On $\varphi_{1,2}$-countable compactness and filters

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ABSTRACT. In this work the author investigates some relations between $\varphi_{1,2}$-countable compactness, filters, sequences and $\varphi_{1,2}$-closure operators.

2000 AMS Classification: 54A20, 54D30.
Keywords: Countable compactness, filter, convergence, operation, unification.

1. Introduction

Many generalizations of the notion of compact space have been defined in the literature, including those of quasi H-closed space, S-closed space, rs-compact space, feebly compact space, countably S-closed space, countably rs-compact space, and many more. Some of these concepts have been characterized in terms of filters and nets, and this has lead to such notions as r-convergence, RC-convergence, SR-convergence, r-accumulation point, RC-accumulation point and SR-accumulation point of filters and filterbases.

The notion of an operation on a topological space is a useful tool when attempting to unify such concepts, and in earlier studies we have defined $\varphi_{1,2}$-countably compact sets, $\varphi_{1,2}$-convergence of a filter and $\varphi_{1,2}$-accumulation points of a filter, and used these to obtain some such unifications.

In the present work we will study the relations between $\varphi_{1,2}$-countable compactness, filters, sequences and $\varphi_{1,2}$-closure operators.

There are several different definitions of an operation in the literature. We have used the one first given in [12] for fuzzy topological spaces.

In a topological space $(X, \tau)$, int, cl, scl, pcl etc. will stand for the interior, closure, semi-closure, pre-closure operations, and so on. For a subset $A$ of $X$, $A^\circ$, $\bar{A}$ will also be used to denote the interior and closure of $A$, respectively.

*Dedicated to the memory of Professor Doğan Çoker.*
Definition 1.1. Let \((X, \tau)\) be a topological space. A mapping \(\varphi : P(X) \rightarrow P(X)\) is called an operation on \((X, \tau)\) if \(\varphi(\emptyset) = \emptyset\) and \(A^o \subseteq \varphi(A), \forall A \in P(X)\).

The class of all operations on a topological space \((X, \tau)\) will be denoted by \(O(X, \tau)\).

For \(\varphi_1, \varphi_2 \in O(X, \tau)\) we set \(\varphi_1 \leq \varphi_2 \iff \varphi_1(A) \subseteq \varphi_2(A), \forall A \in P(X)\).

The operations \(\varphi, \tilde{\varphi}\) are dual if \(\tilde{\varphi}(A) = X \setminus \varphi(X \setminus A), \forall A \in P(X)\).

An operation \(\varphi \in O(X, \tau)\) is called monotonous if \(\varphi(A) \subseteq \varphi(B)\) whenever \(A \subseteq B (A, B \in P(X))\).

Definition 1.2. Let \(\varphi \in O(X, \tau)\). Then \(A \subseteq X\) is called \(\varphi\)-open if \(A \subseteq \varphi(A)\). Dually, \(B \subseteq X\) is called \(\varphi\)-closed if \(X \setminus B\) is \(\varphi\)-open.

Clearly, \(X\) and \(\emptyset\) are both \(\varphi\)-open and \(\varphi\)-closed, while each open set is a \(\varphi\)-open set for any \(\varphi \in O(X, \tau)\).

If \((X, \tau)\) is a topological space, \(\varphi \in O(X, \tau)\), then \(\varphi O(X)\), \(\varphi C(X)\) will denote respectively the set of \(\varphi\)-open, \(\varphi\)-closed subsets of \(X\). For \(x \in X\) we set \(\varphi O(X, x) = \{U \in \varphi O(X) \mid x \in U\}\).

For \(\varphi_2, \varphi_1 \in \varphi O(X)\) sufficient, generally not necessarily, conditions for \(\varphi_1 O(X) \subseteq \varphi_2 O(X)\) are \(\varphi_2 \geq \varphi_1\) or \(\varphi_2 \geq i\) [21]. Here \(i\) is the identity operation.

Definition 1.3. For the operations \(\varphi_1, \varphi_2 \in O(X, \tau)\), \(\varphi_2\) is called regular with respect to \(\varphi_1 O(X)\) if for each \(x \in X\) and \(U, V \in \varphi_1 O(X, x)\), there exists a \(W \in \varphi_1 O(X, x)\) such that \(\varphi_2(W) \subseteq \varphi_2(U) \cap \varphi_2(V)\).

Clearly, if \(\varphi_1 O(X)\) is closed under finite intersection and \(\varphi_2\) is monotonous, then \(\varphi_2\) is regular w.r.t. \(\varphi_1 O(X)\).

Definition 1.4. Let \(\varphi_1, \varphi_2 \in O(X, \tau), A \subseteq X, x \in X\). Then:

(a) \(x \in \varphi_1, \varphi_2 \text{\ int } A\) iff there exists a \(U \in \varphi_1 O(X, x)\) such that \(\varphi_2(U) \subseteq A\).
(b) \(x \in \varphi_1, \varphi_2 \text{\ cl } A\) \iff \(\varphi_2(U) \cap A \neq \emptyset\) for each \(U \in \varphi_1 O(X, x)\).
(c) \(A\) is \(\varphi_1, \varphi_2\)-open \iff \(A \subseteq \varphi_1, \varphi_2 \text{\ int } A\).
(d) \(A\) is \(\varphi_1, \varphi_2\)-closed \iff \(\varphi_1, \varphi_2 \text{\ cl } A \subseteq A\).

For any set \(A\) we have \(X \setminus \varphi_1, \varphi_2 \text{\ int } A = \varphi_1, \varphi_2 \text{\ cl } (X \setminus A)\) and \(A\) is \(\varphi_1, \varphi_2\)-open iff \(X \setminus A\) is \(\varphi_1, \varphi_2\)-closed.

