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Fuzzy quasi-metric spaces

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Abstract. We generalize the notions of fuzzy metric by Kramosil and Michalek, and by George and Veeramani to the quasi-metric setting. We show that every quasi-metric induces a fuzzy quasi-metric and, conversely, every fuzzy quasi-metric space generates a quasi-metrizable topology. Other basic properties are discussed.

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1. Introduction

In [9], Kramosil and Michalek introduced and studied an interesting notion of fuzzy metric space which is closely related to a class of probabilistic metric spaces, the so-called (generalized) Menger spaces. Later on, George and Veeramani started, in [3] (see also [5]), the study of a stronger form of metric fuzziness. In particular, it is well known that every metric induces a fuzzy metric in the sense of George and Veeramani, and, conversely, every fuzzy metric space in the sense of George and Veeramani (and also of Kramosil and Michalek) generates a metrizable topology ([4], [6], [9], [11], [13]).

On the other hand, it is also well known that quasi-metric spaces constitute an efficient tool to discuss and solve several problems in topological algebra, approximation theory, theoretical computer science, etc. (see [10]).

In this paper, we introduce two notions of fuzzy quasi-metric space that generalize the corresponding notions of fuzzy metric space by Kramosil and Michalek, and by George and Veeramani to the quasi-metric context. Several basic properties of these spaces are obtained. We show that every quasi-metric induces a fuzzy quasi-metric and, conversely, every fuzzy quasi-metric generates a quasi-metrizable topology. With the help of these results one can easily derive many properties of fuzzy quasi-metric spaces.

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Our basic references for quasi-uniform and quasi-metric spaces are [2] and [10].

Let us recall that a quasi-pseudo-metric on a set X is a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

Following the modern terminology (see Section 11 of [10]), by a quasi-metric on X we mean a quasi-pseudo-metric d on X that satisfies the following condition: $d(x, y) = d(y, x) = 0$ if and only if $x = y$. If the quasi-pseudo-metric d satisfies: $d(x, y) = 0$ if and only if $x = y$, then we say that d is a T_1 quasi-metric on X.

A quasi-(pseudo-)metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-(pseudo-)metric on X. The notion of a T_1 quasi-metric space is defined in the obvious manner.

Each quasi-pseudo-metric d on X generates a topology τ_d on X which has as a base the family of open d-balls $\{B_d(x, r) : x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

Observe that if d is a quasi-metric, then τ_d is a T_0 topology, and if d is a T_1 quasi-metric, then τ_d is a T_1 topology.

A topological space (X, τ) is said to be quasi-metrizable if there is a quasi-metric d on X such that $\tau = \tau_d$. In this case, we say that d is compatible with τ, and that τ is a quasi-metrizable topology.

Given a quasi-(pseudo-)metric d on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-(pseudo-)metric on X, called the conjugate of d. Finally, the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo-)metric on X.

2. Definitions and basic results

According to [13], a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

Definition 2.1. *A KM-fuzzy quasi-pseudo-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, +\infty)$ such that for all $x, y, z \in X$:*

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, x, t) = 1$ *for all* $t > 0$;
(KM3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ *for all* $t, s \geq 0$;
(KM4) $M(x, y, \_) : [0, +\infty) \to [0, 1]$ *is left continuous.*

Definition 2.2. *A KM-fuzzy quasi-metric on X is a KM-fuzzy quasi-pseudo-metric $(M, *)$ on X that satisfies the following condition:*

(KM2') $x = y$ *if and only if* $M(x, y, t) = M(y, x, t) = 1$ *for all* $t > 0$.

If $(M, *)$ is a KM-fuzzy quasi-pseudo-metric on X satisfying:

(KM2") $x = y$ *if and only if* $M(x, y, t) = 1$ *for all* $t > 0$, *we say that* $(M, *)$ *is a* T_1 *KM-fuzzy quasi-metric on* X.

Definition 2.3. *A KM-fuzzy (pseudo-)metric on* X *is a KM-fuzzy quasi-(pseudo-)metric* $(M, *)$ *on* X *such that for each* $x, y \in X$:

(KM5) $M(x, y, t) = M(y, x, t)$ *for all* $t > 0$.

