obtain important results in convergence of measures such as the theorems of Vitali-Hahn-Saks and Nikodym.

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Besides, the set operations in \( \Sigma(\Omega) \) are well defined and are continuous respect to the metric considered, so that \( (\Sigma(\Omega), \cup, \cap) \) may also be regarded as a topological ring. Generalizations of this can be found in [4].

The purpose of this paper is to notice that the metric space \( \Sigma(\Omega) \) admits a quasi-pseudometric structure which determines in the standard way both the topology and the order given by set-inclusion. We shall as well revisit some topological properties of \( \Sigma(\Omega) \), such as completeness, compactness and connectedness from a quasi-pseudometric perspective.

2. Nikodym-Saks’ complete quasi-pseudometric ordered space.

Given \( A, B \in \Sigma(\Omega) \), we define

\[
q(A, B) := \mu(B \setminus A).
\]

It is immediate to verify that \( q \) is a quasi-pseudometric in \( \Sigma(\Omega) \). By \( q^{-1} \) and \( q^* \) we denote the conjugate quasi-pseudometric and the metric associated to \( q \), respectively, that is

\[
q^{-1}(A, B) = q(B, A) = \mu(A \setminus B),
\]

\[
q^*(A, B) = q(A, B) + q^{-1}(A, B) = \mu(B \setminus A) + \mu(A \setminus B) = \mu(A \cap B).
\]

It is also quite simple to see that the set operations \( \cup \) and \( \cap \) are continuous in the quasi-pseudometric space \( (\Sigma(\Omega), q) \). Also, the mapping \( A \rightarrow \Omega \setminus A \) is a quasi-uniform isomorphism from \( (\Sigma(\Omega), q) \) onto \( (\Sigma(\Omega), q^{-1}) \). In the same manner, one may easily see that \( \mu: (\Sigma(\Omega), q) \rightarrow \mathbb{R} \) is quasi-uniformly upper semicontinuous, i.e., given \( \varepsilon > 0 \), there is \( \delta > 0 \) such that, whenever \( q(A, B) < \delta \), we have \( \mu(B) - \mu(A) < \varepsilon \).

Noticing that set-inclusion \( \subseteq \) is an ordering compatible with the equivalence relation defined in \( \Sigma \), we may regard \( (\Sigma(\Omega), \subseteq) \) as an ordered space. Again, following the terminology of [6], we have the following result.

**Proposition 2.1.** \( (\Sigma(\Omega), q^*, \subseteq) \) is a metric ordered space determined by the quasi-pseudometric \( q \).

**Proof.** It all reduces to see that the graph of the order relation \( \subseteq \) coincides with \( \cap_{\varepsilon > 0} V_{\varepsilon}^{-1} \), where

\[
V_{\varepsilon}^{-1} = \{(A, B) \in \Sigma(\Omega) \times \Sigma(\Omega) : q^{-1}(A, B) < \varepsilon\}.
\]

This is simple, since \( (A, B) \in \cap_{\varepsilon > 0} V_{\varepsilon}^{-1} \) if and only if \( q^{-1}(A, B) = 0 \), which is equivalent to \( q(B, A) = \mu(A \setminus B) = 0 \). That is, \( A \setminus B = \emptyset \) and so \( A \subseteq B \). \( \square \)

Notice that the order defined by set-inclusion coincides with the so called ”specialization order” defined by the quasi-pseudometric \( q \), i.e.,

\[
A \subseteq B \iff q(B, A) = 0 \iff B \in \overline{A},
\]

that is, ”it takes no effort to move from \( B \) to \( A \), so \( A \) must be lower”.

We introduce a couple of definitions by means of which we shall show that Nikodym-Saks’ space is complete from a quasi-pseudometric perspective.
Definition 2.2. In a quasi-pseudometric space \((X, q)\), we say that a subset \(A\) is **quasi-bounded** provided there is an element \(x_0 \in A\) such that the set of reals \(\{q(x, x_0) : x \in A\}\) is bounded.

It is plain that the associated pseudometric space \((X, q^*)\) is bounded if and only if the quasi-pseudometric spaces \((X, q)\) and \((X, q^{-1})\) are both quasi-bounded.

