Compactness in the endograph uniformity

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Communicated by M. Sanchis

Abstract

Given a uniform space \((X, \mathcal{U})\), we denote by \(F^*(X)\) to the family of fuzzy sets \(u\) in \((X, \mathcal{U})\) such that \(u\) is normal and upper semicontinuous. Let \(U_E\) be the endograph uniformity on \(F^*(X)\). In this paper, we mainly characterize totally bounded and compact subsets in the uniform space \((F^*(X), U_E)\).

2020 MSC: 03E72; 94D05; 28A20.

Keywords: fuzzy sets; endograph uniformity; endograph metric; sendograph uniformity; sendograph metric; completeness; compactness.

1. Introduction

Compactness is a fundamental property in both theory and applications [5, 8, 14], and compactness criteria have attracted much attention. The Arzelà-Ascoli theorem(s) provide compactness criteria in classic analysis and topology (see for instance [2]). Characterizations of compactness are useful in theoretical research and practical applications. So many researches are devoted to characterizations of compactness in a variety of fuzzy set spaces endowed with different topologies (see [3] and references within).

Kloeden [9] introduced the endograph metric \(d_E\) on fuzzy sets. Given a metric space \((X, d)\), we denote by \(F(X)\) to the family of fuzzy sets \(u\) in \((X, d)\) such that \(u\) is normal, upper semicontinuous and with compact support. Let \(F^*(X)\) be the completion of \((F(X), d_E)\). In [3], relatively compact subsets in \((F^*(\mathbb{R}^n), d_E)\) (where \(d\) is the usual metric in \(\mathbb{R}^n\)) are characterized via the
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notion of \( \Gamma \)-convergence, which was introduced by Rojas-Medar and Román-Flores [13].

In [6] was introduced the endograph uniformity \( U_E \) on the family \( F^*(X) \) of fuzzy sets \( u \) in the uniform space \( (X,U) \) such that \( u \) is normal and upper semicontinuous. In this paper, we mainly characterize totally bounded and compact subsets in the uniform space \( (F^*(X),U_E) \) (see Theorem 3.1 and 3.6). The latter theorems generalize some results in [4].

We also study totally bounded and compact subsets in the sendograph uniformity \( U_S \) on the family \( F(X) \) of fuzzy sets \( u \) in the uniform space \( (X,U) \) such that \( u \) is normal, upper semicontinuous and has compact support (see Theorem 4.1 and 4.2).

2. Preliminaries

Given a non-empty set \( X \), a fuzzy set \( u \) on \( X \) is a function \( u : X \to [0,1] \).

Let \( \alpha \in (0,1] \). We define the \( \alpha \)-level of \( u \) as the set \( [u]_\alpha = \{ x \in X : u(x) \geq \alpha \} \).

The support of \( u \) is the set \( [u]_0 = \{ x \in X : u(x) > 0 \} \).

Now, let \((X,d)\) be a metric space. Denote by \( K(X) \) (resp. \( C(X) \)) to the family of non-empty compact (resp. closed) subsets of \( X \). Given \( A,B \in K(X) \), we put \( d_\lambda(A,B) = \max \{ d(a,B) : a \in A \} \), where \( d(a,B) = \inf \{ d(a,b) : b \in B \} \). Then \( d_\lambda \) is called the Hausdorff quasi-pseudometric on \( K(X) \). Note that \( d_\lambda(A,B) = 0 \) if and only if \( A \subseteq B \). We recall that the Hausdorff metric on \( K(X) \), denoted by \( d_H \), is defined as \( d_H(A,B) = \max \{ d_\lambda(A,B), d_\lambda(B,A) \} \) for each \( A,B \in K(X) \).

