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A fuzzification of the category of M-valued L-topological spaces

Tomasz Kubiak and Alexander P. Šostak

Abstract. A fuzzy category is a certain superstructure over an ordinary category in which "potential" objects and "potential" morphisms could be such to a certain degree. The aim of this paper is to introduce a fuzzy category $\mathcal{F}TOP(L,M)$ extending the category $TOP(L,M)$ of M-valued L- topological spaces which in its turn is an extension of the category $TOP(L)$ of L-fuzzy topological spaces in Kubiak-Šostak's sense . Basic properties of the fuzzy category $\mathcal{F}TOP(L,M)$ and its objects are studied.

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Introducion

The concept of an L-fuzzy topological space, that is of a pair $(X,\mathcal{T})$ where X is a set and $\mathcal{T} : L^X \to L$ is a mapping subjected to certain axioms was introduced (independently) by T.Kubiak [5] and A. Šostak [9] (Actually a prototype of this definition can be traced already in U.Höhle's work [1].) In some cases it seems reasonable to allow different lattices for domain and codomains of $\mathcal{T}$, resp. L and M, thus coming to the concept of an M-valued L-fuzzy topology on X (or an (L,M)-fuzzy topology on X for short), as a mapping $\mathcal{T} : L^X \to M$ subjected to certain axioms. A detailed study of (L,M)-fuzzy topological spaces will be presented in [6], [7].

In a series of papers the second named author considered the concept of a fuzzy category and the problem of fuzzifications of usual categories (see e.g. [11], [12], [13] etc.) Actually, a fuzzy category is an ordinary category modified in such a way, that "potential" objects and "potential" morphisms are such only to a certain degree, and this degree can be any element of the corresponding lattice. The concept of a fuzzy category lead us to the idea of "fuzzification" of some known categories - that is to construct fuzzy categories on the basis of some standard categories. In particular, in [13] we studied fuzzification of some categories related to topology and algebra.

It is the aim of this paper to "fuzzify" the category $TOP(L,M)$ of (L,M)-fuzzy topological spaces. As a tool for this fuzzification we use the structure of a GL-monoid on the codomain lattice (that is lattice M in our cotext), and in particular, the corresponding residuation in it.

The structure of the paper is as follows. After introducing the fuzzy category $\mathcal{F}TOP(L,M)$ and other basic definitions in Section 2 we discuss the lattice properties of the family of (L,M)-fuzzy topologies on a set X for a fixed level α (Section 3). Further, in Section 4, we proceed to the study of powerset operators in the context of (L,M)-fuzzy topologies, which, appear to be a convenient and powerfull tool for the investigation of such structures. In Section 5 we consider basic constructions in the fuzzy category $\mathcal{F}TOP(L,M)$ of (L,M)-fuzzy topological spaces — namely, products, subspaces, direct sums and quotients. Sections 6 and 7 deal with the inner structure of (L,M)-fuzzy topologies. Namely, in Section 6 we discuss relations between a structure which satisfies the axioms of an (L,M)-fuzzy topology at a level α and the corresponding fuzzy interior operator. Further, in Section 7 the relations between this fuzzy interior operator and the corresponding neighbourhood system are discussed.

1. Preliminaries

Let $L = (L_1, \leq_L, \wedge_L, \vee_L, *_L)$ and $M = (M, \leq_M, \wedge_M, \vee_M, *_M)$ be GL-monoids (cf e.g. [2], [3]). Let $\top_L, \top_M$ and $\bot_L, \bot_M$ denote the top and the bottom elements of L and M respectively. In what follows we shall usually omit the subscripts $_L$ and $_M$ since from the context it will be clear in what lattice the operation is applied.

It is well known that every GL-monoid L is residuated, i.e. there exists a further binary operation — implication " $\rightarrowtail$ " connected with $*$ by the Galois coonection:

$$\alpha * \beta \leq \gamma \iff \alpha \leq \beta \rightarrowtail \gamma \quad \forall \alpha, \beta, \gamma \in L.$$

Let X be a set and L^X be the family of all L-subsets of X, i.e. mappings $A : X \to L$. Then all operations on L in an obvious way can be pointwise extended to L^X thus generating the structure of a GL-monoid on L^X. In particular, implication $A \rightarrowtail B \in L^X$ for L-sets $A, B \in L^X$ is defined by $(A \rightarrowtail B)(x) := A(x) \rightarrowtail B(x)$; the top 1_X and the bottom elements 0_X in L^X are defined respectively as $1_X(x) = \top_L \ \forall x \in X$ and $0_X(x) = \bot_L \ \forall x \in X$.

To recall the concept of an *L-valued* or *$L-$fuzzy category* [11, 12], consider an ordinary (classical) category $\mathcal{C}$ and let $\omega : Ob(\mathcal{C}) \to L$ and $\mu : Mor(\mathcal{C}) \to L$ be $L-$fuzzy subclasses of the classes of its objects and morphisms respectively. Now, an *$L-$fuzzy category* can be defined as a triple $(\mathcal{C}, \omega, \mu)$ satisfying the following axioms ([12], cf also [11] in case $* = \wedge$):

1^0 $\mu(f) \le \omega(X) \wedge \omega(Y) \quad \forall X, Y \in Ob(\mathcal{C})$ and $\forall f \in Mor(X,Y)$;
2^0 $\mu(g \circ f) \ge \mu(f) * \mu(g)$ whenever the composition $g \circ f$ is defined;
3^0 $\mu(e_X) = \omega(X)$ where $e_X : X \to X$ is the identity morphism.

2. Basic definitions

Definition 2.1. [M-Fuzzy Category $\mathcal{F}TOP(L,M)$.]
Let $\mathcal{C}_{(L,M)}$ be an (ordinary) category whose objects are pairs $(X,\mathcal{T})$ where X is a set and $\mathcal{T} : L^X \to M$ is a mapping, and whose morphisms $f : (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ are arbitrary mappings $f : X \to Y$.
Given a set X and a mapping $\mathcal{T} : L^X \to M$ we define three fuzzy predicates:

$$\omega_1(\mathcal{T}) = \mathcal{T}(1_X) \quad (\text{ or, equivalently } \omega_1(\mathcal{T}) = \top \rightarrowtail \mathcal{T}(1_X));$$
$$\omega_2(\mathcal{T}) = \bigwedge_{\mathcal{U}\subset L^X, |\mathcal{U}|<\aleph_0}\Big(\bigwedge_{U\in\mathcal{U}}\mathcal{T}(U) \rightarrowtail \mathcal{T}\big(\bigwedge_{U\in\mathcal{U}}U\big)\Big);$$
$$\omega_3(\mathcal{T}) = \bigwedge_{\mathcal{U}\subset L^X}\Big(\bigwedge_{U\in\mathcal{U}}\mathcal{T}(U) \rightarrowtail \mathcal{T}\big(\bigvee_{U\in\mathcal{U}}U\big)\Big).$$

Let

$$\omega(\mathcal{T}) = \omega_1(\mathcal{T}) \wedge \omega_2(\mathcal{T}) \wedge \omega_3(\mathcal{T}).$$

Given $(X,\mathcal{T}_X)$, $(Y,\mathcal{T}_Y)$ and a mapping $f : X \to Y$ we set

$$\nu(f) = \bigwedge_{V\in L^Y}\Big(\mathcal{T}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V))\Big),$$

and

$$\mu(f) = \nu(f) \wedge \omega_X(\mathcal{T}_X) \wedge \omega_Y(\mathcal{T}_Y).$$

A mapping f will be called *continuous* if $\nu(f) = \top$. Actualy this means that $\mathcal{T}_Y(V) \le \mathcal{T}_X(f^{-1}(V))$ for all $V \in L^X$.

It is easy to note that $\mu(e_X) = \omega(X)$. Further, if $f : (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ and $g : (Y,\mathcal{T}_Y) \to (Z,\mathcal{T}_Z)$ are mappings, then

$$\nu(g \circ f) = \bigwedge_{W\in L^Z}\big(\mathcal{T}_Z(W) \rightarrowtail \mathcal{T}_X(f^{-1}(g^{-1}(W)))\Big) \ge$$
$$\ge \bigwedge_{W\in L^Z}\Big(\big(\mathcal{T}_Z(W) \rightarrowtail \mathcal{T}_Y(g^{-1}(W))\big) * \big(\mathcal{T}_Y(g^{-1}(W) \rightarrowtail \mathcal{T}_X(f^{-1}(g^{-1}(W)))\big)\Big)$$
$$\ge \bigwedge_{W\in L^Z}(\mathcal{T}_Z(W) \rightarrowtail \mathcal{T}_Y(g^{-1}(W)) * \bigwedge_{V\in L^Y}\big(\mathcal{T}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V))\big) =$$
$$= \nu(g) * \nu(f),$$

and hence also

$$\mu(g * f) \ge \mu(g) \circ \mu(f).$$

Thus we arrive at a (M-)fuzzy category $\mathcal{F}TOP(L,M) = (\mathcal{C}_{(L,M)}, \omega, \mu)$.
We interpret $\omega(\mathcal{T})$ as the degree to which a mapping $\mathcal{T}$ is an (L,M)-fuzzy topology on X. In case $\omega(\mathcal{T}) \ge \alpha$ we say that $\mathcal{T}$ is an (L,M)-fuzzy α-topology on X. An (L,M)-fuzzy $\top$-topology is just an (L,M)-fuzzy topology [6], [7] (and an L-fuzzy topology in case $M = L$, see e.g. [5], [9], [10]). On the other hand any mapping $\mathcal{T} : L^X \to M$ is an (L,M)-fuzzy $\bot$-topology on a set X.