Definition 2.3. By a **quasi-pseudometric ordered space** we mean a triple \((X, q, \leq)\) such that the quasi-pseudometric \(q\) determines the topological ordered space \((X, q^*, \leq)\) in the sense given in [6]. Thus, we say that the quasi-pseudometric ordered space \((X, q, \leq)\) is **orderly quasi-complete** whenever every quasi-bounded sequence \((x_n)_{n=1}^{\infty}\) satisfies the following two conditions:

1) \((x_n)_{n=1}^{\infty}\) admits a supremum (least upper bound) and an infimum (greatest lower bound) in \((X, \leq)\).

2) For each \(n\), if \(y_n := \inf \{x_j : j \geq n\}\), then
   \[
   q(y_n, x_n) \leq \sum_{j=n}^{\infty} q(x_{j+1}, x_j).
   \]

Following the terminology introduced in [10], if \((X, q)\) is a quasi-pseudometric space, a sequence \((x_n)_{n=1}^{\infty}\) in \(X\) is said to be **right-\(k\)-Cauchy** whenever, given \(\varepsilon > 0\), there is \(k \in \mathbb{N}\) such that, for \(n \geq m \geq k\), we have \(q(x_n, x_m) < \varepsilon\). We say that \((X, q)\) is **right-\(k\)-sequentially complete** provided every right-\(k\)-Cauchy sequence converges.

Proposition 2.4. If \((X, q, \leq)\) is orderly quasi-complete, then \((X, q)\) is right-\(k\)-sequentially complete.

**Proof.** Let \((x_n)_{n=1}^{\infty}\) be a right-\(k\)-Cauchy sequence in \((X, q)\). We define inductively an increasing sequence \((k_j)_{j=1}^{\infty}\) of positive integers such that, for each \(j\), if \(n \geq m \geq k_j\), then \(q(x_n, x_m) < 2^{-j}\). Now, since \((x_{k_j})_{j=1}^{\infty}\) is quasi-bounded, if, for each \(j\), \(y_j := \inf \{x_{k_i} : i \geq j\}\), and \(y := \sup \{y_j : j \geq 1\} = \lim_{j \to \infty} x_{k_j}\), then, for each \(j\), using condition 2 of the former definition,

\[
q(y, x_{k_j}) \leq q(y, y_j) + q(y_j, x_{k_j}) = q(y_j, x_{k_j}) \leq \sum_{i=j}^{\infty} q(x_{k_{i+1}}, x_{k_i}) < \sum_{i=j}^{\infty} 2^{-i} = 2^{-j+1}.
\]

Finally, for \(\varepsilon > 0\), let \(j_0\) be such that \(2^{-j_0+1} < \varepsilon/2\). Then, for \(n \geq k_{j_0}\), we take \(j_1 \geq j_0\) with \(k_{j_1} \geq n\), and so

\[
q(y, x_n) \leq q(y, x_{k_{j_1}}) + q(x_{k_{j_1}}, x_n) \leq 2^{-j_1+1} + 2^{-j_n} < \varepsilon.
\]

\[\square\]

Corollary 2.5. **Nikodym-Saks**’ quasi-pseudometric space \((\Sigma(\Omega), q)\) is right-\(k\)-sequentially complete.
Thus obtaining, after what we did previously, that, if \(B\) have \(k\) is \(\{\Omega, q\}\) Nikodym-Saks’ quasi-pseudometric space Corollary 2.6. Nikodym-Saks’ space can now be reobtained by means of quasi-pseudometrics.

Let \((\Omega, q)\) be a Cauchy sequence in \((\Sigma(\Omega), q)\). For each \(j \in \mathbb{N}\), there is \(k_j \in \mathbb{N}\) such that, if \(n, m \geq k_j\), then \(q^*(A_n, A_m) < 2^{-j}\). So, if \(n, m \geq k_j\), we have

\[
q(A_n, A_m) < 2^{-j}, \quad q^{-1}(A_n, A_m) < 2^{-j},
\]

thus obtaining, after what we did previously, that, if \(B := \lim_{j \to \infty} A_{k_j}\) and \(C := \lim_{j \to \infty} A_{k_j}\), then \((A_n)_{n=1}^{\infty}\) \(q\)-converges to \(B\). Now, since \((A_n)_{n=1}^{\infty}\) is \(q^{-1}\)-right-\(k\)-Cauchy, it follows that \((\Omega \setminus A_n)_{n=1}^{\infty}\) is \(q\)-right-\(k\)-Cauchy and, given that, for each \(j\),

