Dense $S_\delta$-diagonals and linearly ordered extensions

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ABSTRACT. The notion of the $S_\delta$-diagonal was introduced by H. R. Bennett to study the quasi-developability of linearly ordered spaces. In an earlier paper, we obtained a characterization of topological spaces with an $S_\delta$-diagonal and we showed that the $S_\delta$-diagonal property is stronger than the quasi-$G_\delta$-diagonal property. In this paper, we define a dense $S_\delta$-diagonal of a space and show that two linearly ordered extensions of a generalized ordered space $X$ have dense $S_\delta$-diagonals if the sets of right and left looking points are countable.

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1. $S_\delta$-DIAGONALS

We review in this section the definitions of $S_\delta$-set and $S_\delta$-diagonal, and state our results obtained in [3].

The following definition is a generalization of a $G_\delta$-set and was introduced by H. R. Bennett [2] to study the quasi-developability of linearly ordered (topological) spaces.

**Definition 1.1.** Let $X$ be a topological space. A subset $A$ of $X$ is called an $S_\delta$-set if there exists a countable collection \{\(U(1), U(2), \ldots\)\} of open subsets of $X$ such that, for two points $p \in A$ and $q \in X \setminus A$, there exists an $n$ such that $p \in U(n)$ and $q \notin U(n)$.

It is easy to see that a $G_\delta$-set is an $S_\delta$-set. Hence the notion of $S_\delta$-set is a generalization of $G_\delta$-set. See [3] for a description of $S$-normal spaces whose closed subsets are $S_\delta$-sets.

**Definition 1.2.** Let $X$ be a topological space. $X$ has an $S_\delta$-diagonal if the diagonal subset $\Delta_X$ of $X \times X$ is an $S_\delta$-set of $X \times X$, where $\Delta_X$ denotes the
diagonal set \( \{(x, x) : x \in X\} \) in the Cartesian product \( X \times X \). The symbol \(( , )\)
is used to stand for a point of \( X \times X \).

It is useful to show the following lemma that relates to the property \((*)\) given in [2]. \( \mathbb{N} \) denotes the set of natural numbers.

**Lemma 1.3.** [2] Let \( X \) be a topological space. Let \( \{G(n) : n \in \mathbb{N}\} \) be a family of countable collections of open subsets of \( X \). Suppose that, for any three points \( x, y \) and \( z \) with \( y \neq z \), there exists an \( m \in \mathbb{N} \) such that \( x \in \bigcup G(m) \) and that no element of \( G(m) \) contains the set \( \{y, z\} \), where \( \bigcup G(m) \) denotes \( \bigcup \{U : U \in G(m)\} \). Then, there exists a family \( \{F(n) : n \in \mathbb{N}\} \) of countable collections of open subsets of \( X \) such that, for such three points above, there exists an \( m \in \mathbb{N} \) such that \( x \in \bigcup F(m) \) and any two distinct points of \( \{x, y, z\} \) do not belong to the same member of \( F(m) \).

**Theorem 1.4.** [2] Let \( X \) be a topological space. \( X \) has an \( S_\delta \)-diagonal if and only if there exists a family \( \{G(n) : n \in \mathbb{N}\} \) of countable collections of open subsets of \( X \) such that, for three points \( x, y \) and \( z \) with \( y \neq z \), there exists an \( m \in \mathbb{N} \) such that \( x \in \bigcup G(m) \) and any two distinct points of \( \{x, y, z\} \) do not belong to the same member of \( G(m) \).

