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Multivalued function spaces and Atsuji spaces

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Dedicated to Professor John G. Hocking, a great teacher and researcher who initiated me into the beauty and excitement of mathematical research.

Abstract. In this paper we present two themes. The first one describes a transparent treatment of some of the recent results in graph topologies on multi-valued functions. The study includes Vietoris topology, Fell topology, Fell uniform topology on compacta and uniform topology on compacta. The second theme concerns when continuity is equivalent to proximal continuity or uniform continuity.

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1. Graph topologies on multifunctions.

Suppose (X, T_1) and (Y, T_2) are Hausdorff spaces. We set $Z = X \times Y$ and assign the product topology $T = T_1 \times T_2$. 2^Z denotes the family of closed subsets of Z and can be considered as a space $\mathcal{F}$ of all set valued maps on X to 2^Y taking points of X to (possibly empty) closed subsets of Y. We don't distinguish between a function $f \in \mathcal{F}$ and its graph $\{(x, f(x) : x \in X\} \subset Z = X \times Y$. Thus our study includes topologies on the spaces of partial maps.

For each subset E of Z and a compatible LO-proximity δ on Z [21], set

$$E^- = \{A \in 2^Z : A \cap E \neq \varnothing\}$$
$$E^+ = \{A \in 2^Z : A \subset E\}$$
$$E^{++} = \{A \in 2^Z : A \ll E\} \text{ w. r. t. } \delta$$

(Note: $A \ll E$ iff $A \underline{\delta} E^c$ where $\underline{\delta}$ denotes the negation of δ)
$CL(Z)$ denotes the family of all nonempty closed subsets of Z.
$K(Z)$ denotes the family of all nonempty compact subsets of Z.

1.1. **The Vietoris and Fell topologies on $CL(Z)$.** The **Vietoris topology** $\tau(V)$ is generated by $\{E^+ : E^c \in CL(Z)\} \cup \{E^- : E \in T\}$.

The **proximal topology** $\sigma(\delta)$ is generated by $\{E^{++} : E^c \in CL(Z)\} \cup \{E^- : E \in T\}$.

The **Fell topology** $\tau(F)$ is generated by $\{E^+ : E^c \in K(Z)\} \cup \{E^- : E \in T\}$, and if δ is EF or R, then $\tau(F)$ is also generated by $\{E^{++} : E^c \in K(Z)\} \cup \{E^- : E \in T\}$. Thus when δ is EF or R, the Fell topology equals the proximal Fell topology and this explains the reason for several beautiful results for this topology !

The paper [5] deals with only metric proximities, and [10] remains unpublished. It is not widely known that proximal hypertopologies can be studied in more general situations and not merely in metric spaces as one usually finds in the literature. (However, see the recent papers [7, 8]). We note that the Vietoris topology is itself a proximal topology i. e. $\tau(V) = \sigma(\delta_0)$, where the **fine LO-proximity** δ_0 is given by $A\delta_0 B$ iff $clA \cap clB \neq \varnothing$.
The well known Urysohn Lemma says that δ_0 is EF iff Z is normal.
When we wish to refer to hypertopologies on $CL(Y)$, we use the suffix 2 e. g.
$t_2(V)$ denotes the Vietoris topology on $CL(Y)$
$t_2(F)$ denotes the Fell topology on $CL(Y)$
$\sigma_2(\delta_2)$ denotes the proximal topology w. r. t. δ_2 on $CL(Y)$ etc.

1.2. **Weak topologies on $CL(Z)$.** For each of the topologies described above, we also have an associated

- **weak topology** wherein $CL(Z)$ (respectively $K(Z)$) is replaced by $CL(X) \times CL(Y)$ (respectively $K(X) \times K(Y)$) (see [22]) and we attach the letter "w".
- The **weak Vietoris topology** $\tau(wV)$ is generated by
$$\{E^+ : E^c \in CL(X) \times CL(Y)\} \cup \{E^- : E \in T\}.$$
- The **weak Fell topology** $\tau(wF)$ is generated by
$$\{E^+ : E^c \in K(X) \times K(Y)\} \cup \{E^- : E \in T\}.$$

For the family $\mathcal{F}$ it can be proven easily that $\tau(wF) = \tau(F)$. Combining this result with the fact that when the proximity is EF or R, $\tau(F) = \sigma(F)$ we have
$$\tau(wF) = \tau(F) = \sigma(wF) = \sigma(F).$$

1.3. **Uniform topologies.**

(a) Suppose Y has a compatible uniformity $\mathcal{V}$ and $\mathcal{V}_H$ denotes the corresponding Hausdorff uniformity (also called Bourbaki-Hausdorff uniformity) on $CL(Y)$.
A typical basic open set in the **uniform topology on compacta**, $\tau(UCC, \mathcal{V}_H)$ on $\mathcal{F}$ is of the form
$$< f, A, M >= \{g \in \mathcal{F} : \text{for all } x \in A, (f(x), g(x)) \in M\},$$
where $f \in \mathcal{F}$, $A \in K(X)$ and $M \in \mathcal{V}_H$.

