Fuzzy quasi-metric spaces

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ABSTRACT. We generalize the notions of fuzzy metric by Kramosil and Michalek, and by George and Veeramani to the quasi-metric setting. We show that every quasi-metric induces a fuzzy quasi-metric and, conversely, every fuzzy quasi-metric space generates a quasi-metrizable topology. Other basic properties are discussed.


Keywords: Fuzzy quasi-metric space; Quasi-metric; Quasi-uniformity; Bicomplete; Isometry.

1. Introduction

In [9], Kramosil and Michalek introduced and studied an interesting notion of fuzzy metric space which is closely related to a class of probabilistic metric spaces, the so-called (generalized) Menger spaces. Later on, George and Veeramani started, in [3] (see also [5]), the study of a stronger form of metric fuzziness. In particular, it is well known that every metric induces a fuzzy metric in the sense of George and Veeramani, and, conversely, every fuzzy metric space in the sense of George and Veeramani (and also of Kramosil and Michalek) generates a metrizable topology ([4], [6], [9], [11], [13]).

On the other hand, it is also well known that quasi-metric spaces constitute an efficient tool to discuss and solve several problems in topological algebra, approximation theory, theoretical computer science, etc. (see [10]).

In this paper, we introduce two notions of fuzzy quasi-metric space that generalize the corresponding notions of fuzzy metric space by Kramosil and Michalek, and by George and Veeramani to the quasi-metric context. Several basic properties of these spaces are obtained. We show that every quasi-metric induces a fuzzy quasi-metric and, conversely, every fuzzy quasi-metric generates a quasi-metrizable topology. With the help of these results one can easily derive many properties of fuzzy quasi-metric spaces.

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Our basic references for quasi-uniform and quasi-metric spaces are [2] and [10].

Let us recall that a quasi-pseudo-metric on a set $X$ is a nonnegative real valued function $d$ on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

Following the modern terminology (see Section 11 of [10]), by a quasi-metric on $X$ we mean a quasi-pseudo-metric $d$ on $X$ that satisfies the following condition: $d(x, y) = d(y, x) = 0$ if and only if $x = y$. If the quasi-pseudo-metric $d$ satisfies: $d(x, y) = 0$ if and only if $x = y$, then we say that $d$ is a $T_1$ quasi-metric on $X$.

A quasi-(pseudo-)metric space is a pair $(X, d)$ such that $X$ is a (nonempty) set and $d$ is a quasi-(pseudo-)metric on $X$. The notion of a $T_1$ quasi-metric space is defined in the obvious manner.

Each quasi-pseudo-metric $d$ on $X$ generates a topology $\tau_d$ on $X$ which has as a base the family of open $d$-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

Observe that if $d$ is a quasi-metric, then $\tau_d$ is a $T_0$ topology, and if $d$ is a $T_1$ quasi-metric, then $\tau_d$ is a $T_1$ topology.

A topological space $(X, \tau)$ is said to be quasi-metrizable if there is a quasi-metric $d$ on $X$ such that $\tau = \tau_d$. In this case, we say that $d$ is compatible with $\tau$, and that $\tau$ is a quasi-metrizable topology.

Given a quasi-(pseudo-)metric $d$ on $X$, then the function $d^{-1}$ defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-(pseudo-)metric on $X$, called the conjugate of $d$. Finally, the function $d^t$ defined on $X \times X$ by $d^t(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo-)metric on $X$.

2. Definitions and basic results

According to [13], a binary operation $\ast : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t$-norm if $\ast$ satisfies the following conditions: (i) $\ast$ is associative and commutative; (ii) $\ast$ is continuous; (iii) $a \ast 1 = a$ for every $a \in [0, 1]$; (iv) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

**Definition 2.1.** A KM-fuzzy quasi-pseudo-metric on a set $X$ is a pair $(M, \ast)$ such that $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set in $X \times X \times [0, +\infty)$ such that for all $x, y, z \in X$:

- (KM1) $M(x, y, 0) = 0$;
- (KM2) $M(x, x, t) = 1$ for all $t > 0$;
- (KM3) $M(x, z, t + s) \geq M(x, y, t) \ast M(y, z, s)$ for all $t, s \geq 0$;
- (KM4) $M(x, y, \_)_t : [0, +\infty) \rightarrow [0, 1]$ is left continuous.