## 2. Two linearly ordered extensions and notation

Recall that a generalized ordered space (GO-space) is a triple \((X, \tau, <)\), where \(<\) is a linear ordering of the set \( X \) and \( \tau \) a Hausdorff topology on \( X \) having a base of order-convex sets. We will denote by \( \lambda \) the order topology on \((X, <)\). It is known that \( \lambda \subset \tau \). A space of the form \((X, \lambda, <)\) is called a linearly ordered topological space (LOTS). Every LOTS is a GO-space, but not conversely. In fact it is known that the class of GO-spaces coincides with the class of subspaces of LOTS. Given a GO-space \( X \) there are two well-known linearly ordered extensions of \( X \). One of these is \( X^* \) and was defined by D. J. Lutzer [3]. The other one is \( L(X) \) and was studied in [8]. We review here the definitions of those linearly ordered extensions. The intervals in a GO-space or a LOTS are written in the form \([a,b]\), \([a,b[, \]a,b\] and \]a,b[. For example, \([a,b] = \{x : a \leq x \leq b\}\), \([a,b] = \{x : a \leq x < b\}\) and so on. For a GO-space \( X \), we set \( R = \{x \in X : [x, \rightarrow \left\{ \in \tau - \lambda \right\} \text{ and } L = \{x \in X : \left\{ \in , x \right\} \in \tau - \lambda\}, \) where \( \lambda \) denotes the order topology as mentioned above. \( R \) (resp. \( L \)) is called the set of right (resp. left) looking points. Then \( X^* \) is defined as follows:

\[
X^* = (X \times \{0\}) \cup \{(x, k) : x \in R, k < 0, k \in \mathbb{Z}\} \cup \{(x, k) : x \in L, k > 0, k \in \mathbb{Z}\} \subset X \times \mathbb{Z},
\]

where \( \mathbb{Z} \) denotes the set of integers. On the other hand, \( L(X) \) is defined as follows:

\[
L(X) = (X \times \{0\}) \cup \{(x, -1) : x \in R\} \cup \{(x, 1) : x \in L\} \subset X \times \{-1, 0, 1\}.
\]

\( X^* \) and \( L(X) \) are linearly ordered topological spaces equipped with the lexicographic order topologies. We, furthermore, need some technical notation.
for the proof of the theorems in Section 5. For a convex open subset $U$ of a GO-space $X$, we define a convex open subset $\tilde{U}$ of $E(X)$, where $E(X)$ denotes either $X^*$ or $L(X)$. Then eight cases can occur. In the following, the intervals must be considered in $E(X)$.

1. If $a$ is the minimum point of $U$, then we define $\tilde{U}_1 = [(a,0), \rightarrow \in E(X)$.
2. Let $a = \inf U$ and $a \in X \setminus U$. If $E(X) = X^*$, then $\tilde{U}_1 = \{(x,k) \in X^* : a < x\} = [(a, +\infty), \rightarrow \in X^*$, where $(a, +\infty) \in X \times (\mathbb{Z} \cup \{+\infty\})$ and the interval is taken in $X^*$. Likewise, if $E(X) = L(X)$, then $\tilde{U}_1 = \{(x,k) \in L(X) : a < x\} = [(a, 1), \rightarrow \in L(X)$. Note that $(a, 1)$ may not belong to $L(X)$.
3. If there is a gap $u = (A,B)$ such that $u$ is the left end-point of $U$, then we define $\tilde{U}_1 = [(u,0), \rightarrow \in E(X)$.
4. If none of Cases 1–3 occurs, then we define $\tilde{U}_1 = E(X)$.
5. If $b$ is the maximum point of $U$, then we define $\tilde{U}_2 = \] \leftarrow, (b,0)] \in E(X)$.
6. Let $b = \sup U$ and $b \in X \setminus U$. If $E(X) = X^*$, then $\tilde{U}_2 = \{(x,k) \in X^* : x < b\} = [\leftarrow, (b, -\infty)] \in X^*$ (cf. (2)). If $E(X) = L(X)$, then $\tilde{U}_2 = \{(x,k) \in L(X) : x < b\} = [\leftarrow, (b, 1)] \in L(X)$. Note that $(b, 1)$ may not belong to $L(X)$.
7. If there is a gap $v = (A,B)$ such that $v$ is the right end-point of $U$, then we define $\tilde{U}_2 = \] \leftarrow, (v,0)] \in E(X)$.
8. If none of Cases 5–7 occurs, then we define $\tilde{U}_2 = E(X)$.

