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On the use of partial orders in uniform spaces

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Abstract. We investigate the use of nets indexed by preorders in uniform spaces. Nine different Cauchy conditions and four different convergence conditions yield 36 completeness properties, each of which turns out to be equivalent to a known form of completeness. We also use these preordered nets to characterize the functors θ, λ, and ν, which are associated with these completeness properties. In the case of λ we give an example to show that the analogous characterization with predirected nets does not work.

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1. Introduction

We have recently written a paper [3] on completeness properties determined by nets whose directed sets are well ordered; this current paper represents the opposite extreme, namely, the use of preorders as index sets for nets. In the title, we use the term 'partial order' after the fashion of Kelley, who in Chapter 2 of his book [14] does not require the antisymmetry property for orderings. His partial orders need only be transitive and reflexive; and his directed sets have these properties plus upper bounds for finite sets. But over the years, the terminology has changed, and we follow it, using 'preordered' and 'predirected' for these concepts below.

Section 2 deals with the completeness properties which arise. Section 3 shows how these preordered nets may be used to give new definitions for functors discussed in the literature. These new definitions have some advantages over the definitions previously known. In the case of the functor ν the new definition is internal rather than external. It does not require the use of the completion of the space. In the case of λ the new definition does not make use of Ginsberg's and Isbell's quasiuniformities, i.e., filters of coverings which may fail to satisfy the star refinement property.

2. Variations on the property of Completeness

Definition 2.1. In a uniform space $(X,\mathcal{U})$ a filter $\mathcal{F}$ is *weakly Cauchy* if for every $U \in \mathcal{U}$ there is a U-small set $S \subseteq X$ which has non-empty intersection with every member of $\mathcal{F}$. A net $\xi : D \to X$ is *cofinally Cauchy* if for every $U \in \mathcal{U}$ there is a cofinal set $C \subseteq D$ such that $\xi[C]$ is U-small. A space is *cofinally complete* if every cofinally Cauchy net clusters, or, equivalently, if every weakly Cauchy filter clusters. A filter is *stable* if for every $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that for all $F' \in \mathcal{F}$ we have $F \subseteq U[F']$. A net $\xi : D \to X$ is *almost Cauchy* if for any $U \in \mathcal{U}$ there is a $d \in D$ and a set $\mathcal{C}$ of cofinal subsets of D such that for each $C \in \mathcal{C}$, $\xi[C]$ is U-small, and for each $d' \in D$, if $d' \geq d$ then $d' \in \bigcup \mathcal{C}$. A space is *supercomplete* if each almost Cauchy net clusters, or, equivalently, if every stable filter clusters.

Recall that, originally, Isbell called a space supercomplete if it had a complete hyperspace [12, 13]. That condition is equivalent to the definition given here. This author stated this in [2] before discovering that Isbell mentioned this in the last paragraph of [12]. We note that Császár gave this as an open problem in [4].

In [12] and [13], Isbell uses a notion of nets indexed by preordered sets to characterize supercompleteness. In [2] we obtained a slightly different characterization of supercompleteness using these preordered nets. We wish to extend these results here, but because of the many different Cauchy and convergence conditions that are possible we need to develop first a streamlined terminology for all the completeness properties that are generated. To do this we make use of the Alexandroff topology.

For a preordered set $(P, \leq)$, the *Alexandroff topology* [1] is the collection of upper sets, that is, those A for which $x \in A$ and $x \leq y \Rightarrow y \in A$. This is easily seen to be a topology which is generated by all sets of the form $\uparrow x = \{y : x \leq y\}$ for $x \in P$, and in this guise was called the *partial order topology* in Chapter III of the book [17]. When we refer below to open, dense, open dense, or somewhere dense (= not nowhere dense) subsets of a preordered set, we will be assuming that these terms refer to such sets as determined by the Alexandroff topology.

Definition 2.2. Any function from a preordered set to a uniform space will be called a *ponet*. We say a ponet $S : (P, \leq) \to (X,\mathcal{U})$ is *open (dense, open dense, somewhere dense) Cauchy* if for each $U \in \mathcal{U}$ there is an open (dense, open dense, somewhere dense) set $R \subseteq P$ with $S[R] \times S[R] \subseteq U$. In addition, a ponet $S : (P, \leq) \to (X,\mathcal{U})$ satisfies the property *open/open dense Cauchy* if for each $U \in \mathcal{U}$ the union of some collection of open sets $R \subseteq P$ such that $S[R] \times S[R] \subseteq U$, is open dense, and we may define *somewhere dense/open dense*, *somewhere dense/open*, *somewhere dense/dense*, and *dense/open dense* in an analogous manner.

Definition 2.3. We say x is an *open (dense, open dense, somewhere dense) limit* of a ponet $S : (P, \leq) \to (X, \mathcal{U})$ if for every neighborhood O of x, $S^{-1}[O]$ contains an open (dense, open dense, somewhere dense) set.

From these nine Cauchy conditions and four convergence conditions we may define 36 completeness properties in the obvious way. For example, the property *open/open dense—dense completeness* would say that every open/open dense Cauchy ponet has a dense limit. These 36 properties are not distinct, however, as we will show below.

The next two propositions are the only previous results we know about this type of completeness.

Proposition 2.4 (Isbell [12], equivalence of (c) and (a) in the Theorem, 1962)**.** *Open/open dense—open completeness is equivalent to supercompleteness.*

Proposition 2.5 (Burdick [3], equivalence of (c) and (a) in Theorem 2, 1991)**.** *Open/open dense—somewhere dense completeness is equivalent to supercompleteness.*

In the following table we extend this type of characterization to all 36 of the completeness properties just defined. The nine Cauchy conditions correspond to the nine rows and the four convergence conditions correspond to the four columns. We find that these 36 different combinations resolve into just five well known versions of completeness.