**Definition 2.2.** A KM-fuzzy quasi-metric on $X$ is a KM-fuzzy quasi-pseudo-metric $(M, \ast)$ on $X$ that satisfies the following condition:

- (KM2') $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$.

If $(M, \ast)$ is a KM-fuzzy quasi-pseudo-metric on $X$ satisfying:

- (KM2'') $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$,
we say that \((M, ∗)\) is a \(T_1\) KM-fuzzy quasi-metric on \(X\).

**Definition 2.3.** A KM-fuzzy (pseudo-)metric on \(X\) is a KM-fuzzy quasi-(pseudo-)metric \((M, ∗)\) on \(X\) such that for each \(x, y \in X\):

\[(KM5)\] \(M(x, y, t) = M(y, x, t)\) for all \(t > 0\).

**Remark 2.4.** It is clear that every KM-fuzzy metric is a \(T_1\) KM-fuzzy quasi-metric; every \(T_1\) KM-fuzzy quasi-metric is a KM-fuzzy quasi-metric, and every KM-fuzzy quasi-metric is a KM-fuzzy quasi-pseudo-metric.

**Definition 2.5.** A KM-fuzzy quasi-(pseudo-)metric space is a triple \((X, M, ∗)\) such that \(X\) is a (nonempty) set and \((M, ∗)\) is a KM-fuzzy quasi-(pseudo-)metric on \(X\).

The notions of a \(T_1\) KM-fuzzy quasi-metric space and of a KM-fuzzy (pseudo-)metric space are defined in the obvious manner. Note that the KM-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of Kramosil and Michalek.

If \((M, ∗)\) is a KM-fuzzy quasi-(pseudo-)metric on a set \(X\), it is immediate to show that \((M^{-1}, ∗)\) is also a KM-fuzzy quasi-(pseudo-)metric on \(X\), where \(M^{-1}\) is the fuzzy set in \(X \times X \times [0, +∞)\) defined by \(M^{-1}(x, y, t) = M(y, x, t)\). Moreover, if we denote by \(M'\) the fuzzy set in \(X \times X \times [0, +∞)\) given by \(M'(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}\), then \((M', ∗)\) is, clearly, a KM-fuzzy (pseudo-)metric on \(X\).

**Proposition 2.6.** Let \((X, M, ∗)\) be a KM-fuzzy quasi-pseudo-metric space. Then, for each \(x, y \in X\) the function \(M(x, y, \cdot)\) is nondecreasing.

**Proof.** Let \(x, y \in X\) and \(0 \leq t < s\). Then \(M(x, y, s) \geq M(x, x, s - t) \ast M(x, y, t) = M(x, y, t)\). \(\square\)

Given a KM-fuzzy quasi-pseudo-metric space \((X, M, ∗)\) we define the open ball \(B_M(x, r, t)\), for \(x \in X\), \(0 < r < 1\), and \(t > 0\), as the set \(B_M(x, r, t) := \{y \in X : M(x, y, t) > 1 - r\}\). Obviously, \(x \in B_M(x, r, t)\).

By Proposition 2.6, it immediately follows that for each \(x \in X\), \(0 < r_1 \leq r_2 < 1\) and \(0 < t_1 \leq t_2\), we have \(B_M(x, r_1, t_1) \subseteq B_M(x, r_2, t_2)\). Consequently, we may define a topology \(τ_M\) on \(X\) as

\[τ_M := \{A \subseteq X : \text{for each } x \in A \text{ there are } r \in (0, 1), t > 0, \text{ with } B_M(x, r, t) \subseteq A\}.