A pair $(X,\mathcal{T})$ where $\mathcal{T}$ is an (L,M)-fuzzy $\perp$-topology will be referred to as an (L,M)-fuzzy $\perp$-topological space.

Remark 2.2. Applying ω_3 to $\mathcal{U}=\varnothing$ we get $\omega_3(\mathcal{T}) \leq \top \rightarrowtail \mathcal{T}(0_X) = \mathcal{T}(0_X)$.

Remark 2.3.

- $\omega_1(\mathcal{T}) = \top$ iff $\mathcal{T}(1_X) = \top$;
- $\omega_2(\mathcal{T}) = \top$ iff $\forall U_1, U_2 \in L^X$ it holds $\mathcal{T}(U_1 \wedge U_2) \geq \mathcal{T}(U_1) \wedge \mathcal{T}(U_2)$;
- $\omega_3(\mathcal{T}) = \top$ iff $\forall \mathcal{U} \subset L^X$ it holds $\mathcal{T}(\bigvee_{U\in\mathcal{U}} U) \geq \bigwedge_{U\in\mathcal{U}} \mathcal{T}(U)$.

Thus the fuzzy predicates $\omega_1, \omega_2, \omega_3$ are fuzzifications of the corresponding axioms of an (L,M)-fuzzy topology, cf [6], [7]. Fuzzy predicate ν can be viewed as a version of fuzzification of the axiom of continuity while μ "touch it up" in order to take into account the "defectiveness of topologiness" of $\mathcal{T}$.

Remark 2.4. [The case of an idempotent α.]
Let $\alpha \in L$ be idempotent, i.e. $\alpha * \alpha = \alpha$, and let $\mathcal{F}_\alpha TOP(L,M)$ denote the subcategory of $\mathcal{F}TOP(L,M)$ whose objects $(X,\mathcal{T})$ and morphisms f satisfy conditions $\omega(\mathcal{T}) \geq \alpha$ and $\mu(f) \geq \alpha$. Then $\mathcal{F}_\alpha TOP(L)$ is obviously a usual (crisp) category. In particular, $\mathcal{F}_\top TOP(L,M) = TOP(L,M)$.

Definition 2.5. *Given an object $(X,\mathcal{T})$ of $\mathcal{F}TOP(L,M)$, we define a mapping $\Sigma_{\mathcal{T}} := \Sigma : L^X \to M$ by setting $\Sigma(A) = \mathcal{T}(A \rightarrowtail 0_X)$ for every $A \in L^X$. The mapping Σ thus defined is called* the degree of closedness *in the space $(X,\mathcal{T})$.*

Proposition 2.6. [*Basic properties of* Σ]

(1) $\sigma_1(\Sigma) := \Sigma(0_X) = \mathcal{T}(1_X)$ *and hence* $\sigma_1(\Sigma) = \omega_1(\mathcal{T})$;

(2) $\sigma_2(\Sigma) := \bigwedge_{\mathcal{A}\subset L^X, |\mathcal{A}|<\aleph_0} \Big(\bigwedge_{A\in\mathcal{A}} \Sigma(A) \rightarrowtail \Sigma\big(\bigvee_{A\in\mathcal{A}} A\big)\Big) \geq \omega_2(\mathcal{T})$

(3) $\sigma_3(\Sigma) := \bigwedge_{\mathcal{A}\subset L^X} \Big(\bigwedge_{A\in\mathcal{A}} \Sigma(A) \rightarrowtail \Sigma\big(\bigwedge_{A\in\mathcal{A}} A\big)\Big) \geq \omega_3(\mathcal{T})$.

Proof.

$$\sigma_1(\Sigma) = \Sigma(0_X) = \mathcal{T}(0_X \rightarrowtail 0_X) = \mathcal{T}(1_X) = \omega_1(\mathcal{T});$$

$$\begin{aligned}
\sigma_2(\Sigma) :&= \bigwedge_{\substack{\mathcal{A}\subset L^X \\ |A|<\aleph_0}} \Big(\bigwedge_{A\in\mathcal{A}} \Sigma(A) \rightarrowtail \Sigma\big(\bigvee_{A\in\mathcal{A}} A\big)\Big) = \\
&= \bigwedge_{\substack{\mathcal{A}\subset L^X \\ |A|<\aleph_0}} \Big(\bigwedge_{A\in\mathcal{A}} \mathcal{T}(A \rightarrowtail 0_X) \rightarrowtail \mathcal{T}\big((\bigvee_{A\in\mathcal{A}} A) \rightarrowtail 0_X)\big)\Big) = \\
&= \bigwedge_{\substack{\mathcal{A}\subset L^X \\ |A|<\aleph_0}} \Big(\bigwedge_{A\in\mathcal{A}} \mathcal{T}(U_A) \rightarrowtail \mathcal{T}\big(\bigwedge_{A\in\mathcal{A}} U_A\big)\Big) \geq \\
&\geq \bigwedge_{\substack{\mathcal{U}\subset L^X \\ |\mathcal{U}|<\aleph_0}} \Big(\bigwedge_{U\in\mathcal{U}} \mathcal{T}(U) \rightarrowtail \mathcal{T}\big(\bigwedge_{U\in\mathcal{U}} U\big)\Big) = \omega_2(\mathcal{T}),
\end{aligned}$$

where $U_A := A \rightarrowtail 0_X$. In a similar way,

$$\begin{aligned}\sigma_3(\Sigma) &:= \bigwedge_{\mathcal{A}\subset L^X}\Big(\bigwedge_{A\in\mathcal{A}}\Sigma(A)\rightarrowtail\Sigma(\bigwedge_{A\in\mathcal{A}}A)\Big) = \\ &= \bigwedge_{\mathcal{A}\subset L^X}\Big(\bigwedge_{A\in\mathcal{A}}\mathcal{T}(A\rightarrowtail 0_X)\rightarrowtail\mathcal{T}(\bigwedge_{A\in\mathcal{A}}(A\rightarrowtail 0_X)\Big) = \\ &= \bigwedge_{\mathcal{A}\subset L^X}\Big(\bigwedge_{A\in\mathcal{A}}\mathcal{T}(U_A)\rightarrowtail\mathcal{T}(\bigvee_{A\in\mathcal{A}}U_A)\Big) \geq \\ &\geq \bigwedge_{\mathcal{U}\subset L^X}(\bigwedge_{U\in\mathcal{U}}\mathcal{T}(U)\rightarrowtail\mathcal{T}(\bigvee_{U\in\mathcal{U}}U)).\end{aligned}$$

□

Reasoning in a similar way it is easy to establish the following

Proposition 2.7. *Given a mapping $\Sigma : L^X \to M$ let M-valued predicates $\sigma_1(\Sigma)$, $\sigma_2(\Sigma)$ and $\sigma_3(\Sigma)$ be defined as in Proposition 2.6, and let $\mathcal{T} := \mathcal{T}_\Sigma$ be defined by $\mathcal{T}(A) = \Sigma(A \rightarrowtail 0_X)$. Then $\omega_1(\mathcal{T}) = \sigma_1(\Sigma)$, $\omega_2(\mathcal{T}) \geq \sigma_2(\Sigma)$, $\omega_3(\mathcal{T}) \geq \sigma_3(\Sigma)$.*

In case when L is an MV-algebra the L-powerset L^X also is an MV-algebra, and hence $(A \rightarrowtail 0_X) \rightarrowtail 0_X = A$ for every $A \in L^X$. Therefore it follows:

Proposition 2.8. *If L is an MV-algebra, then $\mathcal{T}_{\Sigma_\mathcal{T}} = \mathcal{T}$ and $\Sigma_{\mathcal{T}_\Sigma} = \Sigma$. In particular the structures $\mathcal{T}$ and Σ mutually define one another. Besides, $\sigma_1\big(\Sigma_\mathcal{T}\big) = \omega_1(\mathcal{T})$, $\sigma_2\big(\Sigma_\mathcal{T}\big) = \omega_2(\mathcal{T})$, $\sigma_3\big(\Sigma_\mathcal{T}\big) = \omega_3(\mathcal{T})$.*

3. Lattice properties of (L, M)-fuzzy α-topologies

Let $\alpha \in M$ be fixed and let $\mathfrak{T}_\alpha(X) := \mathfrak{T}_\alpha(L, M, X)$ be the family of all (L, M)-fuzzy α-topologies on a set X.