The 36 Completeness Properties

C = Complete SC = Supercomplete

CC = Cofinally Complete PF = Paracompact and Fine

I = Indiscrete

	Somewhere Dense Limit	Open Limit	Dense Limit	Open Dense Limit
Open Dense Cauchy	C	C	C	C
Open/ Open Dense Cauchy	SC	SC	I	I
Dense/ Open Dense Cauchy	SC	I	SC	I
Open Cauchy	CC	PF	I	I
Dense Cauchy	CC	I	PF	I

The 36 Completeness Properties (continued)

	Somewhere Dense Limit	Open Limit	Dense Limit	Open Dense Limit
Somewhere Dense/ Open Dense Cauchy	SC	I	I	I
Somewhere Dense/Open Cauchy	CC	I	I	I
Somewhere Dense/Dense Cauchy	CC	I	I	I
Somewhere Dense Cauchy	CC	I	I	I

We will only prove some of the results contained in the table above—enough so that the key ideas are demonstrated.

First of all it is clear that all these properties imply completeness. Likewise, dense—somewhere dense completeness clearly implies cofinal completeness. Going the other way we have the following two results.

Proposition 2.6. *Completeness implies open dense—open dense completeness.*

Proof. Suppose that $(X,\mathcal{U})$ is complete and $S : P \to X$ is an open dense Cauchy ponet. Let Q be the set of open dense subsets of P and let $\mathcal{Q} = \{(D,x) \mid x \in D \in Q\}$. Define a preorder on $\mathcal{Q}$ by $(D_1,x) \leq (D_2,y)$ if $D_2 \subseteq D_1$. Then $\mathcal{Q}$ is predirected. Define $F : \mathcal{Q} \to P$ by $F(D,x) = x$. Then $S \circ F : \mathcal{Q} \to X$ is a Cauchy net. Let x be a limit of $S \circ F$. Then for every neighborhood O of x, $S^{-1}[O]$ contains an open dense set. So x is an open dense limit of S. □

Proposition 2.7. *A cofinally complete space is somewhere dense—somewhere dense complete.*

Proof. Suppose that $(X,\mathcal{U})$ is cofinally complete and $S : P \to X$ is a somewhere dense Cauchy ponet. Let Q be the set of open dense subsets of P and let $\mathcal{Q} = \{(D,x) \mid x \in D \in Q\}$. Define a preorder on $\mathcal{Q}$ by $(D_1,x) \leq (D_2,y)$ if $D_2 \subseteq D_1$. Then $\mathcal{Q}$ is predirected. Define $F : \mathcal{Q} \to P$ by $F(D,x) = x$. Then $S \circ F : \mathcal{Q} \to X$ is a cofinally Cauchy net. Let x be a cluster point of $S \circ F$. Then for every neighborhood O of x, $S^{-1}[O]$ intersects every open dense subset of P so it must be somewhere dense. So x is a somewhere dense limit of S. □

We note that there is a symmetry between open and dense in the table. Here is one example of this.

Proposition 2.8. *A space is open—open complete if and only if it is dense—dense complete.*

Proof. Suppose $(X,\mathcal{U})$ is open—open complete. Suppose $S : P \to X$ is dense Cauchy. Let Q be the collection of dense subsets of P and let $\mathcal{Q} = \{(D,x) \mid x \in D \in Q\}$. Define $(D_1,x) \leq (D_2,y)$ if $D_2 \subseteq D_1$. Define $F : \mathcal{Q} \to P$ by $F(D,x) = x$. Then $S \circ F$ is open Cauchy. Let x be an open limit of $S \circ F$. Then for every neighborhood O of x, $S^{-1}[O]$ is dense. So x is a dense limit of S.

Suppose $(X,\mathcal{U})$ is dense—dense complete. Suppose $S : P \to X$ is open Cauchy. Let Q be the collection of dense subsets of P and let $\mathcal{Q} = \{(D,x) \mid x \in D \in Q\}$. Define $(D_1,x) \leq (D_2,y)$ if $D_2 \subseteq D_1$. Define $F : \mathcal{Q} \to P$ by $F(D,x) = x$. Then $S \circ F$ is dense Cauchy. Let x be a dense limit of $S \circ F$. Then for every neighborhood O of x, $S^{-1}[O]$ intersects every member of Q so it contain as open set. So x is an open limit of S. □

Sometimes the method of the last proof doesn't do the whole job, but the symmetry still holds.

Proposition 2.9. *For a uniform space the following are equivalent:*

(1) *Supercompleteness.*
(2) *Somewhere dense/open dense—somewhere dense completeness.*
(3) *Dense/open dense—somewhere dense completeness.*
(4) *Open/open dense—somewhere dense completeness.*
(5) *Dense/open dense—dense completeness.*
(6) *Open/open dense—open completeness.*

Proof. (1) implies (2). Suppose $(X,\mathcal{U})$ is supercomplete. Suppose $S : P \to X$ is somewhere dense/open dense Cauchy. Let Q be the set of open dense subsets of P and let $\mathcal{Q} = \{(D,x) \mid x \in D \in Q\}$. Define $(D_1,x) \leq (D_2,y)$ if $D_2 \subseteq D_1$. Then $\mathcal{Q}$ is predirected. Define $F : \mathcal{Q} \to P$ by $F(D,x) = x$. Then $S \circ F$ is an almost Cauchy net. Let x be a cluster point of $S \circ F$. Then for every neighborhood O of x, $S^{-1}[O]$ intersects every member of Q so it must be somewhere dense. So x is a somewhere dense limit of S.

(2) implies (3). Trivial.

(3) implies (5). A somewhere dense limit of a dense/open dense ponet will always be a dense limit.