Moreover, for each \(x \in X\) the collection of open balls \(\{B_M(x, 1/n, 1/n) : n = 2, 3, \ldots\}\) is a local base at \(x\) with respect to \(τ_M\). It is clear, that if \((X, M, ∗)\) is a KM-fuzzy quasi-metric (respectively, a \(T_1\) KM-fuzzy quasi-metric, a KM-fuzzy metric), then \(τ_M\) is a \(T_0\) (respectively, a \(T_1\), a Hausdorff) topology.

The topology \(τ_M\) is called the topology generated by the KM-fuzzy quasi-pseudo-metric space \((X, M, ∗)\).

Similarly to the proof of Result 3.2 and Theorem 3.11 of [3], one can show the following results.
Proposition 2.7. Let \((X, M, \ast)\) be a KM-fuzzy quasi-pseudo-metric space. Then, each open ball \(B_M(x, r, t)\) is an open set for the topology \(\tau_M\).

Proposition 2.8. A sequence \((x_n)\) in a KM-fuzzy quasi-pseudo-metric space \((X, M, \ast)\) converges to a point \(x \in X\) with respect to \(\tau_M\) if and only if \(\lim_n M(x, x_n, t) = 1\) for all \(t > 0\).

Definition 2.9. A GV-fuzzy quasi-pseudo-metric on a set \(X\) is a pair \((M, \ast)\) such that \(\ast\) is a continuous t-norm and \(M\) is a fuzzy set in \(X \times X \times (0, +\infty)\) such that for all \(x, y, z \in X, t, s > 0:\)

(GV1) \(M(x, y, t) > 0\);
(GV2) \(M(x, x, t) = 1\);
(GV3) \(M(x, z, t + s) \geq M(x, y, t) \ast M(y, z, s)\);
(GV4) \(M(x, y, \_ : (0, +\infty) \rightarrow (0, 1]\) is continuous.

Definition 2.10. A GV-fuzzy quasi-metric on \(X\) is a GV-fuzzy quasi-pseudo-metric \((M, \ast)\) on \(X\) such that for all \(t > 0:\)

(GV2') \(x = y\) if and only if \(M(x, y, t) = M(y, x, t) = 1\).

If \((M, \ast)\) is a GV-fuzzy quasi-pseudo-metric on \(X\) such that for all \(t > 0:\)

(GV2") \(x = y\) if and only if \(M(x, y, t) = 1\),
we say that \((M, \ast)\) is a \(T_1\) KM-fuzzy quasi-metric on \(X\).

Definition 2.11. A GV-fuzzy (pseudo-)metric on \(X\) is a GV-fuzzy quasi-(pseudo-)metric \((M, \ast)\) on \(X\) such that for all \(x, y \in X, t > 0:\)

(KM5) \(M(x, y, t) = M(y, x, t)\).

Remark 2.12. It is clear that every GV-fuzzy metric is a \(T_1\) GV-fuzzy quasi-metric; every \(T_1\) GV-fuzzy quasi-metric is a GV-fuzzy quasi-metric, and every GV-fuzzy quasi-metric is a GV-fuzzy quasi-pseudo-metric.

Definition 2.13. A GV-fuzzy quasi-(pseudo-)metric space is a triple \((X, M, \ast)\) such that \(X\) is a (nonempty) set and \((M, \ast)\) is a GV-fuzzy quasi-(pseudo-)metric on \(X\).

The notions of a \(T_1\) GV-fuzzy quasi-metric space and of a GV-fuzzy metric space are defined in the obvious manner. Note that the GV-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of George and Veeramani.

Remark 2.14. Note that if \((M, \ast)\) is a GV-fuzzy quasi-(pseudo-)metric on \(X\), then the fuzzy sets in \(X \times X \times (0, +\infty)\), \(M^{-1}\) and \(M'\) given by \(M^{-1}(x, y, t) = M(y, x, t)\) and \(M'(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}\), are, as in the KM-case, a GV-fuzzy quasi-(pseudo-)metric and a GV-fuzzy (pseudo-)metric on \(X\), respectively.