Definition 1.5. [1] A subfamily \(U\) of the power set of a non-empty set \(X\) is called a supratopology on \(X\) if \(\emptyset, X \in U\) and \(U\) is closed under arbitrary unions.

If \(U\) is a supratopology on \(X\), then the pair \((X, U)\) is called a supratopological space.

The notions of base, first and second countability for a supratopology may be defined as for topological spaces [2].

If the operation \(\varphi \in O(X, \tau)\) is monotonous, then \(\varphi O(X)\) is a supratopology.

Theorem 1.6. [22] Let \(\varphi_1, \varphi_2 \in O(X, \tau)\). Then:
Then:

Definition 2.1. Let $\mathcal{A}$ be a subset of $X$.

Example 1.7. For the operations

- $\varphi_1 = \text{int}$, $\varphi_2 = \text{cl} \circ \text{int}$, $\varphi_3 = \text{cl}$, $\varphi_4 = \text{scl}$, $\varphi_5 = \text{i}$, $\varphi_6 = \text{int} \circ \text{cl}$,

defined on a topological space we have:

- $\varphi_1 \leq \varphi_2 \leq \varphi_3$ and $\varphi_1 \leq \varphi_6 \leq \varphi_4 \leq \varphi_3$.
- $\varphi_1 O(X) = \tau$.
- $\varphi_2 O(X) = SO(X)$ is the family of semi-open sets.
- $\varphi_3 O(X) = \varphi_5 O(X) = P(X)$ is the power set of $X$.
- $\varphi_6 = PO(X)$ is the family of pre-open sets.
- $\varphi_1, O(X) = \tau_0$ is the topology of all $\theta$-open sets.
- $\varphi_2, O(X) = SO(X)$ is the family of semi-$\theta$-open sets.
- $\varphi_1, O(X) = \tau_s$ is the semi regularization topology of $X$.
- $\varphi_2, O(X) = SO(X)$ is the family of all $\theta$-semi-open sets.
- The operations $\varphi_1, \varphi_3$ and $\varphi_2, \varphi_6$ are dual to one another.

All these operations are regular w.r.t. $\varphi_1 O(X)$.

2. $\varphi_1,2$-countable compactness

Definition 2.1. [21] Let $\varphi_1, \varphi_2 \in O(X, \tau)$, $X \in A \subseteq P(X)$ and $A \in P(X)$.

Then:

(a) If each countable $\mathcal{A}$-cover $U$ of $A$ has a finite subfamily $U'$ such that $A \subseteq \bigcup \{\varphi_2(U) \mid U \in U'\}$, then we say that $A$ is $(\mathcal{A}, \varphi_2)$-countably compact relative to $X$ (for short, a $(\mathcal{A}, \varphi_2)$-C.C. set).

(b) We call a $(\mathcal{A}, \tau)$-C.C. set a $\mathcal{A}$-C.C. set.

(c) If we take $A = \varphi_1 O(X)$ in (a) we say that $A$ is a $\varphi_1,2$-C.C. set.

If we take $A = \varphi_1,2 O(X)$ in (b) we say that $A$ is a $\varphi_1,2$-C.C. set.

If $X$ is $\varphi_1,2$-C.C. $(\varphi_1,2 O(X)$-C.C.) relative to itself, then $X$ will be called a $\varphi_1,2$-C.C. $(\varphi_1,2 O(X)$-C.C.) space.

We remark that the condition $X \in A$ is added here, and in our earlier papers, to guarantee the existence of an $\mathcal{A}$-cover or of a countable $\mathcal{A}$-cover of a subset of $X$. However, all the results still hold without this condition.
One may define $\varphi_{1,2}$-compact, $A$-compact, $\varphi_{1,2}$-Lindelöf and $A$-Lindelöf sets in a similar way [20, 23].

We assume that all the operations $\varphi_i$, $i = 1, 2, \ldots$ are defined on $(X, \tau)$ whenever they are used.

**Example 2.2.** Let $A \subseteq X$.

1. If $\varphi_1 = \text{int}$, $\varphi_2 = \tau$, then $A$ is a $\varphi_{1,2}$-C.C. set iff $A$ is countably compact.
2. If $\varphi_1 = \text{int}$, $\varphi_2 = \text{cl}$, then $A$ is a $\varphi_{1,2}$-C.C. set iff $A$ is feebly compact relative to $X$ [16], and $X$ is $\varphi_{1,2}$-C.C. iff $X$ is feebly compact (or, equivalently, lightly compact). $X$ is $H(1)$-closed [16] iff it is a Hausdorff first countable $\varphi_{1,2}$-C.C. space with respect to these operations.
3. If $\varphi_1 = \text{cl} \circ \text{int}$, $\varphi_2 = \text{cl}$, then $X$ is $\varphi_{1,2}$-C.C. iff it is countably $S$-closed [6].
4. If $\varphi_1 = \text{int}$, $\varphi_2 = \text{int} \circ \text{cl}$, then $X$ is strongly $H(1)$-closed [19] iff it is a Hausdorff first countable $\varphi_{1,2}$-C.C. space.
5. If $\varphi_1 = \text{cl} \circ \text{int}$, $\varphi_2 = \text{scl}$, then $X$ is $\varphi_{1,2}$-C.C. iff it is countably $rs$-compact [7].
6. For $\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$, $\varphi_2 = \tau$, we have $\varphi_1 O(X) = \varphi_2 O(X) = \tau^\circ$. Hence, $X$ is countably $\alpha$-compact [13] iff it is $\varphi_{1,2}$-C.C. iff it is $\varphi_{1,2} O(X)$-C.C. iff it is $\varphi_1 O(X)$-C.C.