Theorem 3.1. *$\mathfrak{T}_\alpha(X)$ is a complete lattice.*

Proof. First, notice that $\mathcal{T}_{dis} : L^X \to M$ defined by $\mathcal{T}_{dis}(U) = \top$ for all $U \in L^X$ (the so called discrete (L, M)-fuzzy topology) is the top element of $\mathfrak{T}_\alpha(X)$ and $\mathcal{T}_{ind} : L^X \to M$ defined by $\mathcal{T}_{ind}(0_X) = \mathcal{T}_{ind}(1_X) = \alpha$ and $\mathcal{T}_{ind}(U) = \bot$ for $U \in L^X \setminus \{0_X, 1_X\}$ (the so called indiscrete (L, M)-fuzzy topology) is the bottom element of $\mathfrak{T}_\alpha(X)$. Further, let $\mathfrak{T}^0_\alpha(X) \subset \mathfrak{T}_\alpha(X)$ and let $\mathcal{T}_0 : L^X \to M$ be defined by the equality

$$\mathcal{T}_0(U) = \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)}\mathcal{T}(U) \quad \forall U \in L^X.$$

Then

$$\omega_1(\mathcal{T}_0) = \mathcal{T}_0(1_X) = \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)}\mathcal{T}(1_X) = \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)}\mathcal{T}(1_X) \geq \alpha;$$

$$\omega_2(\mathcal{T}_0) = \bigwedge_{\substack{\mathcal{U}\subset L^X\\ |\mathcal{U}|<\aleph_0}} \Big(\bigwedge_{U\in\mathcal{U}} \mathcal{T}_0(U) \rightarrowtail \mathcal{T}_0(\bigwedge_{U\in\mathcal{U}} U)\Big) =$$

$$= \bigwedge_{\substack{\mathcal{U}\subset L^X\\ |\mathcal{U}|<\aleph_0}} \Big(\bigwedge_{U\in\mathcal{U}} \Big(\bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)} \mathcal{T}\Big)(U) \rightarrowtail \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)} (\mathcal{T}(\bigwedge_{U\in\mathcal{U}} U))\Big) \geq$$

$$\geq \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)} \Big(\bigwedge_{\substack{\mathcal{U}\subset L^X\\ |\mathcal{U}|<\aleph_0}} \Big(\bigwedge_{U\in\mathcal{U}} \mathcal{T}(U) \rightarrowtail \mathcal{T}(\bigwedge_{U\in\mathcal{U}} U)\Big)\Big) = \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)} \omega_2(\mathcal{T}) \geq \alpha.$$

Reasoning in a similar way we get:

$$\omega_3(\mathcal{T}_0) = \bigwedge_{\mathcal{U}\subset L^X} \Big(\bigwedge_{U\in\mathcal{U}} \mathcal{T}^0(U) \rightarrowtail \mathcal{T}_0(\bigvee_{U\in\mathcal{U}} U)\Big) \geq$$

$$\geq \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)} \Big(\bigwedge_{\mathcal{U}\in L^X} \Big((\bigwedge_{U\in\mathcal{U}} \mathcal{T}(U)) \rightarrowtail \mathcal{T}(\bigvee_{U\in\mathcal{U}} U)\Big)\Big) = \bigwedge_{\mathcal{T}\in\mathfrak{T}^0_\alpha(X)} \omega_3(\mathcal{T}) \geq \alpha.$$

Thus $\mathcal{T}_0 \in \mathfrak{T}^0_\alpha(X)$ and hence $\mathcal{T}_0$ is indeed the minimal element of $\mathfrak{T}^0_\alpha(X)$ in $\mathfrak{T}_\alpha(X)$. □

The previous theorem allows also to write an explicite formula for the supremum of a subset $\mathfrak{T}^0_\alpha(X) \subset \mathfrak{T}_\alpha(X)$. Namely

$$\sup \mathfrak{T}^0_\alpha(X) = \bigwedge\{\mathcal{T} \in \mathfrak{T}_\alpha(X) \mid \mathcal{T} \geq \mathcal{T}_\lambda \ \forall \mathcal{T}_\lambda \in \mathfrak{T}^0_\alpha(X)\}.$$

Remark 3.2. Let $\mathcal{S} : L^X \to M$ be a mapping and let the mapping $\mathcal{T}_\mathcal{S} : L^X \to M$ be defined by

$$\mathcal{T}_\mathcal{S} = \bigwedge\{\mathcal{T} : \mathcal{T} \in \mathfrak{T}_\alpha(X) \text{ and } \mathcal{T} \geq \mathcal{S}\},$$

where as before $\mathfrak{T}_\alpha(X) := \mathfrak{T}_\alpha(L, M, X)$. From Theorem 3.1 it follows that $\mathcal{T}_\mathcal{S}$ is an (L, M)-fuzzy α-topology, besides it is the smallest one ($\leq$) of all (L, M)-fuzzy α-topologies which are greater or equal than $\mathcal{S}$. In this case $\mathcal{S}$ is called a *subbase* of the (L, M)-fuzzy α-topology $\mathcal{T}_\mathcal{S}$.

Proposition 3.3. [*Level decomposition of* (L, M)*-fuzzy -topologies*]
Let $\mathcal{T} : L^X \to M$ *be an* (L, M)*-fuzzy* α*-topology and assume that* $\gamma \in M$ *is such that* $\gamma * \alpha = \gamma$. *Further, let* $\mathcal{T}_\gamma = \{U \mid \mathcal{T}(U) \geq \gamma\}$. *Then* $\mathcal{T}_\gamma$ *is a (Chang-Goguen) L-topology on* X. *In particular, if* α *is idempotent, then* $\mathcal{T}_\alpha$ *is a (Chang-Goguen) L-topology on* X.

Proof. Since $\omega_1(\mathcal{T}) = \top$ it follows that $\mathcal{T}(1_X) \geq \alpha \geq \gamma$, and $1_X \in \mathcal{T}_\gamma$.
Let $U_1, \ldots, U_n \in \mathcal{T}_\gamma$. Then, since $\omega_2(\mathcal{T}) \geq \alpha$, it holds

$$\gamma \rightarrowtail \mathcal{T}(U_1 \wedge \ldots \wedge U_n) \geq \mathcal{T}(U_1) \wedge \ldots \wedge \mathcal{T}(U_n) \rightarrowtail \mathcal{T}(U_1 \wedge \ldots \wedge U_n) \geq \alpha$$

and hence $\mathcal{T}(U_1 \wedge \ldots \wedge U_n) \geq \alpha * \gamma = \gamma$.
In a similar way, taking into account that $\omega_3(\mathcal{T}) \geq \alpha$, it is easy to verify that if $U_i \in \mathcal{T}_\gamma$ for all $i \in \mathcal{I}$, then $\mathcal{T}\big(\bigvee_{i\in\mathcal{I}} U_i\big) \geq \alpha * \gamma = \gamma$.
□

Theorem 3.4. *Let $\mathcal{S} : L^X \to M$ be an (L, M)-fuzzy β-topology where $\alpha * \beta = \alpha$ Then the mapping $\mathcal{T} : L^X \to M$ defined by $\mathcal{T}(U) = \alpha \rightarrowtail \mathcal{S}(U)$ for every $U \in L^X$ is an (L, M)-fuzzy topology on X.*

Proof. Notice first that in this case $\alpha = \alpha * \beta \leq \beta$, and hence $\mathcal{S}(1_X) \geq \alpha$. Therefore $\omega_1(\mathcal{T}) = \mathcal{T}(1_X) = \alpha \rightarrowtail \mathcal{S}(1_X) = \alpha \rightarrowtail \alpha = \top$.

To verify axioms 2 and 3 for $\mathcal{T}$ notice first that for every $\gamma \in M$ it holds $\alpha \rightarrowtail \gamma * \beta = \alpha \rightarrowtail \gamma$. Indeed,

$$\alpha \rightarrowtail \gamma = \bigvee\{\lambda \mid \lambda * \alpha \leq \gamma\} \leq$$

$$\leq \bigvee\{\lambda \mid \lambda * \alpha * \beta \leq \gamma * \beta\} = \bigvee\{\lambda \mid \lambda * \alpha \leq \gamma * \beta\} = \alpha \rightarrowtail \gamma * \beta.$$

The converse inequality is obvious.

We proceed as follows. Since $\omega_2(\mathcal{S}) \geq \beta$, we get

$$\mathcal{S}(\bigwedge_{i=1}^{n} U_i) \geq \bigwedge_{i=1}^{n} \mathcal{S}(U_i) * \beta$$

and hence

$$\alpha \rightarrowtail \mathcal{S}\big(\bigwedge_{i=1}^{n} U_i\big) \geq \alpha \rightarrowtail \bigwedge_{i=1}^{n} \mathcal{S}(U_i) * \beta = \alpha \rightarrowtail \bigwedge_{i=1}^{n} \mathcal{S}(U_i) = \bigwedge_{i=1}^{n} \big(\alpha \rightarrowtail \mathcal{S}(U_i)\big);$$

thus $\mathcal{T}(\bigwedge_{i=1}^{n} U_i) \geq \bigwedge_{i=1}^{n} \mathcal{T}(U_i)$.