Remark 2.4. It is clear that every KM-fuzzy metric is a T_1 KM-fuzzy quasi-metric; every T_1 KM-fuzzy quasi-metric is a KM-fuzzy quasi-metric, and every KM-fuzzy quasi-metric is a KM-fuzzy quasi-pseudo-metric.

Definition 2.5. *A KM-fuzzy quasi-(pseudo-)metric space is a triple* $(X, M, *)$ *such that* X *is a (nonempty) set and* $(M, *)$ *is a KM-fuzzy quasi-(pseudo-)metric on* X.

The notions of a T_1 *KM-fuzzy quasi-metric space* and of a *KM-fuzzy (pseudo-)metric space* are defined in the obvious manner. Note that the KM-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of Kramosil and Michalek.

If $(M, *)$ is a KM-fuzzy quasi-(pseudo-)metric on a set X, it is immediate to show that $(M^{-1}, *)$ is also a KM-fuzzy quasi-(pseudo-)metric on X, where M^{-1} is the fuzzy set in $X \times X \times [0, +\infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$. Moreover, if we denote by M^i the fuzzy set in $X \times X \times [0, +\infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, then $(M^i, *)$ is, clearly, a KM-fuzzy (pseudo-)metric on X.

Proposition 2.6. *Let* $(X, M, *)$ *be a KM-fuzzy quasi-pseudo-metric space. Then, for each* $x, y \in X$ *the function* $M(x, y, \_)$ *is nondecreasing.*

Proof. Let $x, y \in X$ and $0 \le t < s$. Then $M(x, y, s) \ge M(x, x, s - t) * M(x, y, t) = M(x, y, t)$. □

Given a KM-fuzzy quasi-pseudo-metric space $(X, M, *)$ we define the open ball $B_M(x, r, t)$, for $x \in X$, $0 < r < 1$, and $t > 0$, as the set $B_M(x, r, t) := \{y \in X : M(x, y, t) > 1 - r\}$. Obviously, $x \in B_M(x, r, t)$.

By Proposition 2.6, it immediately follows that for each $x \in X$, $0 < r_1 \le r_2 < 1$ and $0 < t_1 \le t_2$, we have $B_M(x, r_1, t_1) \subseteq B_M(x, r_2, t_2)$. Consequently, we may define a topology τ_M on X as

$$\tau_M := \{A \subseteq X : \text{for each } x \in A \text{ there are } r \in (0, 1), t > 0, \text{ with } B_M(x, r, t) \subseteq A\}.$$

Moreover, for each $x \in X$ the collection of open balls $\{B_M(x, 1/n, 1/n) : n = 2, 3, ...\}$, is a local base at x with respect to τ_M. It is clear, that if $(X, M, *)$ is a KM-fuzzy quasi-metric (respectively, a T_1 KM-fuzzy quasi-metric, a KM-fuzzy metric), then τ_M is a T_0 (respectively, a T_1, a Hausdorff) topology.

The topology τ_M is called the topology *generated* by the KM-fuzzy quasi-pseudo-metric space $(X, M, *)$.

Similarly to the proof of Result 3.2 and Theorem 3.11 of [3], one can show the following results.

Proposition 2.7. *Let $(X, M, *)$ be a KM-fuzzy quasi-pseudo-metric space. Then, each open ball $B_M(x, r, t)$ is an open set for the topology τ_M.*

Proposition 2.8. *A sequence $(x_n)_n$ in a KM-fuzzy quasi-pseudo-metric space $(X, M, *)$ converges to a point $x \in X$ with respect to τ_M if and only if $\lim_n M(x, x_n, t) = 1$ for all $t > 0$.*

Definition 2.9. *A GV-fuzzy quasi-pseudo-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0, +\infty)$ such that for all $x, y, z \in X$, $t, s > 0$:*

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, x, t) = 1$;
(GV3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(GV4) $M(x, y, \_) : (0, +\infty) \to (0, 1]$ *is continuous.*

Definition 2.10. *A GV-fuzzy quasi-metric on X is a GV-fuzzy quasi-pseudo-metric $(M, *)$ on X such that for all $t > 0$:*

(GV2') $x = y$ *if and only if* $M(x, y, t) = M(y, x, t) = 1$.