\[
q(\Omega \setminus A_n, \Omega \setminus A_m) = q^{-1}(A_n, A_m) < 2^{-j}, \quad n, m \geq k_j,
\]

we have that \((\Omega \setminus A_n)_{n=1}^{\infty}\) \(q\)-converges to \(\lim_{j \to \infty} (\Omega \setminus A_{k_j}) = \Omega \setminus C\). Hence \((A_n)_{n=1}^{\infty}\) \(q^{-1}\)-converges to \(C\). Hence, since \(B \subseteq C\), and taking limits in

\[
\mu(C \setminus B) = q(B, C) \leq q(B, A_n) + q(A_n, C) = q(B, A_n) + q^{-1}(C, A_n),
\]

it follows that \(\mu(B \setminus C) = \mu(C \setminus B) = 0\). That is, \(B = C\) in \(\Sigma(\Omega)\), and so \((A_n)_{n=1}^{\infty}\) converges in \((\Sigma(\Omega), q^*)\).

After [6, p.84], we recall that a quasi-uniformity \(U\) in a space \(X\) is convex with respect to the order \(\leq\) whenever, given \(U \in \text{U}\), there is \(V \in \text{U}\) such that \(V \subseteq U\), and, for each \(x \in X\), \(V(x) = \{y \in X : (x, y) \in V\}\) is convex respect to \(\leq\), i.e., \(a \leq c \leq b, a, b \in V(x)\), imply \(c \in V(x)\).

After Proposition 4.19 of [6, p.84], in light of our previous result, we know that \((\Sigma(\Omega), q^*)\) is a convex metric space in the sense before defined. Nevertheless, we cannot conclude, as it happens in many metric convex spaces, that every ball \(V_{q^*}^*(A) = \{X \in \Sigma(\Omega) : q^*(A, X) < \varepsilon\}\) has to be a convex set, as our next result proves.

**Proposition 2.7.** Let \(\Omega = [0, 1]\) and let \(\lambda\) represent the Lebesgue measure. Then, for each \(0 < \varepsilon < 1/2\), there is a measurable set \(A\) such that the ball \(V_{\varepsilon}^*(A)\) is not convex with respect to set-inclusion.
Proof. Let \( \varepsilon/2 < a < 1 - \frac{3\varepsilon}{2} \). We consider the following measurable sets
\[
A = [a, a + \frac{11\varepsilon}{8}], \quad X = [a, a + \frac{\varepsilon}{8}] \cup [a + \frac{\varepsilon}{4}, a + \frac{\varepsilon}{2}] \cup [a + \varepsilon, a + \frac{11\varepsilon}{8}],
\]
\[
Y = [a - \frac{\varepsilon}{2}, a + \frac{3\varepsilon}{2}], \quad Z = [a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}] \cup [a + \varepsilon, a + \frac{11\varepsilon}{8}]
\]
\[
q^*(A, X) = \lambda(X \setminus A) + \lambda(A \setminus X) = \lambda(A \setminus X) = \frac{\varepsilon}{8} + \frac{\varepsilon}{2} = \frac{5\varepsilon}{8} < \varepsilon,
\]
\[
q^*(A, Y) = \lambda(Y \setminus A) + \lambda(A \setminus Y) = \lambda(Y \setminus A) = \frac{\varepsilon}{2} + \frac{\varepsilon}{8} = \frac{5\varepsilon}{8} < \varepsilon,
\]
\[
q^*(A, Z) = \lambda(Z \setminus A) + \lambda(A \setminus Z) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Hence, we have \( X \subseteq Z \subseteq Y, \quad X, Y \in V_\varepsilon^*(A) \), but \( Z \notin V_\varepsilon^*(A) \). \( \square \)

The following result will be needed afterwards. Let us recall first that a subset \( F \) of an ordered set \( (X, \leq) \) is said to be inductive whenever every totally ordered subset of \( F \) has an upper bound in \( X \).

**Proposition 2.8.** Every non-empty closed subset of \( (\Sigma(\Omega), q^*) \) is inductive with respect to set-inclusion.