**Definition 2.3.** Let $\mathcal{F}$ be a filter (or filterbase) on $X$, $(x_n)$ a sequence in $X$ and $a \in X$. We say that:

(a) $\mathcal{F}$, $\varphi_{1,2}$-accumulates to $a$, if $a \in \bigcap\{\varphi_{1,2}\text{cl} F \mid F \in \mathcal{F}\}$ [20].
(b) $\mathcal{F}$, $\varphi_{1,2}$-converges to $a$, if for each $U \in \varphi_1 O(X, a)$, there exists $F \in \mathcal{F}$ such that $F \subseteq \varphi_2(U)$ [20].
(c) $(x_n)$, $\varphi_{1,2}$-accumulates to $a$, if for each $U \in \varphi_1 O(X, a)$ and for each $n$, there exists an $n_0$ such that $n_0 \geq n$ and $x_{n_0} \in \varphi_2(U)$.
(d) $(x_n)$, $\varphi_{1,2}$-converges to $a$, if for each $U \in \varphi_1 O(X, a)$, there exists an $n_0$ such that for each $n \geq n_0$, $x_n \in \varphi_2(U)$.

**Example 2.4.** Let $\mathcal{F}$ be a filter (or filterbase) on $X$ and $a \in X$.

1. If $\varphi_1 = \text{int}$, $\varphi_2 = \tau$, then $\mathcal{F}$, $\varphi_{1,2}$-converges to $a$ iff $\mathcal{F}$ converges to $a$ in $(X, \tau)$ and $\mathcal{F}$, $\varphi_{1,2}$-accumulates to $a$ iff $\mathcal{F}$ accumulates to $a$ (or $a$ is an adherent point of $\mathcal{F}$) in $(X, \tau)$.
2. If $\varphi_1 = \text{int}$, $\varphi_2 = \text{cl}$, then $\mathcal{F}$, $\varphi_{1,2}$-converges to $a$ iff $\mathcal{F}$, $r$-converges [10] (or equivalently $\Theta$-converges [9], almost converges [3] to $a$, and $\mathcal{F}$, $\varphi_{1,2}$-accumulates to $a$ iff $a$ is an $r$-accumulation point [10] (or an almost adherent point [3]) of $\mathcal{F}$.
3. For $\varphi_1 = \text{cl} \circ \text{int}$, $\varphi_2 = \text{cl}$, it can be seen that, $\mathcal{F}$, $\varphi_{1,2}$-converges ($\varphi_{1,2}$-accumulates) to $a$ iff $\mathcal{F}$, $r$-converges (rc-accumulates) to $a$ [9], since $\{V \mid V \in \tau, a \in V\} = \{U \mid U \in \text{SO}(X), a \in U\}$. At the same time, $\mathcal{F}$, $\varphi_{1,2}$-converges ($\varphi_{1,2}$-accumulates) to $a$ iff $\mathcal{F}$, $s$-converges (s-accumulates) to $a$ [4].
4. If $\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$, $\varphi_2 = \tau$, then $\mathcal{F}$, $\varphi_{1,2}$-converges ($\varphi_{1,2}$-accumulates) to $a$ iff $\mathcal{F}$, $\alpha$-converges ($\alpha$-accumulates) to $a$ [14].
(5) If \( \varphi_1 = \text{cl} \circ \text{int}, \varphi_2 = \text{scl} \), it can be easily seen that \( \mathcal{F}, \varphi_{1,2} \)-converges \((\varphi_{1,2}\text{-accumulates})\) to \( a \) iff \( \mathcal{F}, SR \)-converges \((SR\text{-accumulates})\) to \( a \) \([5]\).

(6) For \( \varphi_1 = \text{cl} \circ \text{int}, \varphi_2 = \text{int} \circ \text{scl}, \) then we see that \( \mathcal{F}, \varphi_{1,2} \)-converges \((\varphi_{1,2}\text{-accumulates})\) to \( a \) iff \( \mathcal{F}, RS \)-converges \((RS\text{-accumulates})\) to \( a \) \([15]\).

(7) If \( \varphi_1 = \text{int}, \varphi_2 = \text{int} \circ \text{cl} \) then \( \mathcal{F}, \varphi_{1,2} \)-converges \((\varphi_{1,2}\text{-accumulates})\) to \( a \) iff \( \mathcal{F}, \delta \)-converges \((\delta\text{-accumulates})\) to \( a \) \([19]\).

Similar characterizations of the various notions of convergence and accumulation point for sequences and nets given in the literature can be easily given, and we omit the details.

**Theorem 2.5.** Let \( A \subseteq X \) and \( \mathcal{F} = \{F_n | n \in \mathbb{N}\} \) be a countable filterbase which meets \( A \). If some sequence satisfying \( x_n \in (\bigcap_{i=1}^{n} F_i) \cap A \) for each \( n \), \( \varphi_{1,2} \)-accumulates to some point \( a \in X \), then the filterbase \( \mathcal{F}, \varphi_{1,2} \)-accumulates to \( a \).

Conversely if for any sequence \((x_n)\) in \( A \) the countable filterbase \( \mathcal{F} = \{\{x_n | m \geq n\} | n \in \mathbb{N}\} \) which consists of the tails of the sequence \((x_n)\), \( \varphi_{1,2} \)-accumulates to some point \( a \in X \), then the sequence \((x_n)\), \( \varphi_{1,2} \)-accumulates to \( a \).