(b) In [12] and [16] function space topologies akin to uniform topologies were studied. The range space was not necessarily uniformizable. Here we introduce a similar concept. Suppose $CL(Y)$ is assigned some hypertopology τ_2. Suppose W is a *symmetric* nbhd. of the diagonal in $(CL(Y) \times CL(Y), \tau_2 \times \tau_2)$.

For each $f \in \mathcal{F}$ and $A \in \mathcal{K}(X)$ we set

$$W^*(f, A) = \{g \in \mathcal{F} : \text{ for all } x \in A, (f(x), g(x)) \in W\}.$$

The topology on $\mathcal{F}$ generated by

$$\{W^*(f, A) : f \in \mathcal{F}, A \in \mathcal{K}(X) \text{ and } W \text{ a symmetric } \tau_2 \times \tau_2 \text{ nbhd. of the diagonal in } CL(Y) \times CL(Y)\}$$

is called the τ_2**-uniform topology on compacta** $\tau(UCC, \tau_2)$.

In case $(CL(Y), \tau_2)$ is uniformizable and we restrict W's to symmetric entourages, we do get a uniform topology. This is true as in (a) above or in the case of a locally compact space Y with the Fell topology $\tau_2(F)$ on $CL(Y)$. In this case $(2^Y, \tau_2(F))$ is compact Hausdorff and has a unique compatible uniformity $\mathcal{U}_F$. McCoy calls $\tau(UCC, \tau_2(F)) = \tau(UCC, \mathcal{U}_F)$ "**the Fell uniform topology (on compact sets)**". We use $\mathcal{U}$ to denote the restriction of $\mathcal{U}_F$ to Y.

Theorem 1.1 (cf. [15]). *Suppose X and Y are Hausdorff. Then*

(a) $\tau(F) \subset \tau(wV) \subset \tau(V)$.
If further, Y is locally compact then,
(b) $\tau(F) \subset \tau(UCC, \mathcal{U}_F) \subset \tau(UCC, \mathcal{V}_H)$.

Those interested in more details are referred to [3] for hypertopologies, [21] for proximities, [17] and [19] for function space topologies.

2. Embedding theorems and applications.

One of the most valuable results in function space topologies for single valued functions is the embedding of the range space in the function space via the constant functions. (cf. Theorem 2.1.1, Page 15 in [17]) In this section we prove similar results for multifunctions which are of fundamental importance in our work.

Theorem 2.1. *Suppose $\mathcal{C} = \{X \times E : E \in CL(Y)\}$ is the family of all constant multifunctions. The map $j : 2^Y \to 2^Z$ defined by $j(E) = (X \times E) \in \mathcal{C}$, is a bijection and it is easy to show that the following are embeddings:*

(a) $j : (2^Y, \tau_2(F)) \to (2^Z, \tau(F))$.
(b) $j : (2^Y, \tau_2(V)) \to (2^Z, \tau(wV))$.
(c) $j : (2^Y, \mathcal{U}_F) \to (2^Z, \tau(UCC, \mathcal{U}_F))$.
(d) $j : (2^Y, \mathcal{V}_H) \to (2^Z, \tau(UCC, \mathcal{V}_H))$.

Theorems 1.1 and 2.1 give us the following in which vertical lines show embeddings:

Theorem 2.2.

(a) *If X and Y are Hausdorff, then*

$$\begin{array}{lccccc} CL(Y): & \tau_2(F) & \subset & \tau_2(V) & & \\ \big| \, j & \big| & & \big| & & \\ \mathcal{C} \subseteq F: & \tau(F) & \subset & \tau(wV) & \subset & \tau(V) \end{array}$$

(b) *If X is Hausdorff and Y is locally compact Hausdorff, then*

$$\begin{array}{lccccc} CL(Y): & \tau_2(F) & = & \tau_2(\mathcal{U}_F) & \subset & \tau_2(\mathcal{V}_H) \\ \big| \, j & \big| & & \big| & & \big| \\ \mathcal{C} \subset \mathcal{F}: & \tau(F) & \subset & \tau(UCC,\mathcal{U}_F) & \subset & \tau(UCC,\mathcal{V}_H) \end{array}$$

Now we show how 2.2 enables us to prove some of McCoy's results in a simple manner without any further work.