From $\omega_3(\mathcal{S}) \geq \beta$, reasoning in a similar way as above, we conclude that

$$\mathcal{S}\big(\bigvee_{i \in \mathcal{I}} U_i\big) \geq \bigwedge_{i \in \mathcal{I}} \mathcal{S}(U_i) * \beta,$$

and hence

$$\mathcal{T}(\bigvee_{i \in \mathcal{I}} U_i) \geq \bigwedge_{i \in \mathcal{I}} \mathcal{T}(U_i)$$

for any family $\{U_i \mid i \in \mathcal{I}\} \subset L^X$. □

Corollary 3.5. *If $\mathcal{S} : L^X \to M$ is an (L, M)-fuzzy α-topology and α is idempotent, then the mapping $\mathcal{T} : L^X \to M$ defined by $\mathcal{T}(U) := \alpha \rightarrowtail \mathcal{S}(U)$ for every $U \in L^X$ is an (L, M)-fuzzy topology. If $\mathcal{S} : L^X \to M$ is an (L, M)-fuzzy topology then for every α the mapping $\mathcal{T}(U) = \alpha \rightarrowtail \mathcal{S}(U)$ is an (L, M)-fuzzy topology.*

Theorem 3.6. *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a mapping, $\omega(\mathcal{T}_X) \geq \beta$, $\omega(\mathcal{T}_Y) \geq \alpha$ where $\beta * \alpha = \alpha$, and let $\mathcal{S}_Y : L^Y \to M$ be a subbase of $\mathcal{T}_Y$. Then the following conditions are equivalent:*

1^0 $\mathcal{T}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V)) \geq \alpha \quad \forall V \in L^Y$;
2^0 $\mathcal{S}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V)) \geq \alpha \quad \forall V \in L^Y$.

In particular, these conditions are equivalent in case when $\alpha \leq \beta$ and α is idempotent.

Proof. Since $\mathcal{T}_Y(V) \geq \mathcal{S}_Y(V)$, it holds

$$\mathcal{T}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V)) \leq \mathcal{S}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V))$$

and hence $1^0 \Longrightarrow 2^0$. Conversely, if $\mathcal{S}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V)) \geq \alpha$ for all $V \in L^Y$, then

$$\mathcal{S}_Y(V) \leq \alpha \rightarrowtail \mathcal{T}_X(f^{-1}(V)) \ \forall V \in L^X.$$

Let now $\mathcal{T}'(V) := \mathcal{T}_X(f^{-1}(V))$. It is easy to verify that $\mathcal{T}'$ is an (L,M)-fuzzy β-topology since $\mathcal{T}_X$ is an (L,M)-fuzzy β-topology. Further, let $\mathcal{T}'' : L^Y \to M$ be defined by $\mathcal{T}''(V) := \alpha \rightarrowtail \mathcal{T}'(V)$. Then by Theorem 3.4 $\mathcal{T}''$ is an (L,M)-fuzzy topology on Y. Moreover, $\mathcal{S}_Y(V) \leq \mathcal{T}''(V)$. Thus, since $\mathcal{T}_Y$ is an (L,M)-fuzzy α-topology generated by subbase $\mathcal{S}_Y$, it follows that $\mathcal{S}_Y(V) \leq \mathcal{T}_Y(V) \leq \mathcal{T}''(V)$, and hence

$$\mathcal{T}_Y(V) \leq \alpha \rightarrowtail \mathcal{T}_X(f^{-1}(V)) \Longrightarrow \alpha \leq \mathcal{T}_Y(V) \rightarrowtail \mathcal{T}_X(f^{-1}(V)) \quad \forall V \in L^Y.$$

□

Question 3.7. Do the statements of Corollary 3.5 and Theorem 3.6 hold also in case $\alpha = \beta$ but without assumption of idempotency of α ?

4. Power-set operators and (L,M)-fuzzy α-topologies

Let X, Y be sets, and let $F : L^Y \to L^X$ be a mapping preserving arbitrary joins and meets. In particular, $F(1_Y) = 1_X$ and $F(0_Y) = 0_X$.

Definition 4.1. (cf e.g. [8]) *The powerset operator $F^{\rightarrow} : M^{(L^Y)} \to M^{(L^X)}$ of a mapping $F : L^Y \to L^X$ is defined by the equality*

$$F^{\rightarrow}(\mathcal{T}_Y)(U) = \bigvee\{\mathcal{T}_Y(V) : F(V) = U\}, \forall U \in L^X$$

for every $\mathcal{T}_Y : L^Y \to M$,

Definition 4.2. (cf e.g [8]) *The powerset operator $F^{\leftarrow} : M^{(L^X)} \to M^{(L^Y)}$ of a mapping $F : L^Y \to L^X$ is defined by the equality*

$$F^{\leftarrow}(\mathcal{T}_X)(V) = \mathcal{T}_X\big(F(V)\big) \ \forall V \in L^Y$$

for every $\mathcal{T}_X : L^X \to M$,

The following two theorems show that the powerset operators $F^{\rightarrow}$ and $F^{\leftarrow}$ do not diminish the topologiness degree of the mappings $\mathcal{T}_Y$ and $\mathcal{T}_X$ respectively.

Theorem 4.3. *If M is completely distributive, then*

$$\omega(F^{\rightarrow}(\mathcal{T}_Y)) \geq \omega(\mathcal{T}_Y) := \alpha.$$

(Actually, $\omega_1(F^{\rightarrow}(\mathcal{T}_Y)) \geq \omega_1(\mathcal{T}_Y)$, $\omega_2(F^{\rightarrow}(\mathcal{T}_Y)) \geq \omega_2(\mathcal{T}_Y)$ and $\omega_3(F^{\rightarrow}(\mathcal{T}_Y)) \geq \omega_3(\mathcal{T}_Y)$.)

Proof. Since $F^{\rightarrow}(\mathcal{T}_Y)(1_X) = \bigvee\{\mathcal{T}_Y(V) \mid F(V) = 1_X\} \geq \mathcal{T}_Y(1_Y) \geq \alpha$, it follows that

$$\omega_1(F^{\rightarrow}(\mathcal{T}_Y)) \geq \alpha.$$

To verify that $\omega_2(F^{\rightarrow}(\mathcal{T}_Y)) \geq \alpha$ fix some $U_1, \ldots U_n \in L^X$ and let $U_0 := \bigwedge_{i=1}^n U_i$. We have to show that

$$\Big(\bigwedge_{i=1}^{n} F^{\rightarrow}(\mathcal{T}_Y)(U_i)\Big) \rightarrowtail F^{\rightarrow}(\mathcal{T}_Y)(U_0) \geq \alpha.$$

If for some $i \in \{1, \ldots, n\}$ there does not exist $V_i \in L^Y$ such that $F(V_i) = U_i$, then from the definition of $F^{\rightarrow}(\mathcal{T}_Y)$ it is clear that $F^{\rightarrow}(\mathcal{T}_Y)(U_i) = \bot$ and hence the inequality is obvious. Assume therefore that for each $i = 1, \ldots, n$ some $V_i \in L^Y$ is fixed such that $U_i = F(V_i)$. Then, since $\omega_2(\mathcal{T}_Y) \geq \alpha$, and since

$$F(\bigwedge_{i=1}^{n} V_i) = \bigwedge_{i=1}^{n} F(V_i) = \bigwedge U_i = U_0$$

it follows that

$$\bigwedge_{i=1}^{n} \mathcal{T}_Y(V_i) \rightarrowtail \mathcal{T}_Y(\bigwedge_{i=1}^{n} V_i) \geq \bigwedge_{i=1}^{n} \mathcal{T}_Y(V_i) \rightarrowtail \bigvee_{\substack{V_0 \in L^Y \\ F(V_0)=U_0}} \mathcal{T}_Y(V_0) \geq \alpha.$$

This holds for any choice of $V_i \in L^Y$, $i \in \{1, \ldots, n\}$, satisfying $F^{\rightarrow}(V_i) = U_i$, and therefore taking into account that L is infinitely distributive, we conclude that

$$\bigwedge_{i=1}^{n} \big(F^{\rightarrow}(\mathcal{T}_Y)(U_i)\big) \rightarrowtail F^{\rightarrow}(\mathcal{T}_Y)(U_0) = \bigwedge_{i=1}^{n} \Big(\bigvee_{\substack{F(V_i)=U_i \\ V_i \in L^Y}} \mathcal{T}_Y(V_i)\Big) \rightarrowtail \bigvee_{\substack{F(V_i)=U_i \\ V_0 \in L^Y}} \mathcal{T}_Y(V_0)\Big)$$

$$= \Big(\bigvee_{F(V_i)=U_i} \bigwedge_{i=1}^{n} \mathcal{T}_Y(V_i)\Big) \rightarrowtail \bigvee_{\substack{F(V_0)=U_0 \\ V_0 \in L^Y}} \mathcal{T}_Y(V_0) \geq \alpha.$$

To verify the third inequality, $\omega_3(F^{\rightarrow}(\mathcal{T}_Y)) \geq \alpha$, fix a family $\mathcal{U} = \{U_i \mid i \in \mathcal{I}\}$, and let $U_0 := \bigvee_{i\in\mathcal{I}} U_i$. We have to show that

$$\bigwedge_{i\in\mathcal{I}} F^{\rightarrow}(\mathcal{T}_Y)(U_i) \rightarrowtail F^{\rightarrow}(\mathcal{T}_Y)(\bigvee_{i\in\mathcal{I}} U_i) \geq \alpha.$$

Let for each $i \in \mathcal{I}$ an L-set $V_i \in L^Y$ be fixed such that $F(V_i) = U_i$. (As in the previous situation it is sufficient to assume that such choice of $V_i \in L^Y$ for all $i \in \mathcal{I}$ is possible.) Then

$$\bigwedge_{i\in\mathcal{I}} \mathcal{T}_Y(V_i) \rightarrowtail \mathcal{T}_Y(\bigvee_{i\in\mathcal{I}} V_i) \geq \alpha,$$

that is

$$\bigwedge_{i\in\mathcal{I}} \mathcal{T}_Y(V_i) \geq \mathcal{T}_Y(\bigvee_{i\in\mathcal{I}} V_i) * \alpha.$$