We set $\tilde{U} = \tilde{U}_1 \cap \tilde{U}_2$. $\tilde{U}$ is called the convex open set associated with $U$. Let $U$ be an open set of a GO-space $X$. Then $U$ is decomposed into a union of open convex subsets $\{U_\alpha : \alpha \in A\}$. In this case, we define $\tilde{U} = \bigcup\{\tilde{U}_\alpha : \alpha \in A\}$, where $\tilde{U}_\alpha$ is the open set associated with $U_\alpha$. Then $\tilde{U}$ is an open subset of $E(X)$, and called the open set associated with $U$.

3. $S_3$-diagonals in linearly ordered extensions

The following theorems are proved in our paper [6]. Let $X$ be a GO-space. It is easily seen that $X^*$ contains $X$ as a closed subset and $L(X)$ contains $X$ as a dense subset. See [4, 8] for further information about $X^*$ and $L(X)$. In both cases, $X$ and $X \times \{0\}$ are identified by the correspondence of $x$ to $(x,0)$.

**Theorem 3.1.** [6] Let $X$ be a generalized ordered space with an $S_3$-diagonal. If $R \cup L$ is countable, then $X^*$ has an $S_3$-diagonal.

To prove a similar theorem concerning $L(X)$, it is necessary to assume the existence of sequences in $X$ that witnesses first-countability for points of $R \cup L$.

**Theorem 3.2.** [6] Let $X$ be a GO-space with an $S_3$-diagonal. Assume that, for every point $s \in L$, there exists a decreasing sequence $\{x(s,n) : n \in \mathbb{N}\}$ in $X$ such that $\inf\{x(s,n)\} = s$, and, for every point $s \in R$, there exists an increasing sequence $\{y(s,n) : n \in \mathbb{N}\}$ in $X$ such that $\sup\{y(s,n)\} = s$. If $R \cup L$ is countable, then $L(X)$ has an $S_3$-diagonal.
4. Dense $S_\delta$-diagonals

The following definition gives an analogy to the dense $G_\delta$-diagonal in [II].

Definition 4.1. A Hausdorff space $X$ has a dense $S_\delta$-diagonal if there exists a dense subset $D$ of $\Delta_X$ such that $D$ is an $S_\delta$-subset of $X \times X$, where $\Delta_X$ denotes the diagonal subset of the Cartesian product $X \times X$.

We show the following theorem that is analogous to a result concerning spaces that have a dense $G_\delta$-diagonal [II].

Theorem 4.2. Let $X$ be a Hausdorff space. Then $X$ has a dense $S_\delta$-diagonal if and only if $X$ has a dense subset $Y$ such that $Y$ is an $S_\delta$-subset of $X$ and $Y$ has an $S_\delta$-diagonal.

Proof. If $D \subset \Delta_X$ is a dense $S_\delta$-set in $X \times X$, then $D \cap \Delta_X$ is a dense $S_\delta$-set in $\Delta_X$. Now the map $h : \Delta_X \to X$ defined by $h(x,x) = x$ is a homeomorphism, and the homeomorphic image of a dense $S_\delta$-set is a dense $S_\delta$-set.

Conversely, suppose $Y$ is a dense $S_\delta$-subset of $X$. Then $h^{-1}(Y)$ is a dense $S_\delta$-subset of $\Delta_X$. The rest is easily verified. □

5. Theorems Concerning Dense $S_\delta$-Diagonals of Linearly Ordered Extensions

Theorem 5.1. Let $X = (X, \tau)$ be a GO-space with a dense $S_\delta$-diagonal. If $R \cup L$ is countable, then $X^*$ has a dense $S_\delta$-diagonal.

We first show the following lemma.

Lemma 5.2. Let $X$ be a GO-space and $X^*$ the linearly ordered extension of $X$. For a subspace $Y$ of $X$, set $Z = Y \cup (X^* \setminus X)$. If $Y$ is dense in $X$ and $X^*$ is a dense subset of $X^*$, then $Z$ is a dense subset of $X^*$ and an $S_\delta$-subset of $X^*$.