Theorem 2.3 (cf. [15], Proposition 4.5)**.** *$\tau(UCC,\mathcal{V}_H) \subset \tau(UCC,\mathcal{U}_F)$ if and only if Y is compact. Moreover, $\mathcal{U} = \mathcal{V}$.*

(We note that McCoy assumes that X is also locally compact and Y is completely metrizable.)

Proof. The result follows from 2.2(b) and the known facts that $\tau_2(F) = \tau_2(\mathcal{V}_H)$ iff Y is compact and a compact Hausdorff space has a unique compatible uniformity. □

McCoy shows that $\tau(F) = \tau(UCC,\mathcal{U}_F)$ if and only if X is discrete. ([15, Proposition 4.6]) We have not yet found a "transparent" proof.

Theorem 2.4 (cf. [15], Proposition 4.9)**.** *$\tau(UCC,\mathcal{V}_H) \subset \tau(V)$ if and only if X is discrete and $(Y,\mathcal{V})$ is totally bounded.*

Proof. The result follows from 2.2 and the known fact that $\tau_2(\mathcal{V}_H) \subset \tau_2(V)$ iff $(Y,\mathcal{V})$ is totally bounded. (We note that compactness of Y follows since McCoy assumes that X is completely metrizable.) □

Theorem 2.5 (cf. [15], Proposition 4.10)**.** *$\tau(V) \subset \tau(UCC,\mathcal{V}_H)$ if and only if X is compact and $(Y,\mathcal{V})$ is Atsuji.*

Proof. We note that on $C(X,Y)$, the space of single-valued continuous functions, $\tau(V)$ equals the graph topology ([20]) and is finer than $\tau(UCC,\mathcal{V}_H)$ which equals the compact-open topology. They are equal iff X is compact ([18]). The result then follows from the above result and the known fact : $\tau_2(V) \subset \tau_2(\mathcal{V}_H)$ iff $(Y,\mathcal{V})$ is Atsuji.

(We note that a locally compact metric space is Atsuji iff it is a topological sum of a compact space and a discrete space) □

Theorem 2.6 (cf. [15], Proposition 4.12)**.** *$\tau(V) \subset \tau(UCC,\mathcal{U}_F)$ if and only if X and Y are compact.*

Proof. From 2.2 it follows that on $\mathcal{C}$, $\tau(wV) \subset \tau(UCC,\mathcal{U}_F) \Leftrightarrow \tau_2(V) \subset \tau_2(F) \Leftrightarrow Y$ is compact. The result then follows from 2.5. □

3. Atsuji spaces.

In the first course of Analysis we learn that a continuous function from a compact metric space to an arbitrary metric space is uniformly continuous. Example of a non-compact domain such as an infinite discrete space, shows that equivalence of continuity and uniform continuity does not characterize compactness. There are analogous results in uniform and proximity spaces. There is a considerable literature dealing separately with the three cases of **ATSUJI SPACES** i. e. metric, uniform and proximity spaces in which continuity is equivalent to uniform or proximal continuity. Here we tackle the three cases together, compare them and describe the results briefly. (For details see [9])

From now onwards we suppose that (X, T_1) and (Y, T_2) are Tychonoff spaces. Depending on the situation, the topology T_1 is induced by a metric d, or a uniformity $\mathcal{U}$, or an EF-proximity δ. We also suppose that Y has a compatible uniformity and an EF-proximity. We use the notation:

$C(X, Y)$ is the set of all continuous functions from X to Y.

$U(X, Y)$ is the set of all uniformly continuous functions from X to Y.

$P(X, Y)$ is the set of all proximally continuous functions from X to Y.

When $Y = \mathbb{R}$, we use the standard notations $C(X), C^*(X), P(X), P^*(X)$, etc.

X' denotes the set of all limit points of X.

4. Proximity.

Suppose (A_n) and (B_n) are sequences of sets in an EF-proximity space (X, δ).

(a) (A_n) is **proximally discretely separated** by (B_n) iff for each $n \in \mathbb{N}$, $A_n \ll B_n$ w. r. t. δ and (B_n) is discrete.