**Proof.** Let \( \mathcal{F} = \{F_n | n \in \mathbb{N}\} \) be a countable filterbase which meets \( A \). Then \( \mathcal{F}' = \{\bigcap_{i=1}^{n} F_i | n \in \mathbb{N}\} \) is a decreasing countable filterbase which meets \( A \) and generates the same filter as \( \mathcal{F} \). Take \( x_n \in (\bigcap_{i=1}^{n} F_i) \cap A \) for each \( n \), and let \((x_n)\), \( \varphi_{1,2} \)-accumulate to \( a \). Then, for each \( U \in \varphi_1 O(X, a) \) and for each \( n \), \( \emptyset \neq \varphi_2(U) \cap (\bigcap_{i=1}^{n} F_i) \cap A \subseteq \varphi_2(U) \cap (\bigcap_{i=1}^{n} F_i) \), hence \( \varphi_2(U) \cap F_n \neq \emptyset \). So, \( \mathcal{F}, \varphi_{1,2} \)-accumulates to \( a \).

Conversely let \((x_n)\) be a sequence in \( A \), and let \( \mathcal{F} = \{T_n | n \in \mathbb{N}\} \) be the countable filterbase consisting of the tails of \((x_n)\), which \( \varphi_{1,2} \)-accumulate to some point \( a \) and meets \( A \). Then for each \( U \in \varphi_1 O(X, a) \) and for each \( n \), \( \varphi_2(U) \cap T_n \neq \emptyset \). This means that \( a \) is a \( \varphi_{1,2} \)-accumulation point of \((x_n)\). \( \square \)

**Corollary 2.6.** Let \( A \subseteq X \). Each countable filterbase which meets \( A \), \( \varphi_{1,2} \)-accumulates to some point of \( A \) iff each sequence in \( A \), \( \varphi_{1,2} \)-accumulates to some point of \( A \).

**Theorem 2.7.** Let \( A \subseteq X \). If each countable filterbase which meets \( A \), \( \varphi_{1,2} \)-accumulates to some point of \( A \), then \( A \) is a \( \varphi_{1,2} \)-C.C. set.

**Proof.** Let \( A \subseteq \bigcup \mathcal{U}, \mathcal{U} = \{U_n | n \in I\} \), \( I \) countable and \( U_n \in \varphi_1 O(X) \). Assume that for each finite subset \( J \) of \( I \) we have \( A \not\subseteq \bigcup_{i \in J} \varphi_2(U_i) \). Then \( A \cap (X \setminus \bigcup_{i \in J} \varphi_2(U_i)) \neq \emptyset \). The family \( \mathcal{F} = \{X \setminus \bigcup_{i \in J} \varphi_2(U_i) | J \subseteq I, J \text{ finite}\} \) is a countable filterbase which meets \( A \). So, \( A \cap (\bigcap \{\varphi_2 F | F \in \mathcal{F}\}) \neq \emptyset \).

Let \( \mathcal{F}, \varphi_{1,2} \)-accumulate to \( a \in A \). There exists an \( i_0 \in I \) such that \( a \in U_{i_0} \). Now \( X \setminus \varphi_2(U_{i_0}) \in \mathcal{F}, \varphi_2(U_{i_0}) \cap (X \setminus \varphi_2(U_{i_0})) \neq \emptyset \). This contradiction completes the proof. \( \square \)

However, the converse of the above theorem need not hold. For operations \( \varphi_1 = \text{int}, \varphi_2 = \text{cl} \) in \((X, \tau)\), each countable filterbase \( \varphi_{1,2} \)-accumulates in
Proof. Let $(X, \tau)$ be a second countable supratopological space and $A \subseteq X$. If $A$ is a \( \varphi_{1,2} \)-C.C. set then each filterbase which meets $A$, \( \varphi_{1,2} \)-accumulates to some point of $A$.

**Theorem 2.9.** Let $\varphi_1$ be monotonous, $(X, \varphi_1O(X))$ be a second countable supratopological space and $A \subseteq X$. If $A$ is a \( \varphi_{1,2} \)-C.C. set then each filterbase which meets $A$, \( \varphi_{1,2} \)-accumulates to some point of $A$.

**Proof.** Let the supratopology $\varphi_1O(X)$ have a countable base, $A$ be a \( \varphi_{1,2} \)-C.C. set and $\mathcal{F}$ a filterbase which meets $A$.

Assume that $A \cap (\bigcap \{ \varphi_{1,2} \text{cl } F \mid F \in \mathcal{F} \}) = \emptyset$. For any $x \in A$, there exists a $U_x \in \varphi_1O(X,x)$ and an $F_x \in \mathcal{F}$ such that $\varphi_2(U_x) \cap F_x = \emptyset$. Now, $\mathcal{U} = \{ U_x \mid x \in A \}$ is a $\varphi_1$-open cover of $A$. Since $\varphi_1O(X)$ has a countable base, $\mathcal{U}$ has a countable subfamily which covers $A$. Since $A$ is a \( \varphi_{1,2} \)-C.C. set, there exists a finite subfamily $\{ U_{x_1}, U_{x_2}, \ldots, U_{x_n} \}$ of $\mathcal{U}$ such that $A \subseteq \bigcup_{i=1}^{n} \varphi_2(U_{x_i})$. Now, $(\bigcup_{i=1}^{n} \varphi_2(U_{x_i})) \cap (\bigcap_{i=1}^{n} F_{x_i}) = \emptyset$, so $A \cap (\bigcap_{i=1}^{n} F_{x_i}) = \emptyset$. This contradiction completes the proof. \( \square \)

**Corollary 2.10.** Under the assumptions of Theorem 2.9., the following are equivalent.

1. $A$ is a \( \varphi_{1,2} \)-C.C. set.
2. $A$ is a \( \varphi_{1,2} \)-compact set.
3. Each countable filterbase which meets $A$, \( \varphi_{1,2} \)-accumulates to some point of $A$.