Proof. To see that $Z$ is dense in $X^*$, let $x \in X^* \setminus Z = X \setminus Y$ and $V$ be a neighborhood of $x$ in $X^*$, where $X$ is identified with $X \times \{0\}$ as usual. Since $V \cap X$ is a neighborhood of $x$ in $X$, it follows that $V \cap X \cap Y \neq \emptyset$. Since $V \cap X \cap Y \subset V \cap Z$, it follows that $V \cap Z \neq \emptyset$. Hence $Z$ is a dense subspace of $X^*$. To show the last part, let $\{U(n) : n \in \mathbb{N}\}$ be a countable collection of open subsets of $X$ such that, for $y \in Y$ and $x \in X \setminus Y$, there exists an $m \in \mathbb{N}$ such that $y \in U(m)$ and $x \notin U(m)$. For every $n \in \mathbb{N}$, let $\hat{U}(n)$ be the open subset associated with $U(n)$ as in Section 3. Set $\hat{U}(0) = X^* \setminus X$. Then it is obvious that $\hat{U}(0)$ is an open subset of $X^*$. We show that the countable collection $\{\hat{U}(n) : n \geq 0\}$ of open subsets of $X^*$ assures that $Z$ is an $S_\delta$-subset of $X^*$. Let $z \in Z$ and $x \in X^* \setminus Z = X \setminus Y$.

Case 1. Let $z \in X^* \setminus X$. Then it is easy to see that $z \in \hat{U}(0)$ and $x \notin \hat{U}(0)$.

Case 2. Let $z \in Y$. Since $x \notin Y$, there exists an $m \in \mathbb{N}$ such that $z \in U(m)$ and $x \notin U(m)$. By the definition of $\hat{U}(m)$, it follows that $z \in \hat{U}(m)$ and $x \notin \hat{U}(m)$. This completes the proof. □
Now we shall prove Theorem 5.1.

**Proof of Theorem 5.1.** By Theorem 4.2, there exists a dense subspace \( Y \) of \( X \) such that \( Y \) is an \( S_3 \)-subset of \( X \) and that \( Y \) has an \( S_3 \)-diagonal. Let \( \{ \mathcal{G}(n) : n \in \mathbb{N} \} \) be a family of countable collections of open subsets of \( Y \) such that, for three points \( x, y, z \) of \( Y \) with \( y \neq z \), there exists an \( m \in \mathbb{N} \) such that \( x \in \bigcup \mathcal{G}(m) \) and no element of \( \mathcal{G}(m) \) contains \( \{y, z\} \). The existence of the above family is guaranteed by Theorem 1.4. For an open subset \( V \) of \( Y \), there exists an open set \( V_X \) of \( X \) such that \( V_X \cap Y = V \). Let \( V_X \) be the open subset of \( X^* \) associated with \( V_X \) as explained in Section 2. Set \( V_Z = V_X \cap Z \), where \( Z \) is as mentioned in Lemma 5.2.

It is clear that \( V_Z \) is open in \( Z \) and that \( V_Z \cap Y = V \). For every \( n \in \mathbb{N} \), set \( \tilde{\mathcal{G}}(n) = \{ V_Z : V \in \mathcal{G}(n) \} \) and \( \tilde{\mathcal{G}}(0) = \{ \{x\} : x \in X^* \setminus X \} \). Let \( S = R \cup L = \{ s_i : i \in \mathbb{N} \} \) be an enumeration of the countable set \( S \). Let \( (s_i, k) \in X^* \setminus X \). Set \( \tilde{\mathcal{G}}_+(s_i, k) = \{ \{(s_i, k), \rightarrow [\cap Z\} \} \) and \( \tilde{\mathcal{G}}_-(s_i, k) = \{ \leftarrow [(s_i, k)[\cap Z\} \) \}, where these intervals are considered in \( X^* \). By virtue of Lemma 5.2 and Theorem 4.2, it is sufficient to show that a family of those countable collections of open subsets of \( Z \) witnesses the \( S_3 \)-diagonal of \( Z \). To see this, let \( x, y, z \) be three points of \( Z \) with \( y \neq z \). We may assume without loss of generality that \( y < z \).