Thus, condition (GV2') above is equivalent to the following:

\(M(x, x, t) = 1\) for all \(x \in X\) and \(t > 0\), and \(M'(x, y, t) < 1\) for all \(x \neq y\) and \(t > 0\).

Remark 2.15. Obviously, each GV-fuzzy quasi-(pseudo-)metric \((M, \ast)\) can be considered as a KM-fuzzy quasi-(pseudo-)metric by defining \(M(x, y, 0) = 0\) for
all \(x, y \in X\). Therefore, each GV-fuzzy quasi-pseudo-metric space generates a topology \(\tau_M\) defined as in the KM-case, and Propositions 2.6, 2.7 and 2.8 above remain valid for GV-fuzzy quasi-pseudo-metric spaces.

**Example 2.16** (compare Example 2.9 of [3]). Let \((X, d)\) be a quasi-metric space. Denote by \(a \cdot b\) the usual multiplication for every \(a, b \in [0, 1]\), and let \(M_d\) be the function defined on \(X \times X \times (0, +\infty)\) by

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}.
\]

Then \((X, M_d, \cdot)\) is a GV-fuzzy quasi-metric space called standard fuzzy quasi-metric space and \((M_d, \cdot)\) is the fuzzy quasi-metric induced by \(d\). Furthermore, it is easy to check that \((M_d)^{-1} = M_{d^{-1}}\) and \((M_d)^t = M_d^t\). Finally, from Proposition 2.8 and Remark 2.15, it follows that the topology \(\tau_{d, \cdot}\) generated by \(d\), coincides with the topology \(\tau_{M_d}\) generated by the induced fuzzy quasi-metric \((M_d, \cdot)\).

**Definition 2.17.** We say that a topological space \((X, \tau)\) admits a compatible KM (resp. GV)-fuzzy quasi-metric if there is a KM (resp. GV)-fuzzy quasi-metric \((M, \ast)\) on \(X\) such that \(\tau = \tau_M\).

It follows from Example 2.16 that every quasi-metrizable topological space admits a compatible GV-fuzzy quasi-metric. In Section 3 we shall establish that, conversely, the topology generated by a KM-fuzzy quasi-metric space is quasi-metrizable.

### 3. Quasi-metrizability of the topology of a fuzzy quasi-metric space

A slight modification of the proof of Theorem 1 of [6], permits us to show the following result.

**Lemma 3.1.** Let \((X, M, \ast)\) be a KM-fuzzy quasi-metric space. Then \(\{U_n : n=2, 3, \ldots\}\) is a base for a quasi-uniformity \(U_M\) on \(X\) compatible with \(\tau_M\), where

\[
U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}, \text{ for } n = 2, 3, \ldots
\]

Moreover the conjugate quasi-uniformity \((U_M)^{-1}\) coincides with \(U_{M^{-1}}\) and it is compatible with \(\tau_{M^{-1}}\).

From Example 2.16, Lemma 3.1 and the well-known result that the topology generated by a quasi-uniformity with a countable base is quasi-pseudo-metrizable ([2]), we immediately deduce the following.

**Theorem 3.2.** For a topological space \((X, \tau)\) the following are equivalent.

1. \((X, \tau)\) is quasi-metrizable.
2. \((X, \tau)\) admits a compatible GV-fuzzy quasi-metric.
3. \((X, \tau)\) admits a compatible KM-fuzzy quasi-metric.

**Remark 3.3.** It is almost obvious that the uniformity \(U_M\), coincides with the uniformity \((U_M)^{\ast} := U_M \lor (U_M)^{-1}\)
There exist many different notions of quasi-uniform and quasi-metric completeness in the literature (see [10]). Then, by Lemma 3.1 and Remark 3.3, one can define in a natural way the corresponding notions of completeness in a fuzzy setting and easily deduce several properties taking into account the well-known completeness properties of quasi-uniform and quasi-metric spaces (compare with [6], where these ideas are used to study completeness in the fuzzy metric case).