**Proof.** In [20], it is shown that $A$ is a \( \varphi_{1,2} \)-compact set iff each filterbase which meets $A$, \( \varphi_{1,2} \)-accumulates to some point of $A$. Since each \( \varphi_{1,2} \)-compact set is a \( \varphi_{1,2} \)-C.C. set, the proof is now clear from Theorem 2.7. \( \square \)
Theorem 2.11. Let $\varphi_1, \varphi_2$ be monotonous and suppose that the conditions (\ast) and (\ast\ast) hold. If the supratopology $\varphi_1 O(X)$ has a countable base $B(\varphi_1 O(X))$, then $B' = \{\varphi_2(U) \mid U \in B(\varphi_1 O(X))\}$ is a countable base for the supratopology $\varphi_{1,2}O(X)$.

Proof. Under the given conditions, $B = \{\varphi_2(U) \mid U \in \varphi_1 O(X)\}$ is a base for the supratopology $\varphi_{1,2}O(X)$ and $B' \subseteq B \subseteq \varphi_{1,2}O(X)$. Let $V \in \varphi_{1,2}O(X)$ and $x \in V$. There exists a $U \in \varphi_1 O(X, x)$ such that $\varphi_2(U) \subseteq V$. Hence, $x \in U \subseteq \varphi_2(U) \subseteq V$. There exists a $U' \in B(\varphi_1 O(X))$ such that $x \in U' \subseteq U$. Hence, we have $x \in \varphi_2(U') \subseteq \varphi_2(U) \subseteq V$ and $\varphi_2(U') \in B'$.

Theorem 2.12. Let (\ast) and (\ast\ast) hold and let $B = \{\varphi_2(U) \mid U \in \varphi_1 O(X)\}$. Then the following are equivalent for any subset $A$ of $X$.

(a) $A$ is a $\varphi_{1,2}$-compact set.
(b) $A$ is a $B$-compact set.
(c) $A$ is both a $\varphi_{1,2}$-Lindelöf set and a $\varphi_{1,2}$-C.C. set.
(d) $A$ is both a $B$-Lindelöf set and a $B$-C.C. set.

Proof. Under the given conditions, $A$ is a $\varphi_{1,2}$-compact set iff it is $B$-compact set [20], $A$ is a $\varphi_{1,2}$-Lindelöf set iff it is a $B$-Lindelöf set [23], $A$ is a $\varphi_{1,2}$-C.C. set iff it is a $B$-C.C. set [22]. Hence (b) $\iff$ (d) is now clear, as are the others.

Theorem 2.13. Let $\varphi_1, \varphi_2$ be monotonous and suppose that the conditions (\ast) and (\ast\ast) hold. If the supratopology $\varphi_1 O(X)$ has a countable base $B(\varphi_1 O(X))$, or if $B = \{\varphi_2(U) \mid U \in \varphi_1 O(X)\}$ is countable, then the following are equivalent.

(a) $A$ is a $\varphi_{1,2}$-C.C. set.
(b) $A$ is a $\varphi_{1,2}O(X)$-C.C. set.
(c) $A$ is a $B$-C.C. set.
(d) $A$ is a $\varphi_{1,2}$-compact set.
(e) $A$ is a $\varphi_{1,2}O(X)$-compact set.
(f) $A$ is a $B$-compact set.

Proof. Under the conditions (\ast) and (\ast\ast), (a) $\iff$ (c), (b) $\implies$ (c) and (d) $\iff$ (e) $\iff$ (f) are given in [22] and [20] respectively. If $B$ is a countable base of $\varphi_{1,2}O(X)$, then (c) $\implies$ (b) is clear. In the other case, $B' = \{\varphi_2(U) \mid U \in B(\varphi_1 O(X))\}$ is a countable base of $\varphi_{1,2}O(X)$ and $B' \subseteq B \subseteq \varphi_{1,2}O(X)$. Hence, a $B$-C.C. set will be a $B'$-C.C. set and a $B'$-C.C. set will be a $\varphi_{1,2}O(X)$-C.C. set, so we have again (c) $\implies$ (b). In each case (b) $\iff$ (e) is clear.

Theorem 2.14. Let $\varphi_1$ be monotonous and let $a \in X$ have a countable local base $C_{\varphi_1}(a)$ in the supratopological space $(X, \varphi_1 O(X))$.

1. If $\varphi_2$ is monotonous and regular w.r.t. $\varphi_1 O(X)$, then the family $\mathcal{F} = \{\varphi_2(U) \mid U \in C_{\varphi_1}(a)\}$ is a countable filterbase and $\varphi_{1,2}$-converges to $a$.
2. If $\varphi_1 O(X)$ is a topology and $\varphi_1 O(X) \subseteq \varphi_2 O(X)$, then $C_{\varphi_1}(a)$ is a countable filterbase which $\varphi_{1,2}$-converges to $a$. 

Proof. (1) For \( U, U' \in C_{\varphi_1}(a) \), \( a \in U \cap U' \) and \( U, U' \in \varphi_1O(X) \). Since \( \varphi_2 \) is regular w.r.t. \( \varphi_1O(X) \), there exists a \( V \in \varphi_1O(X,a) \) such that \( \varphi_2(V) \subseteq \varphi_2(U) \cap \varphi_2(U') \). There exists a \( V_c \in C_{\varphi_1}(a) \) such that \( V_c \subseteq V \). Since \( \varphi_2 \) is monotonous, we have \( \varphi_2(V_c) \subseteq \varphi_2(V) \subseteq \varphi_2(U) \cap \varphi_2(U') \). Hence \( F \) is a countable filterbase. Let \( U \in \varphi_1O(X,a) \). There exists a \( V_c \in C_{\varphi_1}(a) \) such that \( U_c \subseteq U \). \( \varphi_2(U_c) \in F \) and, since \( \varphi_2 \) is monotonous \( \varphi_2(U_c) \subseteq \varphi_2(U) \). So, \( F \) is \( \varphi_{1,2} \)-convergent to \( a \).