If $(M, *)$ is a GV-fuzzy quasi-pseudo-metric on X such that for all $t > 0$:

(GV2") $x = y$ *if and only if* $M(x, y, t) = 1$,

we say that $(M, *)$ is a T_1 KM-fuzzy quasi-metric on X.

Definition 2.11. *A GV-fuzzy (pseudo-)metric on X is a GV-fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that for all $x, y \in X$, $t > 0$:*

(KM5) $M(x, y, t) = M(y, x, t)$.

Remark 2.12. It is clear that every GV-fuzzy metric is a T_1 GV-fuzzy quasi-metric; every T_1 GV-fuzzy quasi-metric is a GV-fuzzy quasi-metric, and every GV-fuzzy quasi-metric is a GV-fuzzy quasi-pseudo-metric.

Definition 2.13. *A GV-fuzzy quasi-(pseudo-)metric space is a triple $(X, M, *)$ such that X is a (nonempty) set and $(M, *)$ is a GV-fuzzy quasi-(pseudo-)metric on X.*

The notions of a T_1 *GV-fuzzy quasi-metric space* and of a *GV-fuzzy metric space* are defined in the obvious manner. Note that the GV-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of George and Veeramani.

Remark 2.14. Note that if $(M, *)$ is a GV-fuzzy quasi-(pseudo-)metric on X, then the fuzzy sets in $X \times X \times (0, +\infty)$, M^{-1} and M^i given by $M^{-1}(x, y, t) = M(y, x, t)$ and $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, are, as in the KM-case, a GV-fuzzy quasi-(pseudo-)metric and a GV-fuzzy (pseudo-)metric on X, respectively.

Thus, condition (GV2') above is equivalent to the following:

$M(x, x, t) = 1$ for all $x \in X$ and $t > 0$, and $M^i(x, y, t) < 1$ for all $x \neq y$ and $t > 0$.

Remark 2.15. Obviously, each GV-fuzzy quasi-(pseudo-)metric $(M, *)$ can be considered as a KM-fuzzy quasi-(pseudo-)metric by defining $M(x, y, 0) = 0$ for all $x, y \in X$. Therefore, each GV-fuzzy quasi-pseudo-metric space generates a topology τ_M defined as in the KM-case, and Propositions 2.6, 2.7 and 2.8 above remain valid for GV-fuzzy quasi-pseudo-metric spaces.

Example 2.16 (compare Example 2.9 of [3]). áLet (X, d) be a quasi-metric space. Denote by $a \cdot b$ the usual multiplication for every $a, b \in [0, 1]$, and let M_d be the function defined on $X \times X \times (0, +\infty)$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, \cdot)$ is a GV-fuzzy quasi-metric space called *standard fuzzy quasi-metric space* and $(M_d, \cdot)$ is the fuzzy quasi-metric *induced* by d. Furthermore, it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$ and $(M_d)^i = M_{d^s}$. Finally, from Proposition 2.8 and Remark 2.15, it follows that the topology τ_d, generated by d, coincides with the topology τ_{M_d} generated by the induced fuzzy quasi-metric $(M_d, \cdot)$.

Definition 2.17. *We say that a topological space (X, τ) admits a compatible KM (resp. GV)-fuzzy quasi-metric if there is a KM (resp. GV)-fuzzy quasi-metric $(M, *)$ on X such that $\tau = \tau_M$.*

It follows from Example 2.16 that every quasi-metrizable topological space admits a compatible GV-fuzzy quasi-metric. In Section 3 we shall establish that, conversely, the topology generated by a KM-fuzzy quasi-metric space is quasi-metrizable.

3. Quasi-metrizability of the topology of a fuzzy quasi-metric space

A slight modification of the proof of Theorem 1 of [6], permits us to show the following result.

Lemma 3.1. *Let $(X, M, *)$ be a KM-fuzzy quasi-metric space. Then $\{U_n : n{=}2, 3, ...\}$ is a base for a quasi-uniformity $\mathcal{U}_M$ on X compatible with τ_M, where $U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}$, for $n = 2, 3, ...$*

Moreover the conjugate quasi-uniformity $(\mathcal{U}_M)^{-1}$ coincides with $\mathcal{U}_{M^{-1}}$ and it is compatible with $\tau_{M^{-1}}$.