**Proof.** Let \( F \) be a non-empty closed set in \( (\Sigma(\Omega), q^*) \). Let \( (F_i)_{i \in I} \) be a totally ordered subset of \( F \). Since \( \mu \) is finite, there exists \( \rho = \sup_{i \in I} \mu(F_i) \). We now start an inductive process by taking \( i_1 \in I \) such that \( \mu(F_{i_1}) > \rho - \frac{1}{2\varepsilon} \). Assuming already found \( F_{i_1} \subseteq F_{i_2} \subseteq ... \subseteq F_{i_n} \) in \( F \) such that, for \( j = 1, 2, ..., n \), \( \mu(F_{i_j}) > \rho - \frac{1}{2\varepsilon} \), we proceed to find \( F_{i_{n+1}} \) with the same properties. If \( \mu(F_{i_n}) > \rho - \frac{1}{2\varepsilon} \), then we set \( F_{i_{n+1}} := F_{i_n} \); if \( \mu(F_{i_n}) \leq \rho - \frac{1}{2\varepsilon} \), we find \( F_{i_{n+1}} \) in \( F \) such that \( \mu(F_{i_{n+1}}) > \rho - \frac{1}{2\varepsilon} \), and \( F_{i_{n+1}} \subseteq F_{i_n} \), otherwise, since we are dealing with totally ordered elements, we would have \( F_{i_{n+1}} \subseteq F_{i_n} \), and, \( \mu(F_{i_{n+1}}) \leq \mu(F_{i_n}) \leq \rho - \frac{1}{2\varepsilon} \), which is a contradiction. We have thus constructed an increasing sequence \( (F_{i_n})_{n=1}^\infty \) in \( F \), with \( \mu(F_{i_n}) > \rho - \frac{1}{2\varepsilon}, \quad n \in \mathbb{N} \). We set \( F := \cup_{n=1}^\infty F_{i_n} \).

Then, \( F \in \Sigma(\Omega) \), and, for each \( n \),
\[
q^*(F, F_{i_n}) = \mu(F \setminus F_{i_n}) = \mu(\cup_{j=1}^\infty (F_{i_j} \setminus F_{i_n})) \leq \mu(\cup_{j=n+1}^\infty (F_{i_j} \setminus F_{i_n}))) + \mu(\cup_{j=n+1}^\infty (F_{i_j} \setminus F_{i_n}))
\]
\[
= \mu(\cup_{j=n+1}^\infty (F_{i_j} \setminus F_{i_n})) \leq \mu(\cup_{j=n+1}^\infty (F_{i_j} \setminus F_{i_{j-1}})) = \sum_{j=n+1}^\infty \mu(F_{i_j} \setminus F_{i_{j-1}})
\]
\[
= \sum_{j=n+1}^\infty (\mu(F_{i_j}) - \mu(F_{i_{j-1}})) \leq \sum_{j=n+1}^\infty (\rho - (\rho - \frac{1}{2j-1})) = \sum_{j=n+1}^\infty \frac{1}{2j-1} = \frac{1}{2n-1}.
\]
Hence, \( (F_{i_n})_{n=1}^\infty \) converges to \( F \) in \( (\Sigma(\Omega), q^*) \). Since \( F \) is closed, it follows that \( F \in F \). We show finally that \( F \) is an upper bound for the chain \( (F_i)_{i \in I} \). Give \( i \in I \), we consider two possibilities:

If there is \( n_0 \in \mathbb{N} \) such that \( F_i \subseteq F_{i_{n_0}} \), then it is clear that \( F_i \subseteq F_{i_{n_0}} \subseteq F \).
On the contrary, if \( F_i \) is not contained in \( F_{i_n} \), \( n \in \mathbb{N} \), then again the total ordering guarantees that \( F_{i_n} \subseteq F_i \), \( n \in \mathbb{N} \), and so \( F \subseteq F_i \). Then, since \( \rho \geq \mu(F_i) \geq \mu(F) = \lim_{n} \mu(F_{i_n}) \geq \rho \), we have
\[
\mu(F_i \cap F) = \mu(F_i \setminus F) = \mu(F_i) - \mu(F) = \rho - \rho = 0.
\]
That is, \( F_i = F \). \( \square \)

3. Connectedness and compactness in Nikodym-Saks’ space.

In this section we study the topological properties of connectedness and compactness in the space \((\Sigma(\Omega), q^*)\), observing that such properties are directly related with the degree of atomicity of the measure \( \mu \). We shall introduce again some notation. For \( A \in \Sigma(\Omega) \), by \( \Sigma(A) \) we denote the collection of elements of \( \Sigma(\Omega) \) contained in \( A \), we shall also refer to this collection as a lower interval; similarly \( \Sigma_+(A) \) will stand for all the measurable supersets of \( A \) and we will refer to this as an upper interval. Let us recall that \( E \in \Sigma(\Omega) \) is called an atom when it has positive measure and the lower interval \( \Sigma(E) \) is reduced to \( \{ \emptyset, E \} \). When a measure \( \mu \) does not have any atom then it is said to be non-atomic. It is convenient to recall that a measure can only admit at most a countable amount of disjoint atoms, when \( \Omega \) admits a countable partition formed by atoms, then \( \mu \) is said to be purely atomic.