(2) For \( U, U' \in C_{\varphi_1}(a) \), \( a \in U \cap U' \in \varphi_1O(X,a) \). There exists a \( V_c \in C_{\varphi_1}(a) \) such that \( U_c \subseteq U \cap U' \). Hence \( C_{\varphi_1}(a) \) is a countable filterbase.

Now, let \( V \in \varphi_1O(X,a) \). There exists a \( V_c \in C_{\varphi_1}(a) \) such that \( V_c \subseteq V \). Since \( \varphi_1O(X) \subseteq \varphi_2O(X) \), we have \( V_c \subseteq V \subseteq \varphi_2(V) \). Hence \( C_{\varphi_1}(a) \), \( \varphi_{1,2} \)-converges to \( a \).

Theorem 2.15. Let \( \varphi_1, \varphi_2 \) be monotonous, let \( a \in X \) have a countable local base \( C_{\varphi_1}(a) \) in \( (X, \varphi_1O(X)) \) and also let \( \varphi_2 \) be regular w.r.t. \( \varphi_1O(X) \). For \( A \subseteq X \), \( a \in \varphi_{1,2}cl A \) iff there exists a filter which contains \( A \), has a countable base and \( \varphi_{1,2} \)-converges to \( a \).

Proof. Let \( a \in \varphi_{1,2}cl A \). Then for each \( U \in \varphi_1O(X,a) \), \( \varphi_2(U) \cap A \neq \emptyset \). As in the proof of Theorem 2.14.(1), it is easily seen that \( F_b = \{ \varphi_2(V) \cap A \mid V \in C_{\varphi_1}(a) \} \) is a countable filterbase. The filter \( F \) generated by \( F_b \) contains \( A \), and \( \{ \varphi_2(V) \mid V \in C_{\varphi_1}(a) \} \subseteq F \). Clearly \( F \) is \( \varphi_{1,2} \)-convergent to \( a \).

The other part of the proof is clear from Corollary 3.4. in [20].

Theorem 2.16. Let \( \varphi_1, \varphi_2 \) be monotonous, \( (X, \varphi_1O(X)) \) be a first countable supratopological space, and define \( cl^* : P(X) \rightarrow P(X) \) by \( cl^*(A) = \{ x \mid \text{there exists a filter that contains } A, \text{has a countable base and } \varphi_{1,2} \text{-converges to } x \} \), for each \( A \in P(X) \).

(1) If \( \varphi_2 \) is regular w.r.t. \( \varphi_1O(X) \), then \( cl^*(A) = \varphi_{1,2}cl A \) for each \( A \in P(X) \), and \( cl^* \) defines the topology \( \tau^* = \{ U \subseteq X \mid (X \setminus U)^* \subseteq X \setminus U \} = \varphi_{1,2}O(X) \).

(2) If \( \varphi_2 \) is regular w.r.t. \( \varphi_1O(X) \) and \( \varphi_1O(X) \subseteq \varphi_2O(X) \), then \( cl^* \) defines the topology \( \tau^* = \{ U \subseteq X \mid (X \setminus U)^* = X \setminus U \} = \varphi_{1,2}O(X) \).

(3) If \( \varphi_2 \) is regular w.r.t. \( \varphi_1O(X) \), \( \varphi_1O(X) \subseteq \varphi_2O(X) \), and \( \varphi_2(U) \in \varphi_{1,2}O(X) \) for each \( U \in \varphi_1O(X) \), then the operator \( cl^* \) is a Kuratowski closure operator defining \( \tau^* = \{ U \subseteq X \mid (X \setminus U)^* = X \setminus U \} = \varphi_{1,2}O(X) \).

Hence, if \( \varphi_1, \varphi_2 \) are monotonous and \( (X, \varphi_1O(X)) \) is a first countable topological space, then the \( \varphi_{1,2} \)-closure operator and the topology \( \tau_{\varphi_{1,2}} = \{ U \subseteq X \mid \varphi_{1,2}cl(X \setminus U) \subseteq X \setminus U \} = \varphi_{1,2}O(X) \) can be defined using filters with countable bases.

Proposition 2.17. If \( \varphi_1O(X) \subseteq \varphi_2O(X) \) (hence, if \( \varphi_2 \geq \varphi_1 \) or \( \varphi_2 \geq 1 \)), then \( A \subseteq \varphi_{1,2}cl A \) for each \( A \in P(X) \).
Proposition 2.18. If (**) holds, then $\varphi_2(U) \subseteq \varphi_{1,2}\text{int}(\varphi_2(U))$ (i.e., $\varphi_2(U) \in \varphi_{1,2}O(X)$) for each $U \in \varphi_1O(X)$.

Proof. Let $U \in \varphi_1O(X)$ and $x \in \varphi_2(U)$. Then $x \in \varphi_2(U) \in \varphi_1O(X)$ and $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$. So $x \in \varphi_{1,2}\text{int}(\varphi_2(U))$. □

Corollary 2.19. (a) Under the condition (**), we have, $\varphi_{1,2}\text{cl}(X \setminus \varphi_2(U)) \subseteq X \setminus \varphi_2(U)$ for each $U \in \varphi_1O(X)$.

(b) If $\varphi_1O(X) \subseteq \varphi_2O(X)$ and (***) holds, then $\varphi_{1,2}\text{cl}(X \setminus \varphi_2(U)) = X \setminus \varphi_2(U)$ for each $U \in \varphi_1O(X)$.