(b) In (a) if $\delta = \delta_F$ then we use the term "(A_n) is **discretely normally separated** by (B_n)".

(c) (A_n) is **functionally** (respectively **proximally**) **discrete** iff there is an $f \in C(X)$ (respectively $f \in P(X)$) such that for each $n \in \mathbb{N}$, $f(A_n) = n$.

(d) A sequence (x_n) is **pseudo-Cauchy** iff for each $m \in \mathbb{N}$, there are disjoint sets A, B of $\mathbb{N}$ beyond m such that $\{x_n : n \in A\}\delta\{x_n : n \in B\}$.

Lemma 4.1. *If (A_n) is discretely normally separated by an open family (B_n), then (A_n) is functionally separated by (B_n) i. e. there is an $f \in \mathcal{C}(X)$ such that $f(A_n) = n$ and $f(X - \cup B_n) = 0$.*

Lemma 4.2. *X is normal if and only if every discrete sequence of subsets of X is functionally discrete.*

Lemma 4.3. *Every proximally discrete sequence of sets (A_n) in (X, δ_1) is uniformly discrete w. r. t. some compatible uniformity $\mathcal{U}$ i. e. there exists a $U \in \mathcal{U}$ such that*

$$U(A_n) \cap A_m = \varnothing$$

for $m \neq n$.

We now state the main result in Atsuji spaces :

Theorem 4.4. *For an EF-proximity space (X, δ) the following are equivalent:*

(a) $C(X, Y) = P(X, Y)$ *for each EF-proximity space Y.*
(b) $C(X) = P(X)$.
(c) *If (A_n) is discretely normally separated by an open family (B_n), then (A_n) is proximally separated by (B_n) i. e. there is an $f \in P(X)$ such that $f(A_n) = n$ and $f(X - \cup B_n) = 0$.*
(d) *Each functionally discrete sequence of sets is proximally discrete.*
(e) *Each functionally discrete sequence of sets is uniformly discrete w. r. t. some compatible uniformity.*
(f) *For each pair of disjoint zero sets A, B there exists an $f \in P^*(X)$ such that $f(A) = 0$ and $f(B) = 1$.*
(g) *Each pair of disjoint zero sets are far w. r. t. δ.*
(h) $\delta = \delta_F$.
(i) $C^*(X) = P^*(X)$.

4.1. **Normal space.** It is not widely known that many results in normal spaces are true in Tychonoff spaces with compatible EF-proximities. Urysohn's Lemma of normal spaces is available in an EF-proximity space if we replace disjoint closed sets by sets that are far ! Previous authors have unnecessarily assumed normality which we have shown above is not needed. Now we show how the earlier results follow from ours given above.

Theorem 4.5. *For a normal proximity space (X, d) the following are equivalent:*

(a) $C(X, Y) = P(X, Y)$ *for each EF-proximity space Y.*
(b) $C(X) = P(X)$.
(c) *If (B_n) is a sequence of pairwise disjoint open sets, $clA_n \subset B_n$ for each n, and $\cup clA_n$ is closed, then (A_n) is proximally separated by (B_n).*
(d) *Each discrete sequence of sets is proximally discrete.*
(e) *Each discrete sequence of sets is uniformly discrete w. r. t. some compatible uniformity.*
(f) *For each pair of disjoint closed sets A, B there exists an $f \in P^*(X)$ such that $f(A) = 0$ and $f(B) = 1$.*
(g) *Each pair of disjoint closed sets are far w. r. t. δ.*
(h) $\delta = \delta_0$. *($A \delta_0 B$ iff $clA \cap clB \neq \varnothing$)*
(i) $C^*(X) = P^*(X)$.

The space of all ordinals less than the first uncountable ordinal shows that even with normality, X' need not be compact when $C(X) = P(X)$ for every compatible EF-proximity on X. This space has a unique compatible uniformity or EF-proximity.

4.2. **Uniformity.** Uniform case is a bit tricky since

$$C(X) = U(X) \text{ does not imply that } C(X, Y) = U(X, Y)$$

for each uniform space Y and

$$C^*(X) = P^*(X) \text{ does not imply } C(X) = U(X)!$$

So we have three cases to consider:

(1) the **strong Atsuji** viz. $C(X,Y) = U(X,Y)$ for each uniform space Y,
(2) the **Atsuji** viz. $C(X) = U(X)$,
(3) the **Cech-Atsuji** viz. $C^*(X) = P^*(X)$.