Before characterizing the connectedness of \((\Sigma(\Omega), q^*)\) in terms of atoms, let us notice that the quasi-pseudometric spaces \((\Sigma(\Omega), q)\) and \((\Sigma(\Omega), q^{-1})\) are always connected: Let us suppose that \((\Sigma(\Omega), q)\) admits two disjoint open sets \( A, B \) covering \( \Sigma(\Omega) \). Then, one of them, say \( A \), contains \( \Omega \), so there is \( \delta > 0 \) such that \( V_{\delta}(\Omega) \subseteq A \). But, \( V_{\delta}(\Omega) = \Sigma(\Omega) \). Hence, \( B = \emptyset \).

**Proposition 3.1.** The following are equivalent.

(i) No upper interval \( \Sigma_+(A) \), \( A \neq \emptyset \), is \( q^{-1}\)-open.

(ii) \( \mu \) is non-atomic.

(iii) \( (\Sigma(\Omega), q^*) \) is connected.

(iv) For each \( A \in \Sigma(\Omega) \) and \( \alpha \in [0, \mu(A)] \), there is \( B \in \Sigma(A) \) such that
\[
\mu(B) = \alpha.
\]

**Proof.** (i) \( \Rightarrow \) (ii). Assume that \( E \) is an atom. We show that the upper interval \( \Sigma_+(E) \) is \( q^{-1}\)-open. Given \( A \in \Sigma_+(E) \), let \( \delta = \mu(E) > 0 \). If \( X \in V_{\delta}^{-1}(A) \), then \( q^{-1}(A, X) < \delta \) implies \( \mu(A \setminus X) < \delta \). But,
\[
\mu(E \setminus X) = \mu(E \setminus X \setminus A) + \mu(E \cap A \setminus X) \leq \mu(E \setminus A) + \mu(A \setminus X) < \delta.
\]
Hence, \( \mu(E \setminus X) = 0 \), otherwise, since \( E \) is an atom, we would have \( \mu(E \setminus X) = \mu(E) = \delta \). Thus, \( X \in \Sigma_+(E) \).

(ii) \( \Rightarrow \) (iii). Let us assume that \((\Sigma(\Omega), q^*)\) is disconnected. So, let \( A, B \) be two non-empty disjoint closed sets covering \( \Sigma(\Omega) \). We may suppose without restriction that \( \Omega \in A \). Applying Proposition 4 to the closed set \( B \) and after Zorn’s lemma, there is a maximal element \( M \) in \((B, \subseteq)\). Since \( B \) is also open, there is \( \delta > 0 \) such that \( V_{\delta}(M) \subseteq B \). Now, since \( \Omega \setminus M \) has non-zero measure
Lemma 3.3. If \( A \) and \( B \) are partitions of \( A \), we define the following sets

\[
s_k(A) = \{ A \setminus k \}.
\]

It is plain that, for each \( k, \) the \( s_k(A) \) are disjoint and cover \( A \). Thus, if \( \Sigma(\Omega) \) is connected, so is \( \Sigma(A) \). Now, since \( \mu \) is continuous, it follows that \( \mu(\Sigma(A)) = [0, \mu(A)] \).

(iii) \( \Rightarrow \) (iv). The continuous mapping \( X \rightarrow X \cap A \) maps \( \Sigma(\Omega) \) onto \( \Sigma(A) \). Thus, we may find \( \Sigma(\Omega) \) connected, so is \( \Sigma(A) \). Now, since \( \mu \) is continuous, it follows that \( \mu(\Sigma(A)) = [0, \mu(A)] \).