Applying complete distributivity we get the following chain of (in)equalities:

$$\alpha * F^{\rightarrow}(\mathcal{T}_Y)(\bigvee_{i\in\mathcal{I}} U_i) = \alpha * \bigvee\{\mathcal{T}_Y(\bigvee_{i\in\mathcal{I}} V_i) : F(\vee_i V_i) = \vee_i U_i\} =$$

$$= \bigvee\Big(\alpha*\{\mathcal{T}_Y(\bigvee_{i\in\mathcal{I}} V_i) : F(\vee_i V_i) = \vee_i U_i\}\Big) \geq \bigvee\{\bigwedge_{i\in\mathcal{I}} \mathcal{T}_Y(V_i) : F(\vee_i V_i) = \vee_i U_i\} \geq$$

$$\geq \bigvee\{\bigwedge_{i\in\mathcal{I}} \mathcal{T}_Y(V_i) : F(V_i) = U_i\} = \bigvee\{\bigwedge_{i\in\mathcal{I}} \mathcal{T}_Y(V_i) : V_i \in \mathcal{V}_i := \{V \mid U_i = F(V)\}\} =$$

$$= \bigvee_{\varphi\in\prod_i \mathcal{V}_i} (\bigwedge_{i\in\mathcal{I}} \mathcal{T}_Y(\varphi(i))) = \bigwedge_{i\in\mathcal{I}} \bigvee_{V_i\in\mathcal{V}_i} \mathcal{T}_Y(V_i) = \bigwedge_{i\in\mathcal{I}} \bigvee_{F(V_i)=U_i} \mathcal{T}_Y(V_i) = \bigwedge_{i\in\mathcal{I}} F^{\rightarrow}(\mathcal{T}_Y)(U_i).$$

and hence we obtain the required inequality:

$$F^{\rightarrow}(\mathcal{T}_Y)(\bigvee_{i\in\mathcal{I}} U_i) \rightarrowtail \bigwedge_{i\in\mathcal{I}} F^{\rightarrow}(\mathcal{T}_Y)(U_i) \geq \alpha.$$

□

Theorem 4.4.

$$\omega(F^{\leftarrow}(\mathcal{T}_X)) \geq \omega(\mathcal{T}_X) =: \alpha.$$

(*Actually,* $\omega_1(F^{\leftarrow}(\mathcal{T}_X)) = \omega_1(\mathcal{T}_X), \omega_2(F^{\leftarrow}(\mathcal{T}_X)) \geq \omega_2(\mathcal{T}_X)$ *and* $\omega_3(F^{\leftarrow}(\mathcal{T}_X)) \geq \omega_3(\mathcal{T}_X)$.)

Proof. $\omega_1(F^{\leftarrow}(\mathcal{T}_X)) = F^{\leftarrow}(\mathcal{T}_X)(1_Y) = \mathcal{T}_X(F(1_Y)) = \mathcal{T}_X(1_X) = \omega_1(\mathcal{T}_X) \geq \alpha$.
To verify condition $\omega_2(\mathcal{T}_Y) \geq \alpha$, where $\mathcal{T}_Y := F^{\leftarrow}(\mathcal{T}_X)$, fix $\{V_1,\ldots,V_n\} \subset L^Y$, then

$$\bigwedge_{i=1}^{n} \mathcal{T}_Y(V_i) \rightarrowtail \mathcal{T}_Y\big(\bigwedge_{i=1}^{n} V_i\big) = \bigwedge_{i=1}^{n} F^{\leftarrow}(\mathcal{T}_X)(V_i) \rightarrowtail F^{\leftarrow}(\mathcal{T}_X)\big(\bigwedge_{i=1}^{n} V_i\big)$$

$$= \bigwedge_{i=1}^{n} \mathcal{T}_X\big(F(V_i)\big) \rightarrowtail \mathcal{T}_X\big(\bigwedge_{i=1}^{n} F(V_i)\big) \geq \alpha.$$

Finally, to verify the condition $\omega_3(\mathcal{T}_Y) \geq \alpha$ fix a family $\mathcal{V} = \{V_i \mid i \in \mathcal{I}\} \subset L^Y$. Then

$$\bigwedge_{i\in\mathcal{I}} \mathcal{T}_Y(V_i) \rightarrowtail \mathcal{T}_Y\big(\bigvee_{i\in\mathcal{I}}(V_i)\big) = \bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(\mathcal{T}_X)(V_i) \rightarrowtail F^{\leftarrow}(\mathcal{T}_X)(\bigvee_{i\in\mathcal{I}} V_i) =$$

$$= \bigwedge_{i\in\mathcal{I}} \mathcal{T}_X(F(V_i)) \rightarrowtail \mathcal{T}_X(\bigvee_{i\in\mathcal{I}} F(V_i)) \geq \omega_3(\mathcal{T}_X) \geq \alpha.$$

□

Power-set operators $F^{\leftarrow}$ and $F^{\rightarrow}$ can be applied, in particular, for description of final and initial (L,M)-fuzzy α-topologies. Here are some details:
Let $f : X \to Y$ be a mapping, then by setting $f^{\leftarrow}(V) := f^{-1}(V)$ one defines a mapping $f^{\leftarrow} : L^Y \to L^X$, which obviously, preserves joins and meets, and so one can apply to it theorems 4.3 and 4.4. Namely, one can get the following corollaries from the statements of these theorems and from the definition of power-set operators.

Corollary 4.5. *Let $\mathcal{T}_Y : L^Y \to M$ be a mapping where M is completely distributive, and let $\omega(\mathcal{T}_Y) \geq \alpha$. Then given a mapping $f : X \to Y$, it holds $\omega\Big((f^{\leftarrow})^{\rightarrow}(\mathcal{T}_Y)\Big) \geq \alpha$. Besides, $(f^{\leftarrow})^{\rightarrow}(\mathcal{T}_Y)$ is the weakest (L,M)-fuzzy α-topology (actually, even the weakest (L,M)-fuzzy $\bot$-topology!) on X for which the mapping $f : (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ is continuous (i.e. $\nu(f) = \top$).*

Corollary 4.6. *Let $\mathcal{T}_X : L^X \to M$ be a mapping and $\omega(\mathcal{T}_X) \geq \alpha$. Then given a mapping $f : X \to Y$ it holds $\omega\Big((f^{\leftarrow})^{\leftarrow}(\mathcal{T}_X)\Big) \geq \alpha$. Besides, $(f^{\leftarrow})^{\leftarrow}(\mathcal{T}_X)$ is the strongest (L,M)-fuzzy α-topology (actually, even the strongest (L,M)-fuzzy $\bot$-topology!) on Y for which $f : (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ is continuous (i.e. $\nu(f) = \top$).*

5. Products, subspaces, direct sums and quotients

In this section we shall discuss how basic operations for (L,M)-fuzzy α-topological spaces can be defined.

5.1. **Products.** Let $\mathfrak{X} = \{(X_i,\mathcal{T}_i) : i \in \mathcal{I}\}$ be a family of (L,M)-fuzzy α-topological spaces, where M is completely distributive, let $X = \prod_{i\in\mathcal{I}} X_i$ be the product of the corresponding sets, and let $p_i : X \to X_i$ be the projections. Further, let $\hat{\mathcal{T}_i} := (p_i^{\leftarrow})^{\rightarrow}(\mathcal{T}_i) : L^X \to M$. Then, by Corollary 4.5, $\omega(\hat{\mathcal{T}_i}) \geq \alpha$. Let $\mathcal{S} := \bigvee_{i\in\mathcal{I}} \hat{\mathcal{T}_i}$ and let $\mathcal{T}_X : L^X \to M$ be the (L,M)-fuzzy α-topology generated by the subbase $\mathcal{S} : L^X \to M$. Then, obviously, $\mathcal{T}_X$ is the weakest (L,M)-fuzzy α-topology for which all projections are continuous (i.e. $\nu(p_i) = \top$). Moreover, the pair $(X,\mathcal{T}_X)$ is the product of the family $\mathfrak{X}$ in the fuzzy category $\mathcal{F}TOP(L,M)$ in the following sense:

Given an (L,M)-fuzzy β-topological space $(Z,\mathcal{T}_Z)$ where $\beta*\alpha = \alpha$, and a family of mappings $f_i : (Z,\mathcal{T}_Z) \to (X_i,\mathcal{T}_i)$, $i \in \mathcal{I}$, there exists a unique mapping $h : (Z,\mathcal{T}_Z) \to (X,\mathcal{T}_X)$ such that $p_i \circ h = f_i$ for all $i \in \mathcal{I}$ and

$$\nu(h) \geq \alpha \Longleftrightarrow \bigwedge_{i\in\mathcal{I}} \nu(f_i) \geq \alpha.$$

Indeed, let $h := \triangle_{i\in\mathcal{I}} f_i : Z \to X$ be the diagonal product of mappings f_i, $i \in \mathcal{I}$. If $\nu(h) \geq \alpha$, then for every $i \in \mathcal{I}$

$$\nu(f_i) = \nu(p_i \circ h) \geq \nu(p_i) * \nu(h) \geq \top * \nu(h) \geq \alpha.$$