(2) implies (4). Trivial.

(4) implies (6). A somewhere dense limit of an open/open dense ponet will always be an open limit.

(5) implies (6). Suppose $(X,\mathcal{U})$ is dense/open dense—dense complete. Suppose $S : P \to X$ is open/open dense Cauchy. Let Q be the collection of dense subsets of P and let $\mathcal{Q} = \{(D,x) \mid x \in D \in Q\}$. Define $(D_1,x) \leq (D_2,y)$ if $D_2 \subseteq D_1$. Define $F : \mathcal{Q} \to P$ by $F(D,x) = x$. Then $S \circ F$ is dense/open dense Cauchy. Let x be a dense limit of $S \circ F$. Then for every neighborhood O of

x, $S^{-1}[O]$ intersects every member of Q so it contain an open set. So x is an open limit of S.

(6) implies (1). This follows from Proposition 2.4. □

The method of proof in Proposition 2.8 was used here to prove that (5) implies (6), but it doesn't supply a direct proof that (6) implies (5). If we tried that, at a crucial point we would not be able to say that $S \circ F$ is open/open dense Cauchy. There is an asymmetry here stemming from the fact that while the F^{-1} image of an open set is dense, the F^{-1} image of a dense set merely contains an open set. In Section 3 we will see a breakdown in the symmetry of the results for this very reason (compare Corollary 3.17 with Example 3.15).

The combination of paracompact and fine is equivalent to saying that every open cover is a uniform cover. This is stronger than cofinal completeness ([5], (a) implies (c) in Thereom 1).

Proposition 2.10. *Open—open completeness is equivalent to paracompact and fine.*

Proof. Suppose that $(X,\mathcal{U})$ is open—open complete and that $\mathcal{C}$ is an open cover of X which is not uniform. Let P be the collection of all sets $A \subseteq X$ such that no element of $\mathcal{C}$ contains A as a subset. Let $\mathcal{P} = \{(A, x) \mid x \in A \in P\}$. Define a preorder on $\mathcal{P}$ by saying that $(A, x) \leq (B, y)$ if $B \subseteq A$. Define a ponet $S : \mathcal{P} \to X$ by $S(A, x) = x$.

$\mathcal{P}$ is an open Cauchy ponet. So let x be an open limit. Some $O \in \mathcal{C}$ contains x. But then some $A \in P$ would have to be a subset of O, a contradiction.

Conversely, suppose that $(X,\mathcal{U})$ is paracompact and fine. Let $S : P \to X$ be an open Cauchy ponet with no open limit. Then each point in X would have an open neighborhood O such that $S^{-1}[O]$ would not contain an open set. The collection of these O's is an open cover, therefore a uniform cover. But this contradicts the open Cauchy property. □

Proposition 2.11. *Open/open dense—dense completeness implies that the uniform space has the indiscrete uniformity.*

Proof. Suppose that $(X,\mathcal{U})$ is open/open dense—dense complete. Let X be given the trivial order defined by $x \leq y$ if and only if $x = y$. Then the identity map $\iota : X \to X$ is an open/open dense Cauchy ponet. Let x be a dense limit. Then every neighborhood of x must contain all of X, making $\mathcal{U}$ the indiscrete uniformity. □

3. The functors ν, λ, and θ

Now we turn to a consideration of certain functors which are associated with completeness properties. The functors ν and λ are due to Howes [9] and Ginsberg and Isbell [8], respectively. The latter functor has been utilized by many authors over the last four decades. Our functor θ was announced in the book by Howes [11] without any details. This is the first time we have used it in a paper.

After defining Howes's functor ν, we give several results which illustrate the importance of ν. After that we proceed to a new definition of ν.

Definition 3.1. For each infinite cardinal κ we say a uniform space $(X,\mathcal{U})$ is *κ-bounded* if for every $U \in \mathcal{U}$ there is a subset S of X, of cardinality less than κ, with $U[S] = X$. For each uniform space $(X,\mathcal{U})$ and each infinite cardinal κ let $\mathcal{U}_\kappa$ be the supremum of all the κ-bounded uniformites on X which are coarser than $\mathcal{U}$. For an infinite cardinal κ, a topological space $(X,\mathcal{T})$ is *κ-pseudocompact* if every normal cover of X has a subcover of cardinality less than κ. A topological space is *$[\kappa,\infty)$-compact* if every open cover has a subcover of cardinality less than κ.

The next two definitions and several results following them are due to N. Howes.

Definition 3.2. [9] Given a space $(X,\mathcal{U})$ a new space $(X,\nu\mathcal{U})$ is constructed in the following manner: embed $(X,\mathcal{U})$ in its completion $(Y,\mathcal{V})$, let $\mathcal{V}^*$ be the finest uniformity on Y generating the same topology as $\mathcal{V}$, and then let $\nu\mathcal{U}$ be the restriction of $\mathcal{V}^*$ to X.

It is easy to see that this construction defines a functor (in view of the definitions $\nu(X,\mathcal{U}) = (X,\nu\mathcal{U})$ and $\nu f = f$) from the category of uniform spaces to itself.

Definition 3.3. A space is *preparacompact* [9] if every cofinally Cauchy net has a Cauchy subnet, and it is *almost preparacompact* [10] if every almost Cauchy net has a Cauchy subnet.

Lemma 3.4 (Howes)**.** *A space is preparacompact iff its completion is cofinally complete* [9]*; it is almost preparacompact iff its completion is supercomplete* [10].

In [9] and [10], Howes used the functor ν to answer a question of Tamano as to which spaces had paracompact completions.

Proposition 3.5 (Howes)**.** *For a space $(X,\mathcal{U})$ the following are equivalent:*

(1) *$(X,\mathcal{U})$ has a paracompact completion.*
(2) *$(X,\nu\mathcal{U})$ is preparacompact.*
(3) *$(X,\nu\mathcal{U})$ is almost preparacompact.*

This in turn allowed him to characterize the spaces with Lindelöf completions in several different ways.