Remark 2.20. a) If $\varphi_2$ is the dual operation of $\varphi_2$, then $\{X \setminus \varphi_2(U) \mid U \in \varphi_1O(X)\} = \{\varphi_2(X \setminus U) \mid U \in \varphi_1O(X)\} = \{\varphi_2(K) \mid K \in \varphi_1C(X)\}$.

b) If $\varphi_1$ is monotonous (in which case $\varphi_1O(X)$ is a supratopology), and $\varphi_2(U \cup V) = \varphi_2(U) \cup \varphi_2(V)$ for each $U, V \in \varphi_1O(X)$, then for each finite subfamily $(U_1, U_2, \ldots, U_n)$ of $\varphi_1O(X)$, $\bigcup_{i=1}^{n} U_i \in \varphi_1O(X)$ and $\varphi_2(\bigcup_{i=1}^{n} U_i) = \bigcup_{i=1}^{n} \varphi_2(U_i)$.

Theorem 2.21. Consider the following statements:

(i) $\varphi_1$ is monotonous.

(ii) $\varphi_2$ is monotonous.

(iii) $\varphi_2 \geq 1$ or $\varphi_2 \geq \varphi_1$ (i.e. (*)),

(iv) $\forall U \in \varphi_1O(X)$, $\varphi_2(U) \in \varphi_1O(X)$ and $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$ (i.e. (**)).

(v) For each $U, V \in \varphi_1O(X)$, $\varphi_2(U \cup V) = \varphi_2(U) \cup \varphi_2(V)$.

(vi) $\varphi_2$ is the dual of $\varphi_2$.

and

(a) $A$ is a $\varphi_{1,2}$-C.C. set.

(b) Each countable filterbase $F \subseteq \{X \setminus \varphi_2(U) \mid U \in \varphi_1O(X)\}$ which meets $A$, $\varphi_{1,2}$-accumulates to some point of $A$.

(c) For each countable filterbase $F \subseteq \{X \setminus \varphi_2(U) \mid U \in \varphi_1O(X)\}$ which meets $A$, we have $A \cap (\bigcap F) \neq \emptyset$.

(d) For each decreasing countable filterbase $F \subseteq \{X \setminus \varphi_2(U) \mid U \in \varphi_1O(X)\}$ which meets $A$, we have $A \cap (\bigcap \{\varphi_{1,2}F \mid F \in F\}) \neq \emptyset$.

(e) For each decreasing countable filterbase $F \subseteq \{X \setminus \varphi_2(U) \mid U \in \varphi_1O(X)\}$ which meets $A$, we have $A \cap (\bigcap F) \neq \emptyset$.

(f) If $\Phi$ is any decreasing sequence of countable non-empty $\varphi_1$-closed sets such that for each $F \in \Phi$, $A \cap \varphi_2(F) \neq \emptyset$, then $A \cap (\bigcap \Phi) \neq \emptyset$.

Then,

(1) (b) $\implies$ (d) and (c) $\implies$ (e).

(2) If (iii) holds, then (e) $\implies$ (b) and (e) $\implies$ (d).

(3) If (iii) and (iv) hold, then (c) $\iff$ (b) and (e) $\iff$ (d).

(4) If (iv) holds, then (a) $\implies$ (e).

(5) If (i) and (v) hold, then (d) $\implies$ (b) and (b) $\implies$ (a).

(6) If (ii) and (vi) hold, then (a) $\implies$ (f).

(7) If (i), (iii), (v) and (vi) hold, then (f) $\implies$ (a).
Proof. (1) Immediate.

2) Clear from Proposition 2.17.

3) Clear from Corollary 2.19.

(4) Let $A$ be a $\varphi_{1,2}$-C.C. set, and $F = \{X \setminus \varphi_2(U_i) \mid i \in I\}$, $U_i \in \varphi_1 O(X)$, be a countable filterbase which meets $A$. Assume that $A \cap (\bigcap F) = \emptyset$ and $A \subseteq \bigcup_{i \in I} \varphi_2(U_i)$. Since, $\varphi_2(U) \in \varphi_1 O(X)$, $\varphi_2(\varphi_2(U)) \subseteq \varphi_2(U)$, for each $U \in \varphi_1 O(X)$, and $A$ is a $\varphi_{1,2}$-C.C. set, there exists a finite subset $J$ of $I$ such that, $A \subseteq \bigcup_{j \in J} \varphi_2(U_j) \subseteq \bigcup_{i \in J} \varphi_2(U_i)$. We have $A \cap (\bigcap_{i \in J}(X \setminus \varphi_2(U_i))) = \emptyset$. This contradiction completes the proof.

(5) Let $F \subseteq \{X \setminus \varphi_2(U) \mid U \in \varphi_1 O(X)\}$ be a countable filterbase which meets $A$. Then $F = \{F_n \mid n \in \mathbb{N}\}$, where $F_n = X \setminus \varphi_2(U_n)$, $n \in \mathbb{N}$ and $U_n \in \varphi_1 O(X)$. Let $F'_n = \bigcap_{i=1}^{n} F_i$ for each $n$. Then $F' = \{F'_n \mid n \in \mathbb{N}\}$ is a decreasing countable filterbase, and $F'_n = \bigcap_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (X \setminus \varphi_2(U_i)) = X \setminus \bigcup_{i=1}^{n} \varphi_2(U_i) = X \setminus \varphi_2(\bigcap_{i=1}^{n} U_i)$. Hence, $F' \subseteq \{X \setminus \varphi_2(U) \mid U \in \varphi_1 O(X)\}$.