In this section we only consider the notion of bicompleteness because it provides a satisfactory theory of quasi-uniform and quasi-metric completeness.

Let us recall that a quasi-metric space \((X, d)\) is bicomplete provided that \((X, d_s)\) is a complete metric space. In this case we say that \(d\) is a bicomplete quasi-metric on \(X\).

A metrizable topological space \((X, \tau)\) is said to be completely metrizable if it admits a compatible complete metric. On the other hand, a fuzzy metric space \((X, M, \ast)\) is called complete ([5]) if every Cauchy sequence is convergent, where a sequence \((x_n)_n\) is Cauchy provided that for each \(r \in (0, 1)\) and each \(t > 0\), there exists an \(n_0\) such that \(M(x_n, x_m, t) > 1 - r\) for every \(n, m \geq n_0\). If \((X, M, \ast)\) is a complete fuzzy metric space, we say that \((M, \ast)\) is a complete fuzzy metric on \(X\).

It was proved in [6] that a topological space is completely metrizable if and only if it admits a compatible complete fuzzy metric.

**Definition 4.1.** A KM (resp. GV)-fuzzy quasi-metric space \((X, M, \ast)\) is called bicomplete if \((X, M_i, \ast)\) is a complete fuzzy metric space. In this case, we say that \((M, \ast)\) is a bicomplete KM (resp. GV)-fuzzy quasi-metric on \(X\).

**Proposition 4.2.**

(a) Let \((X, M, \ast)\) be a bicomplete KM-fuzzy quasi-metric space. Then \((X, \tau_M)\) admits a compatible bicomplete quasi-metric.

(b) Let \((X, d)\) be a bicomplete quasi-metric space. Then \((X, M_d, \cdot)\) is a bicomplete GV-fuzzy quasi-metric space.

**Proof.**

(a) Let \(d\) be a quasi-metric on \(X\) inducing the quasi-uniformity \(U_M\). Then \(d\) is compatible with \(\tau_M\). Now let \((x_n)_n\) be a Cauchy sequence in \((X, d)\). Clearly \((x_n)_n\) is a Cauchy sequence in the fuzzy metric space \((X, M^i, \ast)\). So it converges to a point \(y \in X\) with respect to \(\tau_M\). Hence \((x_n)_n\) converges to \(y\) with respect to \(\tau_d\). Consequently \(d\) is bicomplete.

(b) This part is almost obvious because \((M_d)^i = M_d^i\) (see Example 2.16), and thus each Cauchy sequence in \((X, (M_d)^i, \cdot)\) is clearly a Cauchy sequence in \((X, d^i)\).

Extending the classical metric theorem, it was independently proved in [1] and [12], that every quasi-metric space admits a (quasi-metric) bicompletion which is unique up to isometry. Although the problem of completion of fuzzy metric spaces in the sense of Kramosil and Michalek has a satisfactory solution
Theorem 3.29 of [2], that corresponding situation for fuzzy metric spaces in the sense of George and Veeramani is quite different. In fact, it was obtained in [7] an example of a fuzzy metric space \((X, M, *)\) that does not admit completion, i.e. there no exist any complete fuzzy metric space having a dense subspace isometric to \((X, M, *)\). A characterization of those fuzzy metric spaces (in the sense of George and Veeramani) that admit a fuzzy metric completion has recently been obtained in [8].

Although the problem of bicompletion for GV-fuzzy quasi-metric spaces will be discussed elsewhere, we next present some concepts and facts that are basic in solving this problem.

**Definition 4.3.** Let \((X, M, *)\) and \((Y, N, *)\) be two KM (resp. GV)-fuzzy quasi-metric spaces. Then

(a) A mapping \(f\) from \(X\) to \(Y\) is called an isometry if for each \(x, y \in X\) and each \(t > 0\), \(M(x, y, t) = N(f(x), f(y), t)\).