Let \( X \) be a set and let \( A \) and \( B \) be subsets of \( X \times X \), i.e., relations on the set \( X \). The inverse relation of \( A \) will be denoted by \( A^{-1} \), and the composition of \( A \) and \( B \) will be denoted by \( A \circ B \). Thus, we have

\[
A^{-1} = \{(x,y) \in X \times X : (y,x) \in A\}
\]

and

\[
A \circ B = \{(x,y) \in X \times X : \text{there exists } z \in X \text{ such that } (x,z) \in A \text{ and } (z,y) \in B\}.
\]

The symbol \( A^2 \) stands for \( A \circ A \) and \( \Delta_X \) for the diagonal of \( X \), that is, the subset \( \{(x,x) : x \in X\} \) of \( X \times X \). Every set \( A \subseteq X \times X \) that contains \( \Delta_X \) is called an entourage of the diagonal. We will denote by \( D_X \) the family of all entourages of the diagonal of \( X \).

**Definition 2.1.** A uniformity on a non-empty set \( X \) is a subfamily \( \mathcal{U} \) of \( D_X \) which satisfies the following conditions:

- (U1) If \( A \in \mathcal{U} \) and \( A \subseteq B \in D_X \), then \( B \in \mathcal{U} \).
- (U2) If \( A,B \in \mathcal{U} \), then \( U \cap V \in \mathcal{U} \).
- (U3) For every \( A \in \mathcal{U} \), there exists \( B \in \mathcal{U} \) such that \( B^2 \subseteq A \).
- (U4) For every \( A \in \mathcal{U} \), there exists \( B \in \mathcal{U} \) such that \( B^{-1} \subseteq A \).
- (U5) \( \bigcap_{A \in \mathcal{U}} A = \Delta_X \).

A uniform space is a pair \((X,\mathcal{U})\) consisting of a set \( X \) and a uniformity \( \mathcal{U} \) on the set \( X \). Let \((X,\mathcal{U})\) be a uniform space. A family \( \mathcal{B} \subseteq \mathcal{U} \) is called a base...
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for the uniformity \( U \) if for every \( A \in U \), there exists \( B \in \mathcal{B} \) such that \( B \subseteq A \).

The following result is well known and easy to prove.

**Proposition 2.2.** Let \( X \) be a non-empty set. A non-empty family \( \mathcal{B} \) of subsets of \( X \times X \) is a base for some uniformity on \( X \) if and only if it satisfies the following properties:

1. **(BS1)** For any \( A, B \in \mathcal{B} \), there exists \( C \in \mathcal{B} \) such that \( C \subseteq A \cap B \).
2. **(BS2)** For every \( A \in \mathcal{B} \), there exists \( B \in \mathcal{B} \) such that \( B \subseteq A \).
3. **(BS3)** For every \( A \in \mathcal{B} \), there exists \( B \in \mathcal{B} \) such that \( B^\sim \subseteq A \).
4. **(BS4)** \( \bigcap_{A \in \mathcal{B}} A = \Delta_X \).

As usual, a set \( X \) equipped with a topology \( \tau \) is called a *topological space* and it will be denoted by \( (X, \tau) \). It is a well-known fact that every uniformity \( U \) on a set \( X \) induces a topology \( \tau(U) \) on \( X \). To be precise, the topology \( \tau(U) \) is the family \( \{ V \subseteq X : \text{for every } x \in V, \text{there exists } U \in U \text{ such that } U(x) \subseteq V \} \), where \( U(x) = \{ y \in X : (x, y) \in U \} \). In this case, the topological space \( (X, \tau(U)) \) is a Tychonoff space (for the details we refer to the reader to Chapter 8 of the classic text [1]).

We turn to a brief discussion of the hyperspaces that we will consider in this paper. Given a topological space \( (X, \tau) \), the symbols \( \mathcal{C}(X) \) and \( \mathcal{K}(X) \) denote, respectively, the hyperspaces defined by

\[
\mathcal{C}(X) = \{ E \subseteq X : E \text{ is closed and non-empty} \},
\]

\[
\mathcal{K}(X) = \{ E \in \mathcal{C}(X) : E \text{ is compact} \}.
\]

Thus, in the case of a uniform space \( (X, \mathcal{U}) \), \( \mathcal{C}(X) \) (respectively, \( \mathcal{K}(X) \)) denotes the hyperspace of all non-empty closed (respectively, non-empty compact) subsets of \( (X, \tau(U)) \). We will see that \( \mathcal{C}(X) \) and \( \mathcal{K}(X) \) can be endowed with a natural uniformity in this situation.