**Case 1.** If \( \{x, y, z\} \subset Y \), then there exists an \( m \in \mathbb{N} \) such that \( x \in \bigcup \mathcal{G}(m) \) and \( \{y, z\} \not\subset V \) for any \( V \in \mathcal{G}(m) \). Hence it follows that \( x \in \bigcup \tilde{\mathcal{G}}(m) \) and that \( \{y, z\} \not\subset V_Z \) for any \( V_Z \in \tilde{\mathcal{G}}(m) \).

**Case 2.** Let \( x \in Y \) and \( y \) or \( z \) belong to \( Z \setminus Y \).

(i) We assume that \( y \in Z \setminus Y \). Then we can write \( y = (s_i, k) \), where \( k \neq 0 \). If \( x < y \), then \( x \in \]_{\rightarrow, y}\cap Z \] and \( \{y, z\} \not\subset \]_{\rightarrow, y}\]. Hence, by the definition, it follows that \( x \in \bigcup \tilde{\mathcal{G}}_-(s_i, k) \) and that \( \{y, z\} \not\subset V \) for \( V \in \tilde{\mathcal{G}}_-(s_i, k) \). If \( y < x \), then \( x \in \]_{\rightarrow, y}\cap Z \] and \( \{y, z\} \not\subset \]_{\rightarrow, y}\]. Hence it follows that \( x \in \bigcup \tilde{\mathcal{G}}_+(s_i, k) \) and that \( \{y, z\} \not\subset V \) for \( V \in \tilde{\mathcal{G}}_+(s_i, k) \).

(ii) Let \( z \in Z \setminus Y \). Then the proof is analogous to (i).

**Case 3.** Let \( x \in Z \setminus Y \). Then it follows that \( x \in \bigcup \tilde{\mathcal{G}}(0) \) and \( \{y, z\} \not\subset V \) for any \( V \in \tilde{\mathcal{G}}(0) \). Therefore, by virtue of Lemma 1.3 and Theorem 1.4, \( X^* \) has a dense \( S_3 \)-diagonal. This completes the proof. \( \square \)

**Theorem 5.3.** Let \( X = (X, \tau) \) be a GO-space with a dense \( S_3 \)-diagonal. If \( R \cup L \) is countable, then \( L(X) \) has a dense \( S_3 \)-diagonal.

**Proof.** By Theorem 4.2, there exists a dense subspace \( Y \) of \( X \) such that \( Y \) is an \( S_3 \)-subset of \( X \) and \( Y \) has an \( S_3 \)-diagonal. Since \( X \) is dense in \( L(X) \), it follows that \( Y \) is a dense subspace of \( L(X) \). To prove that \( L(X) \) has a dense \( S_3 \)-diagonal, it is sufficient to show, by Theorem 4.2, that \( Y \) is an \( S_3 \)-subset of \( L(X) \). Let \( \{ U(n) : n \in \mathbb{N} \} \) be a countable collection of open subsets of \( X \) such that, for \( y \in Y \) and \( x \in X \setminus Y \), there exists an \( m \in \mathbb{N} \) such that \( y \in U(m) \) and \( x \not\in U(m) \). For every \( n \in \mathbb{N} \), let \( U(n) \) be the open subset of \( L(X) \) associated with \( U(n) \). For \( s_i \in S = R \cup L \) and \( \varepsilon \in \{-1, 1\} \), set \( \hat{U}_+(s_i, \varepsilon) = [(s_i, \varepsilon), \rightarrow [\] \} \) and \( \hat{U}_-(s_i, \varepsilon) = [\leftarrow [(s_i, \varepsilon)[\] \}, where the intervals are considered in \( L(X) \). The
countable collection \( \{\tilde{U}(n), \tilde{U}_+(s_i, \varepsilon), \tilde{U}_-(s_i, \varepsilon) : n \in \mathbb{N}, i \in \mathbb{N}, \varepsilon = \pm 1\} \) of open subsets of \( L(X) \) guarantees that \( Y \) is an \( S_\delta \)-subset of \( L(X) \). To see this, let \( y \in Y \) and \( z \in L(X) \setminus Y \).