(b) \((X, M, *)\) and \((Y, N, *)\) are called isometric if there is an isometry from \(X\) onto \(Y\).

**Definition 4.4.** Let \((X, M, *)\) be a KM (resp. GV)-fuzzy quasi-metric space. A KM (resp. GV)-fuzzy quasi-metric bicompletion of \((X, M, *)\) is a bicomplete KM (resp. GV)-fuzzy quasi-metric space \((Y, N, *)\) such that \((X, M, *)\) is isometric to a \(\tau_N\)-dense subspace of \(Y\).

**Proposition 4.5.** Let \((X, M, *)\) be a KM-fuzzy quasi-metric space and \((Y, N, *)\) a bicomplete KM-fuzzy quasi-metric space. If there is a \(\tau_{M'}\)-dense subset \(A\) of \(X\) and an isometry \(f : (A, M, *) \to (Y, N, *)\), then there exists a unique isometry \(F : (X, M, *) \to (Y, N, *)\) such that \(F|_A = f\).

**Proof.** It is clear that \(f\) is a quasi-uniformly continuous mapping from the quasi-uniform space \((A, \mathcal{U}_M|_{\tau_A \times A})\) to the quasi-uniform space \((Y, \mathcal{U}_N)\). By Theorem 3.29 of [2], \(f\) has a unique quasi-uniformly continuous extension \(F : (X, \mathcal{U}_M) \to (Y, \mathcal{U}_N)\). We shall show that actually \(F\) is an isometry from \((X, M, *)\) to \((Y, N, *)\). Indeed, let \(x, y \in X\) and \(t > 0\). Then, there exist two sequences \((x_n)_n\) and \((y_n)_n\) in \(A\) such that \(x_n \to x\) and \(y_n \to y\) with respect to \(\tau_{M'}\). Thus \(F(x_n) \to F(x)\) and \(F(y_n) \to F(y)\) with respect to \(\tau_N\). Choose \(\varepsilon \in (0, 1)\) with \(\varepsilon < t\). Therefore, there is \(n_\varepsilon\) such that for \(n \geq n_\varepsilon\),

\[
M(x_n, y_n, \varepsilon/2) > 1 - \varepsilon, \quad M(y_n, y, \varepsilon/2) > 1 - \varepsilon,
\]

\[
N(F(x_n), F(y_n), \varepsilon/2) > 1 - \varepsilon, \quad N(F(y_n), F(y), \varepsilon/2) > 1 - \varepsilon.
\]

Thus

\[
M(x, y, t) \geq M(x, x_n, \varepsilon/2) \cdot M(x_n, y_n, t - \varepsilon) \cdot M(y_n, y, \varepsilon/2) \\
\geq (1 - \varepsilon) \cdot N(F(x_n), F(y_n), t - \varepsilon) \cdot (1 - \varepsilon) \\
\geq (1 - \varepsilon) \cdot [(1 - \varepsilon) \cdot N(F(x), F(y), t - 2\varepsilon) \cdot (1 - \varepsilon)] \cdot (1 - \varepsilon).
\]

By continuity of \(*\) and \(\cdot\) and by left continuity of \(N(F(x), F(y), \cdot)\) it follows that \(M(x, y, t) \geq N(F(x), F(y), t)\). Similarly we show that \(N(F(x), F(y), t) \geq M(x, y, t)\). Consequently \(F\) is an isometry from \((X, M, *)\) to \((Y, N, *)\). \(\Box\)
Corollary 4.6. Let \((X, M, \ast)\) be a GV-fuzzy quasi-metric space and \((Y, N, \ast)\) a bicomplete GV-fuzzy quasi-metric space. If there is a \(\tau M\)-dense subset \(A\) of \(X\) and an isometry \(f : (A, M, \ast) \rightarrow (Y, N, \ast)\), then there exists a unique isometry \(F : (X, M, \ast) \rightarrow (Y, N, \ast)\) such that \(F\mid_A = f\).

References


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