Proposition 3.6 (Howes)**.** *For a space $(X,\mathcal{U})$ the following are equivalent:*

(1) *$(X,\mathcal{U})$ has a Lindelöf completion.*
(2) *$(X,\nu\mathcal{U})$ is preparacompact and $\aleph_1$-bounded.*
(3) *$(X,\nu\mathcal{U})$ is almost preparacompact and $\aleph_1$-bounded.*

The following generalizes these results of Howes's.

Lemma 3.7. *For a space* $(X,\mathcal{U})$ *and an infinite cardinal* κ*, the following are equivalent:*

(1) $(X,\mathcal{U})$ *has a* κ*-pseudocompact completion.*
(2) $(X,\nu\mathcal{U})$ *is* κ*-bounded.*

Proposition 3.8. *For a space* $(X,\mathcal{U})$ *and an infinite cardinal* κ*, the following are equivalent:*

(1) $(X,\mathcal{U})$ *has a paracompact,* $[\kappa,\infty)$*-compact completion.*
(2) $(X,\nu\mathcal{U})$ *is* κ*-bounded and preparacompact.*
(3) $(X,\nu\mathcal{U})$ *is* κ*-bounded and almost preparacompact.*

These results are mentioned to demonstrate the usefulness of the functor ν. Therefore we should ask if there are other ways of defining ν, ways which might facilitate the use of the characterizations above. We give new constructions of ν which involve other uniformities on the given set X but do not require adding any more points to X. We feel that the results above, which give properties of the completion of a space, will be more significant if the construction of ν itself does not make use of the completion.

Proposition 3.9. *For a fixed uniform space* $(X,\mathcal{U})$ *and a possibly different uniformity* $\mathcal{U}^*$ *on* X*, the following are equivalent:*

(1) *Any ponet which is open dense Cauchy for* $\mathcal{U}$ *is open dense Cauchy for* $\mathcal{U}^*$*.*
(2) *Any net which is Cauchy for* $\mathcal{U}$ *is Cauchy for* $\mathcal{U}^*$*.*
(3) *Any filter which is Cauchy for* $\mathcal{U}$ *is Cauchy for* $\mathcal{U}^*$*.*
(4) $\mathcal{U}^* \subseteq \nu\mathcal{U}$*.*

Proof. (1) implies (2). Trivial.

(2) implies (3). Elementary.

(3) implies (4). Let the Hausdorff completion of $(X,\mathcal{U})$ be $i : (X,\mathcal{U}) \to (Y,\mathcal{V})$ and the Hausdorff completion of $(X,\mathcal{U}^*)$ be $i^* : (X,\mathcal{U}^*) \to (Z,\mathcal{W})$. It suffices to show that there is a continuous map $f : (Y,\mathcal{T}(\mathcal{V})) \to (Z,\mathcal{T}(\mathcal{W}))$ where $f \circ i = i^*$. The sets Y and Z may be regarded as sets of equivalence classes of Cauchy filters, and the equivalence relation is such that two filters $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if and only if $\mathcal{F} \cap \mathcal{F}'$ is Cauchy. So we may define f by saying that if $\mathcal{F}$ is a representative of the equivalence class $[\mathcal{F}]_{\mathcal{U}} \in Y$, then $f([\mathcal{F}]_{\mathcal{U}}) = [\mathcal{F}]_{\mathcal{U}^*}$. Our remarks above show that f is well-defined.

To show f continuous it suffices to show that if filter $\mathcal{F}$ converges to y in $(Y,\mathcal{V})$ then $f[\mathcal{F}]$ converges to $f(y)$ in $(Z,\mathcal{W})$. This is certainly true if every $F \in \mathcal{F}$ intersects $i[X]$, since the trace of $\mathcal{F}$ on $i[X]$ would be $\mathcal{U}$-Cauchy, therefore $i^{-1}[\mathcal{F}]$ would be a representative of the equivalence class y, and so also of $f(y)$, and so $F = i[i^{-1}[\mathcal{F}]]$ would converge to $f(y)$.

So given $\mathcal{F}$ converging to y in $(Y,\mathcal{V})$, let $\mathcal{G} = \{V[F] \mid V \in \mathcal{V}, F \in \mathcal{F}\}$. $\mathcal{G}$ still converges to y, so it is a Cauchy filter on $(Y,\mathcal{V})$ and every member of $\mathcal{G}$

intersects $i[X]$. So $i^{-1}[\mathcal{G}]$ is Cauchy on $(X,\mathcal{U})$. Since $i^{-1}[\mathcal{G}]$ is a representative of the equivalence class y it is also a representative of the equivalence class $f(y)$. Then for any $W \in \mathcal{W}$, $W[y]$ will contain $G \cap i[X]$ for some $G \in \mathcal{G}$. Suppose $G = V[F]$ for some $F \in \mathcal{F}$. Then every point of F is the limit with respect to $(Y,\mathcal{V})$ of a filter on $G \cap i[X]$ and so by the remark in the last paragraph every point of $f[F]$ is the limit with respect to $(Z,\mathcal{W})$ of a filter on $f[G]$. If W has been chosen to be a closed relation then $W[f(y)]$ contains $f[F]$. This shows that $f[\mathcal{F}]$ converges to $f(y)$.