**Case 1.** Let \( z \in X \setminus Y \). Then there exists an \( m \in \mathbb{N} \) such that \( y \in U(m) \) and \( z \notin U(m) \). Hence it follows that \( y \in \tilde{U}(m) \) and \( z \notin \tilde{U}(m) \).

**Case 2.** Let \( z \in L(X) \setminus X \). We can write \( z = (s_i, \varepsilon) \), where \( \varepsilon \in \{-1, 1\} \). If \( y < z \), then it follows that \( y \in \tilde{U}_-(s_i, \varepsilon) \) and \( z \notin \tilde{U}_-(s_i, \varepsilon) \). If \( z < y \), then it follows that \( y \in \tilde{U}_+(s_i, \varepsilon) \) and \( z \notin \tilde{U}_+(s_i, \varepsilon) \). Hence \( Y \) is an \( S_\delta \)-subset of \( L(X) \).

This completes the proof of Theorem 5.3. \( \square \)

6. **Examples**

**Example 6.1.** Theorems 3.1 and 3.2 do not hold without the assumption of the countability of the set \( R \cup L \). Let us consider the Sorgenfrey line \( X = (\mathbb{R}, S) \). In this case, the right looking points \( R = \mathbb{R} \) is uncountable. Since \( X \) has a \( G_\delta \)-diagonal, \( X \) has an \( S_\delta \)-diagonal. However, \( X^* \) does not have an \( S_\delta \)-diagonal. To prove this, it is sufficient to see that \( X^* \) does not have a quasi-\( G_\delta \)-diagonal [6]. We easily see that there does not exist a family of countable collections of open subsets of \( X^* \) that separates two points of the form \((x, 0)\) and \((x, 1)\), where \( x \in X \).

**Example 6.2.** Theorem 3.2 does not hold without the existence of the sequences for points of \( R \cup L \). To show a counterexample, let \( Y \) be the set of countable ordinals \([0, \omega_1]\) with the discrete topology. The right looking points of \( Y \) comprise the set of limit ordinals. Let \( Y^* \) be the linear extension of \( Y \) defined in Section 2. Let \( X = Y^* \cup \{(\omega_1, 0)\} \), where \( X \) is ordered as \((\omega_1, 0) > \alpha \) for all \( \alpha \in Y^* \), and given the discrete topology. Then \( R = \{(\omega_1, 0)\} \) is a singleton and \( L(X) = Y^* \cup \{(\omega_1, -1)\} \cup \{(\omega_1, 0)\} \), where \( \alpha < (\omega_1, -1) < (\omega_1, 0) \) for all \( \alpha \in Y^* \). There does not exist an increasing sequence in \( Y^* \) that converges to \((\omega_1, -1)\). Furthermore, \( L(X) \) does not have a quasi-\( G_\delta \)-diagonal, because the points of \( X \) and the point \((\omega_1, -1)\) are not separated by a family of countable collections of open subsets of \( L(X) \). Hence \( L(X) \) does not have an \( S_\delta \)-diagonal.

**Example 6.3.** A generalized ordered space does not necessarily have a dense \( S_\delta \)-diagonal. To show this, consider the linearly ordered space \( Z \) that was constructed by H. R. Bennett and D. J. Lutzer [4]. They proved that \( Z \) is not first-countable at any point. \( Z \) is defined as follows:

\[
Z = \{ (\alpha_1, \alpha_2, \ldots, \alpha_n, \omega_1, \omega_1, \ldots) : \alpha_i < \omega_1, 1 \leq i \leq n, \alpha_i = \omega_1, i > n, n \geq 1 \},
\]

with the lexicographic order. Since \( Z \) is densely-ordered, a dense subset \( Y \) of \( Z \) is a LOTS. If \( Y \) has a quasi-\( G_\delta \)-diagonal, \( Y \) is quasi-developable. Since a quasi-developable space is first-countable, \( Y \) does not have a quasi-\( G_\delta \)-diagonal. Therefore, \( Z \) does not have a dense \( S_\delta \)-diagonal.
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References


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