Conversely, let $\nu(f_i) \geq \alpha$ for all $i \in \mathcal{I}$. We have to verify that in this case

$$\mathcal{T}_X(W) \rightarrowtail \mathcal{T}_Z(h^{-1}(W)) \geq \alpha \qquad \forall W \in L^X.$$

According to Theorem 3.6 it is sufficient to verify that

$$\mathcal{S}(W) \rightarrowtail \mathcal{T}_Z(h^{-1}(W)) \geq \alpha \qquad \forall W \in L^X.$$

However, from the definition of $\mathcal{S}$ it is clear that $\mathcal{S}(W) = \mathcal{T}_i(V_i)$ if $W := \tilde{V_i}$ where $\tilde{V_i} = p_i^{-1}(V_i)$ for some $V_i \in L^{X_i}$ and $\mathcal{S}(W) = \bot$ otherwise. Therefore it is sufficient to verify the above inequality for L-sets of the form $\tilde{V}_i$. However, in this case $h^{-1}(\tilde{V}_i) = h^{-1}(p_i^{-1}(V_i)) = f_i^{-1}(V_i)$, and hence the requested inequality can be rewritten as

$$\mathcal{T}_i(V_i) \rightarrowtail \mathcal{T}_Z(f_i^{-1}(V_i)) \geq \alpha$$

which holds according to our assumptions.

5.2. **Subspaces.** Let $(X, \mathcal{T})$ be an (L, M)-fuzzy α-topological space, let $X_0 \subset X$ and let $e : X_0 \to X$ be the embedding mapping. Further, let $\mathcal{T}_0 := (e^{\leftarrow})^{\rightarrow}(\mathcal{T})$. Then according to Corollary 4.5 $\omega(\mathcal{T}_0) \geq \omega(\mathcal{T})$ and hence $(X_0, \mathcal{T}_0)$ is an (L, M)-fuzzy α-topological space. From the construction it is clear that $\nu(e) = \top$. Moreover, it is easy to note that $(X_0, \mathcal{T}_0)$ is a subobject of $(X, \mathcal{T})$ in the following sense:

For every (L, M)-fuzzy $\bot$-topological space $(Z, \mathcal{T}_Z)$ and for every mapping $f : (Z, \mathcal{T}_Z) \to (X_0, \mathcal{T}_0)$ it holds

$$\nu(f) = \nu(e \circ f).$$

Indeed, let $V_0 = e^{-1}(V)$ for some $V \in L^X$. Then

$$\mathcal{T}_0(V_0) \rightarrowtail \mathcal{T}_Z(f^{-1}(V_0)) \geq \mathcal{T}(V) \rightarrowtail \mathcal{T}_Z(f^{-1}(e^{-1}(V))) =$$
$$= \mathcal{T}(V) \rightarrowtail \mathcal{T}_Z((e \circ f)^{-1}(V)) \geq \nu(e \circ f),$$

and hence $\nu(f) \geq \nu(e \circ f)$. The converse inequality is obvious.

5.3. **Coproducts (Direct sums).** Let $\mathfrak{X} = \{(X_i, \mathcal{T}_i) : i \in \mathcal{I}\}$ be a family of (L, M)-fuzzy α-topological spaces, let $X = \oplus_{i \in \mathcal{I}} X_i$ be the disjoint union of the corresponding sets, and let $e_i : X_i \to X$ be the inclusion mapping. Further, let $\mathcal{S}_i := (e_i^{\leftarrow})^{\leftarrow}(\mathcal{T}_i)$. Then by Corollary 4.6 $\omega(\mathcal{S}_i) \geq \alpha$, and hence, according to Theorem 3.1 $\mathcal{T} := \bigwedge_{i \in \mathcal{I}} \mathcal{S}_i$ is an (L, M)-fuzzy α-topology. Besides, it is clear that $\mathcal{T}$ is the strongest (L, M)-fuzzy α-topology for which all inclusions e_i are continuous, i.e. $\nu(e_i) = \top$. Moreover $(X, \mathcal{T})$ is the coproduct of the family $\mathfrak{X}$ in the fuzzy category $\mathcal{FTOP}(L, M)$ in the following sense:

Let $(Z, \mathcal{T}_Z)$ be an (L, M)-fuzzy $\bot$-topological space and let

$$f_i : (X_i, \mathcal{T}_i) \to (Z, \mathcal{T}_Z), \ i \in \mathcal{I},$$

be a family of mappings. Further, let the mapping

$$f : (X, \mathcal{T}) \to (Z, \mathcal{T}_Z)$$

be defined by $f(x) = f_i(x)$ iff $x \in X_i$. Then

$$\nu(f) = \bigwedge_{i \in \mathcal{I}} \nu(f_i).$$

Indeed, since $f_i = f \circ e_i$ and $\nu(e_i) = \top$ for all $i \in \mathcal{I}$, the inequality $\nu(f) \geq \bigwedge_{i \in \mathcal{I}} \nu(f_i)$ is obvious.

Conversely, assume that $\bigwedge_{i \in \mathcal{I}} \nu(f_i) \geq \alpha$. Then

$$\alpha \leq \mathcal{T}_Z(V) \rightarrowtail \mathcal{T}_i(f^{-1}(V)) = \mathcal{T}_Z(V) \rightarrowtail \mathcal{S}_i(e_i(f_i^{-1}(V)) = \mathcal{T}_Z(V) \rightarrowtail \mathcal{S}_i(f^{-1}(V)).$$

Now, taking infimum over all $i \in \mathcal{I}$, we obtain:

$$\mathcal{T}_Z(V) \rightarrowtail \mathcal{T}(f^{-1}(V)) \geq \alpha.$$

5.4. **Quotients.** Let $(X, \mathcal{T}_X)$ be an (L,M)-fuzzy α-topological space and let $f : X \to Y$ be a surjective mapping. Further, let $\mathcal{T}_Y = (f^{\leftarrow})^{\leftarrow}(\mathcal{T}_X)$. Then, according to Corollary 4.6 $\omega(\mathcal{T}_Y) \geq \alpha$ and hence $(Y, \mathcal{T}_Y)$ is an (L,M)-fuzzy α-topological space. It is clear that $\mathcal{T}_Y$ is the strongest (L,M)-fuzzy α-topology for which the mapping f is continuous, i.e. $\nu(f) = \top$. The pair $(Y, \mathcal{T}_Y)$ can be viewed as the quotient of $(X, \mathcal{T}_X)$ under mapping f in the fuzzy category $\mathcal{F}TOP(L,M)$ in the following sense:

Let $(Z, \mathcal{T}_Z)$ be an (L,M)-fuzzy α-topological space and let $g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ be a mapping. Then

$$\nu(g \circ f) = \nu(g).$$

Indeed, the inequality

$$\nu(g \circ f) \leq \nu(g)$$

holds always. To establish the converse inequality let $h = g \circ f$ and let $W \in L^Z$. Then by surjectivity of the mapping f there exists $U \in L^X$ such that $g^{-1}(W) = f(U)$ and, in particular, $U = f^{-1}(g^{-1}(W))$. Hence, by definition of $\mathcal{T}_Y$ we have

$$\mathcal{T}_Y(g^{-1}(W)) = \mathcal{T}_X(f^{-1}(g^{-1}(W))) = \mathcal{T}_X(h^{-1}(W)).$$

It follows from here that

$$\mathcal{T}_Z(W) \rightarrowtail \mathcal{T}_Y(h^{-1}(W)) = \mathcal{T}_Z(W) \rightarrowtail \mathcal{T}(g^{-1}(W))$$

and taking infimum over all $W \in L^Z$ we obtain:

$$\nu(g \circ f) \geq \nu(g).$$

6. Interior operator

Theorem 6.1. *Let $\mathcal{T} : L^X \to M$ be a mapping where M is completely distributive and let $\omega(\mathcal{T}) \geq \alpha$. We define the mapping*

$$\mathrm{Int} := \mathrm{Int}_{\mathcal{T}} : L^X \times M \to L^X$$

by setting:

$$\mathrm{Int}(A, \beta) = \bigvee \{U : U \leq A, \mathcal{T}(U) \geq \beta\} \ \forall A \in L^X, \ \forall \beta \in M.$$

Then:

(1^{int}) $\mathrm{Int}(1_X, \beta) = 1_X \ \forall \beta \leq \alpha$;
(2^{int}) $A \leq A'$, $\beta' \leq \beta \Longrightarrow \mathrm{Int}(A, \beta) \leq \mathrm{Int}(A', \beta')$;
(3^{int}) $\bigwedge_{i=1,\ldots,n} \mathrm{Int}(A_i, \beta) \leq \mathrm{Int}(\bigwedge_{i=1,\ldots,n} A_i, \beta * \alpha) \ \forall \beta \in M$;
(4^{int}) $\mathrm{Int}(A, \bot) = A$.
(5^{int}) $\mathrm{Int}(\mathrm{Int}(A, \beta), \beta * \alpha) \geq \mathrm{Int}(A, \beta) \ \forall \beta \in M$;
(6^{int}) *If* $\mathrm{Int}(A, \beta) = A^0 \ \forall \beta \in M',$ *then* $\mathrm{Int}(A, \vee M') = A^0$.