(4) implies (1). Any ponet which is open dense Cauchy for $\mathcal{U}$ will have an open dense limit in the completion $(\overline{X},\overline{\mathcal{U}})$ of $(X,\mathcal{U})$, by the table in Section 2. This will still be an open dense limit when $\overline{\mathcal{U}}$ is replaced by the fine uniformity for its topology. Therefore the ponet will be open dense Cauchy for $\nu\mathcal{U}$ and so for $\mathcal{U}^*$. □

Corollary 3.10. *For any uniform space $(X,\mathcal{U})$, $\nu\mathcal{U}$ is equal to (1) the supremum of the uniformities $\mathcal{U}^*$ such that every $\mathcal{U}$-Cauchy net is $\mathcal{U}^*$-Cauchy, and (2) the supremum of the uniformities $\mathcal{U}^*$ such that every $\mathcal{U}$-open dense Cauchy ponet is $\mathcal{U}^*$-open dense Cauchy. In each case the supremum is the finest member of the set.*

Corollary 3.11. *A uniform space $(X,\mathcal{U})$ has a paracompact completion if and only if every cofinally Cauchy net in $(X,\nu\mathcal{U})$ has a subnet which is Cauchy for $\mathcal{U}$.*

Observation 3.12. If a space $(X,\mathcal{U})$ is complete then $\nu\mathcal{U}$ is fine; if the space is paracompact (or even just topologically complete) then the converse is true.

Another useful functor is the locally fine coreflection, λ. The reader is referred to [8], [11], [12], [13], and [15] for many properties of this functor. We will characterize λ using ponets as we have for ν.

Definition 3.13. A cover $\mathcal{C}$ of X is called *uniformly locally uniform* if there is a uniform cover $\mathcal{C}'$ such that on each $S \in \mathcal{C}'$, the trace of $\mathcal{C}$ on S is uniform. A space $(X,\mathcal{U})$ is called *locally fine* if every uniformly locally uniform cover is uniform. Given a space $(X,\mathcal{U})$ the uniformity $\lambda\mathcal{U}$ is the coarsest one finer than $\mathcal{U}$ such that $(X,\lambda\mathcal{U})$ is locally fine. λ is constructed by transfinite recursion (see [8] or [13]).

Proposition 3.14. *$\nu\lambda = \nu$. Consequently, for a uniform space $(X,\mathcal{U})$, we have $\nu\mathcal{U}$ finer than $\lambda\mathcal{U}$.*

Proof. It suffices to show that the completion of $\lambda(X,\mathcal{U})$ is homeomorphic to the completion of $(X,\mathcal{U})$ via a homeomorphism which is the identity on X. This follows from Ginsberg's and Isbell's ([8], Theorem 4.4) (λ commutes with completion). □

We wish to prove some characterizations of λ similar to those we have done for ν, but first we observe an example that shows that the obvious net property doesn't hold in this case.

Example 3.15. *Stable filters for* $(X,\mathcal{U})$ *need not be stable for* $\lambda\mathcal{U}$*, and consequently almost Cauchy nets for* $(X,\mathcal{U})$ *need not be almost Cauchy for* $\lambda\mathcal{U}$*, nor do dense/open dense ponets for* $(X,\mathcal{U})$ *need to be dense/open dense for* $\lambda\mathcal{U}$*.* Let $\mathcal{U}$ be the usual uniformity on the reals, and let a filter $\mathcal{F}$ be generated by the sets $F_\epsilon = \bigcup_{n\in\mathbb{Z}}[n-\epsilon, n+\epsilon]$, for $\epsilon > 0$. Then $\mathcal{F}$ is $\mathcal{U}$-stable. Since $\mathcal{U}$ is a complete metric uniformity, $\lambda\mathcal{U}$ is fine [8]. But even if $\mathcal{U}^*$ is a uniformity which makes the function $f : (\mathbb{R},\mathcal{U}^*) \to (\mathbb{R},\mathcal{U})$, where $f(x) = x^2$, uniformly continuous, then $\mathcal{F}$ is not $\mathcal{U}^*$-stable. So $\mathcal{F}$ is not $\lambda\mathcal{U}$-stable.

Proposition 3.16. *For a fixed uniform space* $(X,\mathcal{U})$ *and a possibly different uniformity* $\mathcal{U}^*$ *on* X*, the following are equivalent:*

(1) $\mathcal{U}^* \subseteq \lambda\mathcal{U}$.
(2) *Any ponet which is open/open dense Cauchy for* $\mathcal{U}$ *is open/open dense Cauchy for* $\mathcal{U}^*$*.*
(3) *Any ponet which is open/open dense Cauchy for* $\mathcal{U}$ *is open Cauchy for* $\mathcal{U}^*$*.*
(4) *Any locally fine uniformity which is finer than* $\mathcal{U}$ *is finer than* $\mathcal{U}^*$*.*

Proof. (1) implies (2). Given a ponet which is open/open dense Cauchy for $\mathcal{U}$ we can prove by transfinite induction that it is open/open dense Cauchy for $\lambda\mathcal{U}$. This is the essence of Lemma 40 of Chapter VII of [13].

(2) implies (3). Trivial.

(3) implies (4). Given a locally fine uniformity $\mathcal{V}$ on X with $\mathcal{U} \subseteq \mathcal{V}$, suppose that $\mathcal{U}^* \nsubseteq \mathcal{V}$. Then there is a $\mathcal{U}^*$-uniform cover $\mathcal{C}$ which is not $\mathcal{V}$-uniform. Let P be the collection of all sets $A \subseteq X$ such that $\mathcal{C}$ is not a $\mathcal{V}$-uniform cover of A. Let $\mathcal{P} = \{(A,x) \mid x \in A \in P\}$. Define a preorder on $\mathcal{P}$ by saying that $(A,x) \leq (B,y)$ if $B \subseteq A$. Define a ponet $S : \mathcal{P} \to X$ by $S(A,x) = x$.