(iv) \( \Rightarrow \) (i). Let us assume that \( \mu(\Sigma(\Omega)) = [0, \mu(A)] \), then \( \sum \mu(A) = \delta / 2 \). Let \( X := E \setminus A \). Then, \( \mu(E \setminus X) = \mu(A) = \delta / 2 < \delta \) implies that \( X \in V_{\delta/2}^{-1}(E) \) and consequently \( E \subseteq X \), a contradiction, since \( \mu(E \setminus X) \neq 0 \).

By recalling that a finite countably additive measure \( \lambda \) in \( (\Omega, \Sigma) \) is \( \mu \)-continuous whenever \( \lim_{\mu(X) \to 0} \lambda(X) = 0 \), we have that in this case the identity mapping is well defined and continuous from \( (\Sigma(\Omega), q^*) \) into \( (\Sigma(\Omega), q_\lambda^*) \). Therefore the following result is straightforward.

**Corollary 3.2.** If \( \lambda \) is a finite countably additive \( \mu \)-continuous measure and \( \mu \) is non-atomic, then so is \( \lambda \).

We study in the following the compactness of Nikodym-Saks’ space. As we did in the connectedness part, it is curious to notice that the quasi-pseudometric spaces \( (\Sigma(\Omega), q) \) and \( (\Sigma(\Omega), q^{-1}) \) are always compact; just recall that, for instance, any \( q \)-open cover of \( \Sigma(\Omega) \) must have a member containing \( \Omega \), consequently, this open set has to be \( \Sigma(\Omega) \).

We show next that, in some sense, the compactness of \( (\Sigma(\Omega), q^*) \) does not get along with the connectedness. As a matter of fact, we show that compactness is equivalent to \( \mu \) being purely atomic. We need, in order to do so, to introduce some more notation.

Given a sequence \( \{A_n\}_{n=1}^\infty \) in \( \Sigma(\Omega) \), for each \( n \), if \( (i_1, i_2, \ldots, i_n) \in \{0, 1\}^n \), we define the following sets

\[
A_{(i_1, i_2, \ldots, i_n)} := (\cap A_{j, 1 \leq j \leq n, i_j = 1}) \cap (\cap A_{j, 1 \leq j \leq n, i_j = 0}).
\]

It is plain that, for each \( n, \) the \( A_{(i_1, i_2, \ldots, i_n)} \) are partitions of \( \Omega \) and \( \Omega \setminus A_k, \) respectively.

**Lemma 3.3.** If \( \mu \) is non-atomic, then there is a sequence \( \{A_n\}_{n=1}^\infty \) in \( \Sigma(\Omega) \) such that, for each \( n, \)

\[
\mu(A_{(1, 1, \ldots, 1)}) = \mu(A_{(0, 0, \ldots, 0)}) = \frac{1}{4} + \frac{1}{2^{n+1}} \mu(\Omega),
\]

\[
\mu(A_{(i_1, i_2, \ldots, i_n)}) = \frac{1}{2^{n+1}} \mu(\Omega), \quad (i_1, i_2, \ldots, i_n) \notin \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\}.
\]
Proof. We give an inductive sketch of proof. The first set $A_1$ appears courtesy of the non-atomicity of the measure $\mu$ and Proposition 3.1. Once already obtained $A_1, A_2, \ldots, A_n$, again Proposition 3.1 guarantees the existence of measurable sets $B_{(i_1,i_2,\ldots,i_n)} \subseteq A_{(i_1,i_2,\ldots,i_n)}$, $(i_1,i_2,\ldots,i_n) \in \{0,1\}^n$, such that

$$
\mu(B_{(i_1,i_2,\ldots,i_n)}) = \frac{1}{2^{n+2}} \mu(\Omega), \quad (i_1,i_2,\ldots,i_n) \neq (1,1,\ldots,1),
$$

$$
\mu(B_{(1,1,\ldots,1)}) = \left(\frac{1}{4} + \frac{1}{2^{n+2}}\right) \mu(\Omega).
$$

Defining $A_{n+1} := \cup\{B_{(i_1,i_2,\ldots,i_n)} : (i_1,i_2,\ldots,i_n) \in \{0,1\}^n\}$, the induction process is done. \qed

Proposition 3.4. Nikodym-Saks’ space $(\Sigma(\Omega), q^*)$ is compact if and only if $\mu$ is purely atomic.