From Example 2.16, Lemma 3.1 and the well-known result that the topology generated by a quasi-uniformity with a countable base is quasi-pseudo-metrizable ([2]), we immediately deduce the following.

Theorem 3.2. *For a topological space (X, τ) the following are equivalent.*

(1) (X, τ) is quasi-metrizable.
(2) (X, τ) admits a compatible GV-fuzzy quasi-metric.
(3) (X, τ) admits a compatible KM-fuzzy quasi-metric.

Remark 3.3. It is almost obvious that the uniformity $\mathcal{U}_{M^i}$ coincides with the uniformity $(\mathcal{U}_M)^s := \mathcal{U}_M \vee (\mathcal{U}_M)^{-1}$

4. Bicomplete fuzzy quasi-metric spaces

There exist many different notions of quasi-uniform and quasi-metric completeness in the literature (see [10]). Then, by Lemma 3.1 and Remark 3.3, one can define in a natural way the corresponding notions of completeness in a fuzzy setting and easily deduce several properties taking into account the well-known completeness properties of quasi-uniform and quasi-metric spaces (compare with [6], where these ideas are used to study completeness in the fuzzy metric case).

In this section we only consider the notion of bicompleteness because it provides a satisfactory theory of quasi-uniform and quasi-metric completeness.

Let us recall that a quasi-metric space (X, d) is bicomplete provided that (X, d^s) is a complete metric space. In this case we say that d is a bicomplete quasi-metric on X.

A metrizable topological space (X, τ) is said to be completely metrizable if it admits a compatible complete metric. On the other hand, a fuzzy metric space $(X, M, *)$ is called complete ([5]) if every Cauchy sequence is convergent, where a sequence $(x_n)_n$ is Cauchy provided that for each $r \in (0,1)$ and each $t > 0$, there exists an n_0 such that $M(x_n, x_m, t) > 1 - r$ for every $n, m \geq n_0$. If $(X, M, *)$ is a complete fuzzy metric space, we say that $(M, *)$ is a complete fuzzy metric on X.

It was proved in [6] that a topological space is completely metrizable if and only if it admits a compatible complete fuzzy metric.

Definition 4.1. *A KM (resp. GV)-fuzzy quasi-metric space $(X, M, *)$ is called bicomplete if $(X, M^i, *)$ is a complete fuzzy metric space. In this case, we say that $(M, *)$ is a bicomplete KM (resp. GV)-fuzzy quasi-metric on X.*

Proposition 4.2.

(a) Let $(X, M, *)$ be a bicomplete KM-fuzzy quasi-metric space. Then (X, τ_M) admits a compatible bicomplete quasi-metric.

(b) Let (X, d) be a bicomplete quasi-metric space. Then $(X, M_d, \cdot)$ is a bicomplete GV-fuzzy quasi-metric space.

Proof. (a) Let d be a quasi-metric on X inducing the quasi-uniformity $\mathcal{U}_M$. Then d is compatible with τ_M. Now let $(x_n)_n$ be a Cauchy sequence in (X, d^s). Clearly $(x_n)_n$ is a Cauchy sequence in the fuzzy metric space $(X, M^i, *)$. So it converges to a point $y \in X$ with respect to τ_{M^i}. Hence $(x_n)_n$ converges to y with respect to τ_{d^s}. Consequently d is bicomplete.

(b) This part is almost obvious because $(M_d)^i = M_{d^s}$ (see Example 2.16), and thus each Cauchy sequence in $(X, (M_d)^i, \cdot)$ is clearly a Cauchy sequence in (X, d^s). □

Extending the classical metric theorem, it was independently proved in [1] and [12], that every quasi-metric space admits a (quasi-metric) bicompletion which is unique up to isometry. Although the problem of completion of fuzzy metric spaces in the sense of Kramosil and Michalek has a satisfactory solution

([14]), the corresponding situation for fuzzy metric spaces in the sense of George and Veeramani is quite different. In fact, it was obtained in [7] an example of a fuzzy metric space $(X, M, *)$ that does not admit completion, i.e. there no exist any complete fuzzy metric space having a dense subspace isometric to $(X, M, *)$. A characterization of those fuzzy metric spaces (in the sense of George and Veeramani) that admit a fuzzy metric completion has recently been obtained in [8].