We show that S is open/open dense Cauchy for $\mathcal{V}$. It suffices to show that for any $A \in P$ and $U \in \mathcal{V}$ there is a U-small $B \in P$ with $B \subseteq A$. Suppose not, i.e., every U-small subset of A is $\mathcal{V}$-uniformly covered by $\mathcal{C}$. Consider the cover $\mathcal{C}' = \{C \cup (X - A) \mid C \in \mathcal{C}\}$. The U-small subsets of X are all $\mathcal{V}$-uniformly covered by $\mathcal{C}'$, so by local fineness $\mathcal{C}'$ is a $\mathcal{V}$-uniform cover of X. The trace of $\mathcal{C}'$ on A is the same as the trace of $\mathcal{C}$, so $\mathcal{C}$ is a $\mathcal{V}$-uniform cover of A, a contradiction.

Since S is $\mathcal{V}$-open/open dense Cauchy it is $\mathcal{U}$-open/open dense Cauchy. But it fails to be $\mathcal{U}^*$-open Cauchy since no element of P is contained in any element of $\mathcal{C}$, and this contradicts property (3).

(4) implies (1). This follows since $\lambda\mathcal{U}$ is a locally fine uniformity which is finer than $\mathcal{U}$. □

Corollary 3.17. *For any uniform space* $(X,\mathcal{U})$, $\lambda\mathcal{U}$ *is equal to* (1) *the supremum of the uniformities* $\mathcal{U}^*$ *such that every* $\mathcal{U}$*-open/open dense Cauchy ponet is* $\mathcal{U}^*$*-open/open dense Cauchy, and* (2) *the supremum of the uniformities* $\mathcal{U}^*$ *such that every* $\mathcal{U}$*-open/open dense Cauchy ponet is* $\mathcal{U}^*$*-open Cauchy. In each case the supremum is the finest member of the set.*

The usual definition of λ gives rise to technical difficulties in that the Ginsberg-Isbell derivatives used in the transfinite induction need not be covering uniformities, and then it is tricky to show in the end that their union, i.e., the λ uniform covers, is a covering uniformity. If property (1) in Corollary 3.17 were taken as the definition of $\lambda\mathcal{U}$ instead this wouldn't be a problem. What's more it would be easy to show directly that $\lambda\mathcal{U}$ preserves the open/open dense ponets of $\mathcal{U}$. The next theorem of Isbell's [12] is our chief reason for interest in the functor λ.

Isbell's Theorem *A space* $(X,\mathcal{U})$ *is supercomplete iff it is paracompact and* $\lambda\mathcal{U}$ *is fine.*

The fact that paracompactness plus fineness implies supercompleteness, and that supercomplete implies paracompact, can be proved without use of the functor λ. If we add to these facts the observation that λ preserves the topology of the space (true because it is true for ν) then Isbell's Theorem follows from our Corollary 3.17 and Propostion 2.9. We should not claim too much, however. We have not replaced the traditional development of this subject because we have used previously known properties of λ in the proofs of Propositions 3.14 and 3.16. In particular we used the assumption that λ as traditionally defined is in fact a uniformity to prove that (4) implies (1) in Proposition 3.16.

Let us further point out that Pelant in [15] has shown that the locally fine uniform spaces are all subfine and so λ coincides with the subfine coreflection described in Chapter VII of [13]. This also then provides a relatively straightforward way of defining λ. However, the proof that it coincides with the original definition of λ is quite involved. Our Corollary 3.17 has the combined advantages of being relatively easy to prove and giving a construction of $\lambda\mathcal{U}$ which clearly yields a uniformity.

In view of these properties of ν and λ it would be interesting to have a third functor θ with the following properties:

(1) Cofinal completeness should be equivalent to paracompactness and θ fine.
(2) θ should be coarser than λ.
(3) $\theta^2 = \theta$.
(4) θ shouldn't change the underlying set or the generated topology, and should make the uniformity finer than before.
(5) θ, like ν and λ, should be a coreflection when considered as a functor from the category of all uniform spaces to its image category. (This actually follows from (3) and (4) and the assumption that θ is a functor).

Definition 3.18. A cover $\mathcal{C}$ of a uniform space $(X,\mathcal{U})$ is *uniformly locally finitizible* if there is a $U \in \mathcal{U}$ such that for any $x \in X$, $U[x]$ is covered by some finite subset of $\mathcal{C}$. A uniform space $(X,\mathcal{U})$ is $\aleph_0$-*nearly discrete* if for any uniformity $\mathcal{U}^*$ on X, if every $\mathcal{U}^*$-uniform cover is $\mathcal{U}$-uniformly locally finitizible then $\mathcal{U}^* \subseteq \mathcal{U}$. For a uniform space $(X,\mathcal{U})$ let $\theta^+\mathcal{U}$ be the supremum of the uniformities $\mathcal{U}^*$ on X such that every $\mathcal{U}^*$-uniform cover $\mathcal{C}$ is $\mathcal{U}$-uniformly locally finitizible.

Lemma 3.19. *For a uniform space* $(X,\mathcal{U})$*,* $\theta^+\mathcal{U}$ *is* $\aleph_0$*-nearly discrete and any* $\theta^+\mathcal{U}$*-uniform cover is* $\mathcal{U}$*-uniformly locally finitizible.*

Proposition 3.20. *For a fixed uniform space* $(X,\mathcal{U})$ *and a possibly different uniformity* $\mathcal{U}^*$ *on* X*, the following are equivalent:*

(1) $\mathcal{U}^* \subseteq \theta^+\mathcal{U}$.
(2) *Every cofinally Cauchy net for* $\mathcal{U}$ *is cofinally Cauchy for* $\mathcal{U}^*$.
(3) *Every weakly Cauchy filter for* $\mathcal{U}$ *is weakly Cauchy for* $\mathcal{U}^*$.
(4) *Every somewhere dense Cauchy ponet for* $\mathcal{U}$ *is somewhere dense Cauchy for* $\mathcal{U}^*$.
(5) *Every open Cauchy ponet for* $\mathcal{U}$ *is somewhere dense Cauchy for* $\mathcal{U}^*$.
(6) *Every dense Cauchy ponet for* $\mathcal{U}$ *is somewhere dense Cauchy for* $\mathcal{U}^*$.
(7) *Every* $\aleph_0$*-nearly discrete uniformity which is finer than* $\mathcal{U}$ *is finer than* $\mathcal{U}^*$.