Besides, if $\omega(\mathcal{T}) = \top$, *then* Int *satisfies the following stronger version of the property* (5^{int})*:*

(5_0^{int}) $\mathrm{Int}(\mathrm{Int}(A,\beta),\beta) \geq \mathrm{Int}(A,\beta)$

Conversely, if a mapping $\mathrm{Int} : L^X \times M \to L^X$ *satisfies conditions* (1^{int}) - (6^{int}) *above for a fixed* $\alpha \in M$ *, then the mapping* $\mathcal{T} := \mathcal{T}_{\mathrm{Int}} : L^X \to M$ *defined by the equality*

$$\mathcal{T}(A) = \bigvee\{\beta \in M : \mathrm{Int}(A,\beta) = A\}$$

is an (L,M)*-fuzzy* α*-topology on* X *and besides* $\omega_3(\mathcal{T}_{\mathrm{Int}}) = 1$.
(In the sequel mappings $\mathrm{Int} : L^X \times M \to L^X$ *satisfying the above properties* (1^{int}) - (6^{int}) *for a fixed* $\alpha \in M$ *will be referred to as an* (L,M)*-fuzzy* α*- interior operator.)*
The (L,M)*-fuzzy* α*-topology and the corresponding* (L,M)*-fuzzy* α*-interior operator are related in the following way:*

$$\mathcal{T}_{\mathrm{Int}_\mathcal{T}} * \alpha \leq \mathcal{T} \leq \mathcal{T}_{\mathrm{Int}_\mathcal{T}}$$

and

$$\mathrm{Int}_{\mathcal{T}_{\mathrm{Int}}} \leq \mathrm{Int} \textit{ and } \mathrm{Int}_{\mathcal{T}_{\mathrm{Int}}}(\cdot,\beta * \alpha) \geq \mathrm{Int}(\cdot,\beta)\ \forall \beta \in M.$$

In case $\omega_3(\mathcal{T}) = \top$*, the equalities*

$$\mathcal{T} = \mathcal{T}_{\mathrm{Int}_\mathcal{T}} \textit{ and } \mathrm{Int} = \mathrm{Int}_{\mathcal{T}_{\mathrm{Int}}}$$

hold (cf Theorem 8.1.2 in [4]*).*

Proof. (1) Since $\omega_1(\mathcal{T}) \geq \alpha$, it follows that $\mathcal{T}(1_X) \geq \alpha \geq \beta$ and hence $\mathrm{Int}(1_X,\beta) \geq 1_X$.
(2) Obvious.
(3) Applying infinite distributivity of the lattice M and condition $\omega_2(\mathcal{T}) \geq \alpha$ we have

$$\begin{aligned}\bigwedge_{i=1\ldots n} \mathrm{Int}(A_i,\beta) &= \bigwedge_i \big(\bigvee\{U_i \mid U_i \leq A_i,\ \mathcal{T}(U_i) \geq \beta\}\big) \leq \\ &\leq \bigvee\{\bigwedge_i U_i \mid U_i \leq A_i,\ \mathcal{T}(U_i) \geq \beta\ \} \leq \\ &\leq \bigvee\{V \mid V \leq \bigwedge_i A_i,\ \mathcal{T}(V) \geq \beta * \alpha\} = \\ &= Int(\bigwedge_i A_i, \beta * \alpha).\end{aligned}$$

(4) Obvious.
(5) $\mathrm{Int}(A,\beta) = \bigvee\{U \in L^X \mid U \leq A,\ \mathcal{T}(U) \geq \beta\}$; hence by condition $\omega_3(\mathcal{T}) \geq \alpha$ we have $\mathcal{T}(\mathrm{Int}(A,\beta)) \geq \beta * \alpha$ and therefore $\mathrm{Int}(\mathrm{Int}(A,\beta),\beta * \alpha) \geq \mathrm{Int}(A,\beta)$.
(6) $\mathrm{Int}(A, \bigvee M') = \bigvee\{U \in L^X \mid U \leq A,\ \mathcal{T}(U) \geq \bigvee M'\} =$
$\bigvee\{U \in L^X \mid U \leq A,\ \mathcal{T}(U) \geq \beta\ \forall \beta \in M'\} = A^0$.
Moreover, if $\omega_3(\mathcal{T}) = \top$, then $\mathcal{T}(Int(A,\beta)) \geq \beta$ and hence

$$\mathrm{Int}(\mathrm{Int}(A,\beta),\beta) \geq \mathrm{Int}(A,\beta).$$

Conversely,
(1) if $\mathrm{Int}(1_X, \beta) = 1_X$ for all $\beta \leq \alpha$, then $\mathcal{T}_{\mathrm{Int}}(1_X) \geq \alpha$, and hence $\omega_1(\mathcal{T}_{\mathrm{Int}}) \geq \alpha$.
(2) Let $U_1, \ldots, U_n \in L^X$, and let $\beta_0 := \mathcal{T}_{\mathrm{Int}}(U_1) \wedge \ldots \wedge \mathcal{T}_{Int}(U_n)$. Then for every $\beta \ll \beta_0$ $Int(U_i, \beta) \geq U_i$ and hence by property (3^{int})

$$\mathrm{Int}(\bigwedge_{i=1}^{n} U_i, \beta * \alpha) \geq \bigwedge_{i=1}^{n} Int(U_i, \beta) \geq \bigwedge_{i=1}^{n} U_i.$$

It follows from here that

$$\mathcal{T}_{Int}(\bigwedge_{i=1}^{n} U_i) = \bigvee\{\gamma \mid \mathrm{Int}(\bigwedge_{i=1}^{n} U_i, \gamma) \geq \bigwedge_{i=1}^{n} U_i\} \geq \beta * \alpha$$

for every $\beta \ll \beta_0$ and hence, by complete distributivity of M

$$\mathcal{T}_{\mathrm{Int}}(\bigwedge_{i=1}^{n} U_i) \geq \beta_0 * \alpha.$$

Therefore

$$\bigwedge_{i=1}^{n} \mathcal{T}_{\mathrm{Int}}(U_i) \rightarrowtail \mathcal{T}_{Int}(\bigwedge_{i=1}^{n} U_i) \geq \alpha,$$

for each finite family $\{U_1, \ldots, U_n\} \subset L^X$ and hence $\omega_2(\mathcal{T}) \geq \alpha$.
(3) Let $\mathcal{U} := \{U_i \mid i \in \mathcal{I}\}$ and let $\bigwedge_{i\in\mathcal{I}} \mathcal{T}_{\mathrm{Int}}(U_i) =: \beta_0$. Then for every $i \in \mathcal{I}$ and for every $\beta \ll \beta_0$ it holds $\mathrm{Int}(U_i, \beta) \geq U_i$. Applying ($2^{int}$) we conclude from here that

$$\bigvee_{i\in\mathcal{I}} U_i \leq \bigvee_{i\in\mathcal{I}} \mathrm{Int}(U_i, \beta) \leq \mathrm{Int}(\bigvee_{i\in\mathcal{I}} U_i, \beta)$$

for every $\beta \ll \beta_0$. and hence $\mathcal{T}_{\mathrm{Int}}(\bigvee_{i\in\mathcal{I}} U_i) \geq \beta$. Hence, by complete distributivity of the lattice M we conclude:

$$\mathcal{T}_{\mathrm{Int}}(\bigvee_{i\in\mathcal{I}} U_i) \geq \bigwedge_{i\in\mathcal{I}} \mathcal{T}_{\mathrm{Int}}(U_i).$$

Thus $\omega_3(\mathcal{T}_{\mathrm{Int}}) = \top$ and hence $\omega(\mathcal{T}_{\mathrm{Int}}) \geq \alpha$

To verify the relations between $\mathcal{T}_{\mathrm{Int}_\mathcal{T}}$ and $\mathcal{T}$, take some $U \in L^X$ and let $\mathcal{T}(U) =: \beta$. Then $\mathrm{Int}_\mathcal{T}(U, \beta) \geq U$, and hence $\mathcal{T}_{\mathrm{Int}_\mathcal{T}}(U) \geq \beta$, thus the inequality $\mathcal{T} \leq \mathcal{T}_{\mathrm{Int}_\mathcal{T}}$ is established.

Conversely, let $\mathcal{T}_{\mathrm{Int}_\mathcal{T}}(U) = \bigvee\{\beta \mid \mathrm{Int}_\mathcal{T}(U, \beta) \geq U\} = \beta_0$. Then for each $\beta \ll \beta_0$

$$\mathrm{Int}_\mathcal{T}(U, \beta) = \bigvee\{V | \mathcal{T}(V) \geq \beta\} = U,$$

and hence, in view of the property $\omega_3(\mathcal{T}) \geq \alpha$, we conclude that

$$\mathcal{T}(\mathrm{Int}_\mathcal{T}(U, \beta)) \geq \beta * \alpha.$$

Since this holds for every $\beta \ll \beta_0$ and for every $U \in L^X$ it follows from here that

$$\mathcal{T}_{\mathrm{Int}_\mathcal{T}} * \alpha \leq \mathcal{T}.$$

In particular, if $\omega_3(\mathcal{T}) = \top$, then $\mathcal{T}_{\mathrm{Int}_\mathcal{T}} = \mathcal{T}$.