The strong Atsuji case has a simple solution : $\mathcal{U}$ must be the fine uniformity. Also the Cech-Atsuji case is equivalent to $\delta = \delta_F$ and this is a proximal property. So the only non-trivial case to study is the Atsuji one. ([2]).

Theorem 4.6. *For a uniform space* $(X,\mathcal{U})$ *the following are equivalent:*

(a) $C(X) = U(X)$,
(b) *If* (A_n) *is discretely normally separated by* (B_n), *then* (A_n) *is uniformly separated by* (B_n) *i. e. there is* $U \in \mathcal{U}$ *such that* $U(A_n) \subset B_n$ *for each* $n \in \mathbb{N}$.
(c) *Every functionally discrete sequence of sets is uniformly discrete.*

Theorem 4.7. *For a uniform normal space* $(X,\mathcal{U})$ *the following are equivalent:*

(a) $C(X) = U(X)$,
(b) *If* $clA_n \subset intB_n$ *for each* $n \in \mathbb{N}$, (B_n) *is pairwise disjoint and* $\cup clA_n$ *is closed, then there is a* $U \in \mathcal{U}$ *such that* $U(A_n) \subset B_n$ *for each* $n \in \mathbb{N}$.
(c) *Every discrete sequence of sets is uniformly discrete.*

We now consider several statements which are analogues in the uniform case of the well known equivalent statements in the metric case.

Theorem 4.8. *Consider the following statements concerning a uniform space* $(X,\mathcal{U})$.

(a) $\mathcal{U}$ *is fine.*
(b) $C(X,Y) = U(X,Y)$ *for each uniform space* Y.
(c) $C(X) = U(X)$,
(d) *Every functionally discrete sequence of sets is uniformly discrete.*
(e) *Each pair of disjoint zero sets can be separated by a uniformly continuous function.*
(f) *Each pair of disjoint zero sets can be separated by an entourage.*
(g) $\delta = \delta_F$.
(h) $C^*(X) = U^*(X)$.

Then $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Rightarrow (e) \Leftrightarrow (f) \Leftrightarrow (g) \Leftrightarrow (h)$ *and none of the arrows can be reversed.*

Theorem 4.9. *Consider the following statements concerning a normal uniform space* $(X,\mathcal{U})$.

(a) X' *is compact and for each entourage* $U \in \mathcal{U}$, *there is an entourage* $V \in \mathcal{U}$ *such that* $U(X)^c$ *is* V*-discrete.*
(b) $\mathcal{U}$ *is a Lebesgue uniformity.*
(c) $\mathcal{U}$ *is fine.*

(d) *$C(X,Y) = U(X,Y)$ for each uniform space Y.*
(e) *$C(X) = U(X)$,*
(f) *Every discrete sequence of sets is uniformly discrete.*
(g) *$C^*(X) = U^*(X)$,*
(h) *$\delta = \delta_0$.*
(i) *Disjoint closed sets are uniformly separated.*
(j) *$\tau(V) \subset \tau(\mathcal{U}_H)$ on $CL(X)$.*
(k) *Every pseudo-Cauchy sequence of distinct points has a cluster point.*

Then $(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Rightarrow (e) \Leftrightarrow (f) \Rightarrow (g) \Leftrightarrow (h) \Leftrightarrow (i) \Leftrightarrow (j) \Rightarrow (k)$. $(j) \neq (k)$ is open and none of the arrows can be reversed.

4.3. **Metric space.** Finally we put all our results together and get several known characterizations of metric Atsuji spaces.

Theorem 4.10. *Suppose (X,d) is a metric space, $\mathcal{U}$ is the metric uniformity and δ is the metric proximity. The following are equivalent:*

(a) *X' is compact and for each $\varepsilon > 0$, there is a $\eta > 0$ such that $[S(X,\varepsilon)]^c$ is η- discrete.*
(b) *Every open cover of X is Lebesgue.*
(c) *$\mathcal{U}$ is fine.*
(d) *$C(X,Y) = U(X,Y)$ for each uniform space Y.*
(e) *$C(X) = U(X)$,*
(f) *$C^*(X) = U^*(X)$,*
(g) *$\delta = \delta_0$.*
(h) *Disjoint closed sets are at a positive distance apart.*
(i) *$\tau(V) \subset \tau(\mathcal{U}_H)$ on $CL(X)$.*
(j) *Every pseudo-Cauchy sequence of distinct points has a cluster point.*

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