Proof. (1) implies (2). Given a $\mathcal{U}$-cofinally Cauchy net $S : D \to X$ and a $\mathcal{U}^*$-uniform cover $\mathcal{C}$, take a $\mathcal{U}$-uniform cover $\mathcal{C}'$ such that each member of $\mathcal{C}'$ can be covered by finitely many members of $\mathcal{C}$. There is some $C \in \mathcal{C}'$ such that S is frequently in C, and then among the finitely many members of $\mathcal{C}$ that cover C, S must be frequently in at least one of them.

Equivalence of (2) and (3). Elementary.

(2) implies (4). This uses the methods of Section 2.

(4) implies (5). Trivial.

Equivalence of (5) and (6). This uses the methods of Section 2.

(5) implies (7). Let $\mathcal{V}$ be an $\aleph_0$-nearly discrete uniformity on X with $\mathcal{U} \subseteq \mathcal{V}$. Suppose that $\mathcal{U}^* \not\subseteq \mathcal{V}$. Then there is a $\mathcal{U}^*$-uniform cover $\mathcal{C}$ which is not $\mathcal{V}$-uniform, so it is not $\mathcal{V}$-uniformly locally finitizible. Let P be the collection of all sets $A \subseteq X$ such that $\mathcal{C}$ has no finite subset covering A. Let $\mathcal{P} = \{(A,x) \mid x \in A \in P\}$. Define a preorder on $\mathcal{P}$ by saying that $(A,x) \le (B,y)$ if $B \subseteq A$. Define a ponet $S : \mathcal{P} \to X$ by $S(A,x) = x$.

S is clearly $\mathcal{V}$-open Cauchy so it is $\mathcal{U}$-open Cauchy. To show it is not $\mathcal{U}^*$-somewhere dense Cauchy we observe that for any $A \in P$ and any $C \in \mathcal{C}$, $A - C \in P$.

(7) implies (1). This follows from Lemma 3.19. □

Corollary 3.21. *A uniform space* $(X,\mathcal{U})$ *which satisfies* $\nu\mathcal{U} \subseteq \theta^+\mathcal{U}$ *has a paracompact completion if and only if it is preparacompact.*

We should point out that unlike ν and λ above, θ^+ may change the topology of the space since it always contains all the totally bounded uniformities for the discrete topology on the given set of points.

Definition 3.22. For any uniform space $(X,\mathcal{U})$ define $\theta\mathcal{U} = \theta^+\mathcal{U} \wedge \lambda\mathcal{U}$.

Observation 3.23. $\theta^2 = \theta$ since the same is true for θ^+ and λ. θ preserves topology because the same is true for λ.

The following proposition is an immediate consequence of Propositions 3.16 and 3.20.

Proposition 3.24. *For a fixed uniform space $(X,\mathcal{U})$ and a possibly different uniformity $\mathcal{U}^*$ on X, the following are equivalent:*

(1) $\mathcal{U}^* \subseteq \theta\mathcal{U}$.
(2) *Every cofinally Cauchy net for $\mathcal{U}$ is cofinally Cauchy for $\mathcal{U}^*$ and every ponet which is open/open dense Cauchy for $\mathcal{U}$ is open/open dense Cauchy for $\mathcal{U}^*$.*
(3) *Every somewhere dense Cauchy ponet for $\mathcal{U}$ is somewhere dense Cauchy for $\mathcal{U}^*$ and every ponet which is open/open dense Cauchy for $\mathcal{U}$ is open/open dense Cauchy for $\mathcal{U}^*$.*

Corollary 3.25. *For any uniform space $(X,\mathcal{U})$, $\theta\mathcal{U}$ is equal to* (1) *the supremum of the uniformities $\mathcal{U}^*$ such that every $\mathcal{U}$-open/open dense Cauchy ponet is $\mathcal{U}^*$-open/open dense Cauchy and every cofinally Cauchy net for $\mathcal{U}$ is cofinally Cauchy for $\mathcal{U}^*$, and* (2) *the supremum of the uniformities $\mathcal{U}^*$ such that every $\mathcal{U}$-open/open dense Cauchy ponet is $\mathcal{U}^*$-open/open dense Cauchy and every somewhere dense Cauchy ponet for $\mathcal{U}$ is somewhere dense Cauchy for $\mathcal{U}^*$. In each case the supremum is the finest member of the set.*

Definition 3.26. We will say (see Rice's paper [16]) a space is *uniformly paracompact* if every open cover has a uniformly locally finite refinement.

Proposition 3.27. *For a uniform space $(X,\mathcal{U})$ the following are equivalent:*

(1) $(X,\mathcal{U})$ *is cofinally complete.*
(2) *Any open cover $\mathcal{O}$ of X which is closed under finite unions is $\mathcal{U}$-uniform.*
(3) *X is paracompact and all locally finite collections in X are $\mathcal{U}$-uniformly locally finite.*
(4) $(X,\mathcal{U})$ *is uniformly paracompact.*

Proposition 3.27 is a combination of several results in Fried [6]. See our paper [3] for more details about the history of this result.