Let now $A \in L^X$ and let $\text{Int}(A,\beta) =: W$. Then by the property (5^{int}) of the (L,M)-fuzzy α-interior operator and from the definition of the (L,M)-structure $\mathcal{T}_{\text{Int}} : L^X \to M$ we have $\mathcal{T}_{\text{Int}}(W) \geq \beta$ and hence, taking into account monotonicity and property (4^{int}) of the (L,M)-fuzzy α-interior operator, it follows

$$\text{Int}_{\mathcal{T}_{\text{Int}}}(A, \beta * \alpha) \geq \text{Int}_{\mathcal{T}_{\text{Int}}}(W,\beta) \geq W = \text{Int}(A,\beta),$$

i.e. $\text{Int}_{\mathcal{T}_{\text{Int}}}(\cdot, \beta * \alpha) \geq \text{Int}(\cdot,\beta)$. In particular, if $\omega_3(\mathcal{T}) = \top$, then $\text{Int}_{\mathcal{T}_{\text{Int}}} = \text{Int}$.

Conversely, let $\text{Int}_{\mathcal{T}_{\text{Int}}}(M,\beta) =: W$, then by the definition of $\text{Int}_{\mathcal{T}_{\text{Int}}}$, we conclude that

$$\bigvee \{U | \mathcal{T}_{\text{Int}}(U) \geq \beta, U \leq A\} = W.$$

Taking into account that, as it was already established above, $\omega_3(\mathcal{T}_{\text{Int}}) = \top$ it follows that

$$\beta \leq \mathcal{T}_{\text{Int}}(W) = \bigvee \{\beta' \ | \text{Int}(W,\beta') = W\}.$$

properties (6^{int}) and (2^{int}) we conclude that

$$\text{Int}(A,\beta) \geq \text{Int}(W,\beta) \geq W$$

and hence

$$\text{Int} \geq \text{Int}_{\mathcal{T}_{\text{Int}}},$$

that is

$$\text{Int}_{\mathcal{T}_{\text{Int}(\cdot,\beta)}} \leq \text{Int}(\cdot,\beta) \leq \text{Int}_{\mathcal{T}_{\text{Int}}}(\cdot, \beta * \alpha).$$

In particular, if $\omega_3(\mathcal{T}) = \top$, then $\text{Int} = \text{Int}_{\mathcal{T}_{\text{Int}}}$.

□

7. Neighborhood systems

Let $\text{Int} : L^X \to M$ be an (L,M)-α-fuzzy interior operator, i.e. Int satisifes properies (1^{int}) — (6^{int}).

Theorem 7.1. *Let $\mathcal{N}_{\text{Int}} := \mathcal{N} : X \times L^X \times L \to L$ be defined by the equality*

$$\mathcal{N}(x,U,\beta) = \text{Int}(U,\beta)(x).$$

Then:

($1^{\mathcal{N}}$) $\mathcal{N}(x, 1_X, \beta) = \top \ \forall x \in X$ *if* $\beta \leq \alpha$;
($2^{\mathcal{N}}$) $U \leq U'$, $\beta' \leq \beta \Longrightarrow \mathcal{N}(x,U,\beta) \leq \mathcal{N}(x,U',\beta')$;
($3^{\mathcal{N}}$) $\bigwedge_{i=1}^{n} \mathcal{N}(x,U_i,\beta) \leq \mathcal{N}(x, \bigwedge_{i=1}^{n} U_i, \beta * \alpha)$;
($4^{\mathcal{N}}$) $\mathcal{N}(x,U,0) = \text{Int}(U,0)(x) \ (= U(x))$;.
($5^{\mathcal{N}}$) $\mathcal{N}(x,U,\beta) \leq \bigvee \{\mathcal{N}(x,V,\beta * \alpha) \mid V(y) \leq \mathcal{N}(y,U,\beta) : \forall y \in X\}$;
($6^{\mathcal{N}}$) *if* $U(x) \leq \mathcal{N}(x,U,\beta) \quad \forall x \in X, \ \forall \beta \in M' \subset M$, *then* $U(x) \leq \mathcal{N}(x,U,\vee M')$.

Conversely, if $\mathcal{N} : X \times L^X \times L \to L$ satisfies conditions ($1^{\mathcal{N}}$) — ($6^{\mathcal{N}}$) above, then the mapping $\text{Int}_{\mathcal{N}} := \text{Int} : L^X \times L \to L^X$ defined by

$$\text{Int}(U,\beta)(x) = \mathcal{N}(x,U,\beta)$$

satisfies axioms (1^{int}) – (6^{int}), i.e. is an (L,M)-fuzzy α-interior operator. Moreover, $\text{Int}_{\mathcal{N}_{\text{Int}}} = \text{Int}$ and $\mathcal{N}_{\text{Int}_N} = \mathcal{N}$.

Proof. Let $\mathrm{Int} : L^X \times M \longrightarrow L^X$ be an (L, M)-fuzzy α-interior operator and let $\mathcal{N} := \mathcal{N}_{\mathrm{Int}}$ be defined as above.

$(1^{\mathcal{N}})$: For $\beta \leq \alpha$ by (1^{int}) it holds $\mathcal{N}(x, 1_X, \beta) = \mathrm{Int}(1_X, \beta)(x) = 1_X(x) = \top$.

$(2^{\mathcal{N}})$: If $U \leq U'$ and $\beta' \leq \beta$, then by (2^{int}) it holds

$$\mathcal{N}(x, U, \beta) = \mathrm{Int}(U, \beta)(x) \leq \mathrm{Int}(U', \beta')(x) = \mathcal{N}(x, U', \beta').$$

$(3^{\mathcal{N}})$: Applying (3^{int}) we get:

$$\bigwedge_{i=1,\ldots,n} \mathcal{N}(x, U_i, \beta) = \bigwedge_{i=1,\ldots,n} \mathrm{Int}(U_i, \beta)(x) \leq$$

$$\leq \mathrm{Int}(\bigwedge_{i=1,\ldots,n} U_i, \beta * \alpha)(x) = \mathcal{N}(x, \bigwedge_{i=1,\ldots,n} U_i, \beta * \alpha)$$

$(4^{\mathcal{N}})$ obviously follows from (4^{int}).

$(5^{\mathcal{N}})$: Applying (5^{int}) and denoting $\mathrm{Int}(U, \beta * \alpha) = V$ we get:

$$\begin{aligned}\mathcal{N}(x, U, \beta) &= \mathrm{Int}(U, \beta)(x) \leq \mathrm{Int}(\mathrm{Int}(U, \beta), \beta * \alpha)(x) = \\ &= \mathrm{Int}(V, \beta * \alpha)(x) \leq \bigvee\{\mathrm{Int}(W, \beta * \alpha) \mid \mathrm{Int}(W, \beta * \alpha) \leq V\} = \\ &= \bigvee \mathcal{N}(x, V, \beta * \alpha) \leq \\ &\leq \bigvee\{\mathcal{N}(x, W, \beta * \alpha) | W(y) \leq \mathcal{N}(y, U, \beta)\ \forall y \in X\}.\end{aligned}$$

$(6^{\mathcal{N}})$: Assume that $U(x) \leq \mathcal{N}(x, U, \beta)$ for every $\beta \in M'$ and every $x \in X$. Then $U(x) = \mathrm{Int}(U, \beta)(x)$ and hence $U = \mathrm{Int}(U, \beta)$ for every $\beta \in M'$. Applying property (6^{int}) of the (L, M)-fuzzy α-interior operator we conclude that $U = \mathrm{Int}(U, \bigvee M')$ and hence $U(x) = \mathcal{N}(x, U, \bigvee M')$.

Conversely, let $\mathcal{N} : X \times L^X \times M \to M$ satisfy the properties (1^{int}) — (6^{int}) and let $\mathrm{Int} = \mathrm{Int}_{\mathcal{N}}$ be defined as above. Then Int is the interior operator.
The validity of properties $(1^{int}), (2^{int}), (3^{int}), (4^{int})$ and (6^{int}) is obvious from the definition of $\mathrm{Int}_{\mathcal{N}}$ and the corresponding properties of $\mathcal{N}$.

To show (5^{int}) notice that

$$\begin{aligned}\mathrm{Int}(U, \beta)(x) &= \mathcal{N}(x, U, \beta) \leq \bigvee\{\mathcal{N}(x, W, \beta * \alpha) | W(y) \leq \mathcal{N}(y, U, \beta)\} = \\ &= \bigvee\{\mathrm{Int}(W, \beta * \alpha)(x) | W \in L^X \text{ such that } W(y) \leq \mathrm{Int}(U, \beta)(y)\} \\ &= \mathrm{Int}(\mathrm{Int}(U, \beta), \beta * \alpha)(x)\end{aligned}$$

Finally, the equalities $\mathrm{Int}_{\mathcal{N}_{\mathrm{Int}}} = \mathrm{Int}$ and $\mathcal{N}_{\mathrm{Int}_{\mathcal{N}}} = \mathcal{N}$ are obvious from the definitions. □

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Tomasz Kubiak (tkubiak@amu.edu.pl)
Wydział Matematyki i Informatyki, Adam Mickiewicz University, PL-60-769, Poznań, Poland

Alexander Šostak (sostaks@com.latnet.lv)
Departmant of Mathematics, University of Latvia, LV-1586 Riga, Latvia