Proof. Assuming $(\Sigma(\Omega), q^*)$ is compact, it all reduces to show that every set $A \in \Sigma(\Omega)$, with $\mu(A) > 0$, contains atoms. If this were not so, then the restricted measure $\mu|_A$ would be non-atomic in the restricted Nikodym-Saks’ space $\Sigma(A)$. Applying the former lemma, we would find a sequence $(A_n)_{n=1}^\infty \subseteq \Sigma(A)$ satisfying the conditions there stated. It may be easily seen that the distance $q^*(A_n, A_m) = \frac{1}{4} \mu(A)$, $n \neq m$. Hence, such a sequence cannot admit any Cauchy subsequence, which contradicts the fact that $\Sigma(\Omega)$ is compact.

Conversely, let $(E_n)_{n=1}^\infty$ be an atomic partition of $\Sigma(\Omega)$. We show that $(\Sigma(\Omega), q^*)$ is homeomorphic to Cantor’s space $2^\mathbb{N}$. Consider the one-to-one and onto mapping $T : 2^\mathbb{N} \rightarrow \Sigma(\Omega)$ such that

$$
T(x) := \cup\{E_n : x_n = 1\}, \quad x \in 2^\mathbb{N}.
$$

Given $x \in 2^\mathbb{N}$, $\varepsilon > 0$, let $A = T(x)$. Since

$$
\mu(A) = \sum \left\{ \mu(E_n) : x_n = 1 \right\}, \quad \mu(\Omega \setminus A) = \sum \left\{ \mu(E_n) : x_n = 0 \right\},
$$

there are two finite subsets $F, F'$ of $\mathbb{N}$ such that $x(F) = 1$, $x(F') = 0$, and

$$
\sum \left\{ \mu(E_n) : n \in F \right\} > \mu(A) - \varepsilon, \quad \sum \left\{ \mu(E_n) : n \in F' \right\} > \mu(\Omega \setminus A) - \varepsilon.
$$

It follows that the set $V = \{y \in 2^\mathbb{N} : y(F) = 1, y(F') = 0\}$ is a neighborhood of $x$ such that $\mu(A \uplus T(y)) < \varepsilon$, $y \in V$. Finally, to see that $T^{-1}$ is also continuous, it suffices to notice that, for each $p \in \mathbb{N}$, the intervals $\Sigma^+(E_p)$ and $\Sigma(\Omega \setminus E_p)$ are closed subsets of $\Sigma(\Omega)$. \qed

4. Nikodym-Saks’ space as a topological group.

We know that $\Sigma(\Omega)$ is an abelian nilpotent group and that the symmetric difference $\ominus$ is continuous, thus $\Sigma(\Omega)$ may be regarded also as a topological group. Besides, for each $A \in \Sigma(\Omega)$, $\Sigma(\Omega)$ can be expressed as the topological direct sum of the closed subgroups $\Sigma(A)$ and $\Sigma(\Omega \setminus A)$, i.e., $\Sigma(\Omega) = \Sigma(A) \oplus \Sigma(\Omega \setminus A)$. Our aim in this section is to show that Nikodym-Saks’ space can be decomposed, in a unique way, as the topological direct sum of a connected subgroup.
(the component of the zero element $\emptyset$) plus a compact totally disconnected subgroup.

**Proposition 4.1.** There exists a unique measurable set $M$ such that $\Sigma(\Omega) = \Sigma(M) \oplus \Sigma(\Omega \setminus M)$, with $\Sigma(M)$ connected and $\Sigma(\Omega \setminus M)$ compact.

**Proof.** Let $A$ denote the countable collection, possibly empty, of atoms of $\Sigma(\Omega)$. Since the set $M := \Omega \setminus \left( \cup \{ E : E \in A \} \right)$ contains no atoms, it follows after Proposition 3.1 that the topological group $\Sigma(M)$ is connected. Now, assuming $M \neq \Omega$ (otherwise, $\Sigma(\Omega \setminus M) = \{ \emptyset \}$, clearly compact), we have that the restricted measure $\mu_{|\Omega \setminus M}$ is purely atomic, Proposition 3.4 then shows that $\Sigma(\Omega \setminus M)$ is compact. Notice also that in this case, since $\Sigma(\Omega \setminus M)$ is a copy of a Cantor space, it is totally disconnected. \hfill \Box

**Proposition 4.2.** The connected subgroup $\Sigma(M)$ before obtained coincides with the connected component of $\emptyset$.