In [16], Rice gave several different ways of characterizing uniform paracompactness. Some of his results are suggestive of Isbell's Theorem, but he stops short of defining a functor θ analogous to λ. We will prove a new version of Rice's Theorem 3 which uses only functors into the category of uniformities.

Definition 3.28. Let κ be an infinite cardinal. A space $(X,\mathcal{U})$ is *locally κ-fine* if every $\mathcal{U}$-uniformly locally $\mathcal{U}_\kappa$-uniform cover is a $\mathcal{U}$-uniform cover.

Frolík [7] has already called this *locally p-fine* when $\kappa = \aleph_0$ and *locally e-fine* when $\kappa = \aleph_1$. He states without proof that these properties are coreflective.

Note that among the uniformities $\mathcal{U}^*$ such that every $\mathcal{U}^*$-uniform cover is a $\mathcal{U}$-uniformly locally $\mathcal{U}_\kappa$-uniform cover, there is a finest one.

Definition 3.29. For a uniform space $(X,\mathcal{U})$ let $\theta_{-}\mathcal{U}$ be the supremum of all the uniformities $\mathcal{U}^*$ whose uniform covers are all $\mathcal{U}$-uniformly locally $\mathcal{U}_{\aleph_0}$-uniform.

θ_{-} is a functor but it doesn't always satisfy $\theta_{-}^2 = \theta_{-}$.

Example 3.30. *A space where $\theta_{-}^2\mathcal{U} \neq \theta_{-}\mathcal{U}$.* Let $X = \omega \times \omega \times 2$. Let the uniformity $\mathcal{U}$ be generated by all equivalences relations satisfying the following properties:

(1) For all but finitely many pairs (n, m), $(n, m, 0)$ is related to $(n, m, 1)$.
(2) For all but finitely many n, (n, m, i) is related to any (n, m', i').

The relation R which relates (n, m, i) to (n', m', i') if and only if $i = i'$ is not a member of $\theta_{-}\mathcal{U}$. But the relations R_k which have as their equivalence classes the three sets $\{(n, m, i) \mid n \leq k,\ i = 0\}$, $\{(n, m, i) \mid n \leq k,\ i = 1\}$, and $\{(n, m, i) \mid n > k\}$, are to be found in $\theta_{-}\mathcal{U}$, and it is because of them that R is a member of $\theta_{-}^2\mathcal{U}$.

Lemma 3.31. *For any uniformity $\mathcal{U}$ we have $\theta_{-}\mathcal{U} \subseteq \theta\mathcal{U}$.*

Proposition 3.32. *For any uniform space $(X,\mathcal{U})$ the following are equivalent:*

(1) *$(X,\mathcal{U})$ is cofinally complete.*
(2) *$(X, \theta_{-}\mathcal{U})$ is paracompact and fine.*
(3) *$(X, \theta\mathcal{U})$ is paracompact and fine.*

Proof. (1) implies (2). Suppose $(X,\mathcal{U})$ is cofinally complete. Given a cover $\mathcal{C}$ of X, let $\mathcal{C}'$ be the collection of open sets O such that there is a $U \in \mathcal{U}$ where for any $x \in X$, $U[x]$ is a subset of some member of $\mathcal{C}$. $\mathcal{C}'$ is a cover of X and it is closed under finite unions. So by Proposition 3.27 it is uniform. This shows that any open cover of X is uniformly locally uniform.

Again by Proposition 3.27, every open cover of X has a uniformly locally finite refinement. So every open cover of X has a refinement which is $\mathcal{U}$-uniformly locally $\mathcal{U}_{\aleph_0}$-uniform.

Therefore the fine uniformity is one of the uniformities which we take the supremum of to get $\theta_{-}\mathcal{U}$. Since θ_{-} doesn't change the topology, $\theta_{-}\mathcal{U}$ must be the fine uniformity. The fact that cofinal completeness implies paracompactness completes the proof.

(2) implies (3). θ is trapped between θ_{-} and λ. Therefore if θ_{-} is fine so must θ be fine as well.

(3) implies (1). If a net is cofinally Cauchy for $\mathcal{U}$ it is cofinally Cauchy for $\theta\mathcal{U}$ by Proposition 3.24. Then since paracompact and fine implies $(X, \theta\mathcal{U})$ is cofinally Cauchy, the net must have a cluster point. □

Proposition 3.33. *For a fixed uniform space $(X,\mathcal{U})$ and a possibly different uniformity $\mathcal{U}^*$ on X, the following are equivalent:*

(1) *Any open Cauchy ponet for $\mathcal{U}$ is open Cauchy for $\mathcal{U}^*$.*
(2) *Any dense Cauchy ponet for $\mathcal{U}$ is dense Cauchy for $\mathcal{U}^*$.*

(3) $\mathcal{U}^* \subseteq \mathcal{U}$.

Proof. (1) implies (2). Suppose that $\mathcal{U}^* \nsubseteq \mathcal{U}$. Then there is a cover $\mathcal{C}$ which is $\mathcal{U}^*$-uniform but not $\mathcal{U}$-uniform. Let P be the set of $A \subseteq X$ such that no $C \in \mathcal{C}$ has A as a subset. Proceeding as in several other proofs in this section we can construct a ponet which is $\mathcal{U}$-open Cauchy but not $\mathcal{U}^*$-open Cauchy.

(1) implies (2). Use the methods of Section 2.

(3) implies (1). Trivial. □

Omnibus Theorem *A uniform space is complete* (*supercomplete, cofinally complete, paracompact and fine*) *if and only if the supremum of the uniformities which preserve the open dense* (*the open/open dense, both the open/open dense and the somewhere dense, the open*) *Cauchy ponets is fine and the underlying topological space is topologically complete* (*paracompact, paracompact, paracompact*)*. If this supremum is complete* (*supercomplete, cofinally complete, paracompact and fine*) *then so is the original space.*

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