If we assume that (d) holds then $A \cap (\bigcap\{\varphi_{1,2}\cl F'_n \mid F'_n \in F'\}) \neq \emptyset$. Since $F'_n \subseteq F_n$ for each $n$, we have $\varphi_{1,2}\cl F'_n \subseteq \varphi_{1,2}\cl F_n$. So $A \cap (\bigcap\{\varphi_{1,2}\cl F'_n \mid F'_n \in F'\}) \neq \emptyset$.

Now, let us verify that (b) $\implies$ (a). Let $A \subseteq \bigcup U$, $U \subseteq \varphi_1 O(X)$ and $U = \{U_i \mid i \in I\}$ be countable. Assume that for each finite subset $J$ of $I$, $A \not\subseteq \bigcup_{j \in J} \varphi_2(U_j)$.

Then, $A \cap (X \setminus \bigcup_{j \in J} \varphi_2(U_j)) \neq \emptyset$. From our hypotheses, $\bigcup_{j \in J} U_j \subseteq \varphi_1 O(X)$ and $\varphi_2(\bigcup_{j \in J} U_j) = \bigcup_{j \in J} \varphi_2(U_j)$. So, for each finite subset $J$ of $I$, we have $A \cap (X \setminus \varphi_2(\bigcup_{j \in J} U_j)) \neq \emptyset$. Let $F = \{X \setminus \varphi_2(\bigcup_{j \in J} U_j) \mid J \subseteq I, J$ finite$. Then $F \subseteq \{X \setminus \varphi_2(U) \mid U \in \varphi_1 O(X)\}$ and $F$ is a countable filterbase which meets $A$. There exists an $a \in A$ such that $a \in \bigcap\{\varphi_{1,2}\cl F \mid F \in F\}$ and a $U_a \in U$ such that $a \in U_a$. Now, $X \setminus \varphi_2(U_a) \in F$ and $\varphi_2(U_a) \cap (X \setminus \varphi_2(U_a)) = \emptyset$. This contradiction completes the proof.

(6) Let $\Phi$ be a countable decreasing sequence of nonempty $\varphi_1$-closed sets such that for each $F \in \Phi$, $A \cap \varphi_2(F) \neq \emptyset$. Assume that $A \cap (\bigcap \Phi) = \emptyset$. Then, $A \subseteq \bigcup\{X \setminus F \mid F \in \Phi\}$. Since for each $F \in \Phi$, $X \setminus F \in \varphi_1 O(X)$, and $A$ is a $\varphi_{1,2}$-C.C. set, there exists a finite subfamily $\Phi'$ of $\Phi$ such that $A \subseteq \bigcup\{\varphi_2(X \setminus F) \mid F \in \Phi'\}$. Since $\varphi_2$ is monotonous, $A \subseteq \varphi_2(\bigcup_{F \in \Phi'}(X \setminus F))$. There exists an $F' \in \Phi'$ such that $\bigcup_{F \in \Phi'}(X \setminus F) = X \setminus F'$. Then $A \subseteq \varphi_2(X \setminus F') = X \setminus \varphi_2(F')$, so $A \cap \varphi_2(F') = \emptyset$. This contradiction completes the proof.

(7) Let $U = \{U_n \mid n \in \mathbb{N}\}$ be a countable $\varphi_1$-open cover of $A$. Assume that for each finite subset $J$ of $I$, $A \not\subseteq \bigcup_{j \in J} \varphi_2(U_j)$. In this case, for each finite subset $J$ of $N$, $X \not= \bigcup_{j \in J} U_j$ since, otherwise, we would have $A \subseteq \bigcup_{j \in J} U_j \subseteq \bigcup_{j \in J} \varphi_2(U_j)$ for a finite subset $J$ of $N$.

Let $F_n = X \setminus \bigcup_{i=1}^{n} U_i$ for each $n$. For each $n$, $F_n \neq \emptyset$, $F_n \in \varphi_1 C(X)$ and $A \cap (X \setminus \bigcup_{i=1}^{n} \varphi_2(U_i)) \neq \emptyset$. Now
On $\varphi_{1,2}$-countable compactness

$$A \cap (X \setminus \bigcup_{i=1}^{n} \varphi_2(U_i)) = A \cap (X \setminus \varphi_2(\bigcup_{i=1}^{n} U_i))$$

$$= A \cap (\varphi_2(X \setminus \bigcup_{i=1}^{n} U_i))$$

$$= A \cap \varphi_2(F_n)$$

$$\neq \emptyset.$$ Hence, $A \cap (\bigcap_{n=1}^{\infty} F_n) \neq \emptyset$. But $\bigcap_{n=1}^{\infty} F_n = X \setminus (\bigcup_{n=1}^{\infty} U_n)$ and we obtain that $A \cap (X \setminus \bigcup_{n=1}^{\infty} U_n) = \emptyset$. This contradiction completes the proof. \hfill $\square$

**Example 2.22.**

1. If $\varphi_1 = \text{int}$, $\varphi_2 = \text{cl}$, then $\tilde{\varphi}_2 = \text{int}$ and the conditions (i), (ii), (iii), (v) and (vi) are satisfied.
2. If $\varphi_1 = \text{cl} \circ \text{int}$, $\varphi_2 = \text{scl}$, then conditions (i), (ii), (iii), (iv) and (vi) are satisfied, and $\tilde{\varphi}_2 = \text{semi-interior}$ is the dual of $\varphi_2$.
3. If $\varphi_1 = \text{cl} \circ \text{int}$, $\varphi_2 = \text{cl}$, then $\tilde{\varphi}_2 = \text{int}$ and all the conditions are satisfied.

Many known results, see for example [6,11,17,18,19], and also many new results, may now be obtained by choosing particular operations and combining the above results with the unifications obtained in [20-23].

**References**


Received December 2001
Revised February 2003

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