**Proof.** Let us denote by $C$ the connected component of $\Sigma(\Omega)$ containing $\emptyset$. It is well known that $C$ is a closed topological subgroup of $\Sigma(\Omega)$. Hence, since $\emptyset \in \Sigma(M)$ and $\Sigma(M)$ is connected, it follows that $\Sigma(M) \subseteq C$. We show the reverse inclusion.

Let $A \in C$. Assume $A \setminus M \neq \emptyset$. Hence, after what we have seen before, $\Sigma(A \setminus M)$ is a totally disconnected subgroup. But, the mapping $\tau : C \rightarrow \Sigma(A \setminus M)$ such that $\tau(X) = X \cap (A \setminus M)$ is continuous, and so $\tau(C)$ is connected in $\Sigma(A \setminus M)$. Thus, $\tau(C)$ must be a singleton, but this is not so since $\emptyset, A \setminus M \in \tau(C)$. \hfill \Box

We finish by studying the properties of compactness and connectedness related with their local properties.

**Corollary 4.3.** $\Sigma(\Omega)$ is compact if and only if it is locally compact.

**Proof.** Since $\Sigma(\Omega)$ is Hausdorff, we need only show the sufficiency part. So, if $\Sigma(\Omega)$ is locally compact, we may find $\delta > 0$ such that the closed ball $V^\delta_{\emptyset}(\emptyset) = \{ X \in \Sigma(\Omega) : q^\delta(\emptyset, X) \leq \delta \}$ is compact. Let, as before, $M \in \Sigma(\Omega)$ be such that $\Sigma(M)$ is the connected component of $\emptyset$. It all reduces to see that $M = \emptyset$. If not, since $\Sigma(M)$ is connected, we can find, after Proposition 3.1, $A \in \Sigma(M)$ such that $0 < \mu(A) \leq \delta$, and so $\Sigma(A) \subseteq V^\delta_{\emptyset}(\emptyset)$. Hence $\Sigma(A)$ is compact and, after Proposition 3.4, $A$ must contain atoms. This would imply that, $M$ containing $A$ would also contain atoms, thus contradicting the definition of $M$. \hfill \Box

**Corollary 4.4.** If $\Sigma(\Omega)$ is connected, then it is locally connected.

**Proof.** Let us assume that $\Sigma(\Omega)$ is connected. We first show that every ball centered at $\emptyset$ is connected. This is a simple consequence of the facts

$$V^\delta_{\emptyset}(\emptyset) = \cup \{ \Sigma(X) : X \in V^\delta_{\emptyset}(\emptyset) \}, \quad \emptyset \in \cap \{ \Sigma(X) : X \in V^\delta_{\emptyset}(\emptyset) \},$$
and that, for any \( X \), \( \Sigma(X) \) is connected. Now, being in a topological group, since every ball centered at \( A \) is a translation of the form
\[
V_\delta^*(A) = A \uplus V_\delta^*(\emptyset),
\]
it follows that \( V_\delta^*(A) \) is also connected. \( \square \)

As it was pointed out by Professor Paolo De Lucia one can easily find examples of locally connected disconnected (but not totally disconnected) Nikodym-Saks spaces.

**Example 4.5.** Consider the measure space \((\Omega, \Sigma, \mu)\) given by (\( \lambda \) represents the Lebesgue measure in \([0,1]\) and we denote by \( L \) the class of Lebesgue-measurable subsets of \([0,1]\)):
\[
\begin{align*}
\Omega &= [0,1] \cup \{2\}, \\
\Sigma &= L \cup \{A \cup \{2\} : A \in L\}, \\
\mu | L &= \lambda, \quad \mu(A \cup \{2\}) = \lambda(A) + 1, \quad A \in L.
\end{align*}
\]
Clearly, since \( \{2\} \) is a \( \mu \)-atom, the corresponding Nikodym-Saks’ space \( \Sigma(\Omega) \) is not connected. Notice that it is neither totally disconnected, since \( \Sigma([0,1]) \) is the connected component of \( \emptyset \). And, after recalling that \( \Sigma(\Omega) = \Sigma([0,1]) \oplus \Sigma(\{2\}) \), it can be easily shown that \( \Sigma(\Omega) \) is locally connected.

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**References**


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Jesús Ferrer
Departamento de Análisis Matemático, Universidad de Valencia, Dr. Moliner 50, 46100 Burjasot (Valencia), Spain

E-mail address: Jesus.Ferrer@uv.es