Although the problem of bicompletion for GV-fuzzy quasi-metric spaces will be discussed elsewhere, we next present some concepts and facts that are basic in solving this problem.

Definition 4.3. *Let $(X, M, *)$ and $(Y, N, \star)$ be two KM (resp. GV)-fuzzy quasi-metric spaces. Then*

(a) *A mapping f from X to Y is called an isometry if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$.*

(b) *$(X, M, *)$ and $(Y, N, \star)$ are called isometric if there is an isometry from X onto Y.*

Definition 4.4. *Let $(X, M, *)$ be a KM (resp. GV)-fuzzy quasi-metric space. A KM (resp. GV)-fuzzy quasi-metric bicompletion of $(X, M, *)$ is a bicomplete KM (resp. GV)-fuzzy quasi-metric space $(Y, N, \star)$ such that $(X, M, *)$ is isometric to a τ_{N^i}-dense subspace of Y.*

Proposition 4.5. *áLet $(X, M, *)$ be a KM-fuzzy quasi-metric space and $(Y, N, \star)$ a bicomplete KM-fuzzy quasi-metric space. If there is a τ_{M^i}-dense subset A of X and an isometry $f : (A, M, *) \to (Y, N, \star)$, then there exists a unique isometry $F : (X, M, *) \to (Y, N, \star)$ such that $F|_A = f$.*

Proof. áIt is clear that f is a quasi-uniformly continuous mapping from the quasi-uniform space $(A, \mathcal{U}_M|_{A \times A})$ to the quasi-uniform space $(Y, \mathcal{U}_N)$. By Theorem 3.29 of [2], f has a unique quasi-uniformly continuous extension $F : (X, \mathcal{U}_M) \to (Y, \mathcal{U}_N)$. We shall show that actually F is an isometry from $(X, M, *)$ to $(Y, N, \star)$. Indeed, let $x, y \in X$ and $t > 0$. Then, there exist two sequences $(x_n)_n$ and $(y_n)_n$ in A such that $x_n \to x$ and $y_n \to y$ with respect to τ_{M^i}. Thus $F(x_n) \to F(x)$ and $F(y_n) \to F(y)$ with respect to τ_{N^i}. Choose $\varepsilon \in (0, 1)$ with $\varepsilon < t$. Therefore, there is n_ε such that for $n \geq n_\varepsilon$,

$$\begin{aligned} M(x, x_n, \varepsilon/2) &> 1 - \varepsilon, \quad M(y_n, y, \varepsilon/2) > 1 - \varepsilon, \\ N(F(x_n), F(x), \varepsilon/2) &> 1 - \varepsilon, \quad N(F(y), F(y_n), \varepsilon/2) > 1 - \varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} M(x, y, t) &\geq M(x, x_n, \varepsilon/2) * M(x_n, y_n, t - \varepsilon) * M(y_n, y, \varepsilon/2) \\ &\geq (1 - \varepsilon) * N(F(x_n), F(y_n), t - \varepsilon) * (1 - \varepsilon) \\ &\geq (1 - \varepsilon) * [(1 - \varepsilon) \star N(F(x), F(y), t - 2\varepsilon) \star (1 - \varepsilon)] * (1 - \varepsilon). \end{aligned}$$

By continuity of $*$ and $\star$ and by left continuity of $N(F(x), F(y), \_)$ it follows that $M(x, y, t) \geq N(F(x), F(y), t)$. Similarly we show that $N(F(x), F(y), t) \geq M(x, y, t)$. Consequently F is an isometry from $(X, M, *)$ to $(Y, N, \star)$. □

Corollary 4.6. *Let* $(X, M, *)$ *be a GV-fuzzy quasi-metric space and* $(Y, N, \star)$ *a bicomplete GV-fuzzy quasi-metric space. If there is a* τ_{M^i}*-dense subset* A *of* X *and an isometry* $f : (A, M, *) \to (Y, N, \star)$*, then there exists a unique isometry* $F : (X, M, *) \to (Y, N, \star)$ *such that* $F\,|_A = f$.

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