Let \( (X, \mathcal{U}) \) be a uniform space. For each \( U \in \mathcal{U} \) and each \( A \subseteq X \), let us define \( U(A) = \bigcup_{x \in A} U(x) \). Now, for each \( U \in \mathcal{U} \) consider the families

\[
\mathcal{C}[U] = \{ (A, B) \in \mathcal{C}(X) \times \mathcal{C}(X) : A \subseteq U(B), \ B \subseteq U(A) \},
\]

\[
\mathcal{K}[U] = \{ (A, B) \in \mathcal{K}(X) \times \mathcal{K}(X) : A \subseteq U(B), \ B \subseteq U(A) \}.
\]

Among the most interesting results in the theory of hyperspaces are the following three well-known results.

**Proposition 2.3 ([11]).** If \( (X, \mathcal{U}) \) is a uniform space, then \( \mathcal{K}[U] : U \in \mathcal{U} \) is a base for a uniformity \( \mathcal{K}(\mathcal{U}) \) on \( \mathcal{K}(X) \).

A remarkable result by Michael [11] allows us to describe the topology induced by the uniformity \( \mathcal{K}(\mathcal{U}) \). Let us recall that, for any topological space \( (X, \tau) \), the topology \( \tau \) induces a topology \( \tau_V \) on \( \mathcal{C}(X) \), the so-called Vietoris topology, a base for \( \tau_V \) is the family of all sets of the form

\[
\mathcal{V}(V_1, V_2, \ldots, V_k) = \left\{ B \in \mathcal{C}(X) : B \subseteq \bigcup_{i=1}^{k} V_i \text{ and } B \cap V_i \neq \emptyset \text{ for } i = 1, 2, \ldots, k \right\},
\]

where \( V_1, V_2, \ldots, V_n \) is a finite sequence of non-empty open sets of \( X \).
Theorem 2.4 ([11]). If $(X, \mathcal{U})$ is a uniform space, then the topology induced by $\mathcal{K}(\mathcal{U})$ on $\mathcal{K}(X)$ coincides with the Vietoris topology induced by $\tau(\mathcal{U})$ on $\mathcal{K}(X)$.

Allowing for the previous result, if no confusion can arise, $\mathcal{K}(X)$ will be denote the hyperspace of all non-empty compact subsets of $(X, \tau(\mathcal{U}))$ equipped with the Vietoris topology induced by $\tau(\mathcal{U})$. For the hyperspace $\mathcal{C}(X)$ we have the following.

Proposition 2.5 ([11]). If $(X, \mathcal{U})$ is a uniform space, then $\{\mathcal{C}[U] : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{C}(\mathcal{U})$ on $\mathcal{C}(X)$.

The following result is easy to prove.

Lemma 2.6. Let $(X, \mathcal{U})$ be a uniform space. If $W \in \mathcal{U}$ and $A, B, C, D \in \mathcal{K}(X)$ satisfy $(A, C) \in \mathcal{K}[W]$ and $(B, D) \in \mathcal{K}[W]$, then $(A \cup B, C \cup D) \in \mathcal{K}[W]$.

Let $(X, \mathcal{U})$ be a uniform space. Let us recall that a non-empty subset $A \subseteq X$ is totally bounded in $(X, \mathcal{U})$ if for every $U \in \mathcal{U}$, there exists a finite subset $F \subseteq A$ such that $A \subseteq U(F)$.

Proposition 2.7. Let $(X, \mathcal{U})$ be a uniform space. Then $A \subseteq X$ is totally bounded in $(X, \mathcal{U})$ if and only if for every $U \in \mathcal{U}$, there exists a finite subset $F \subseteq X$ such that $A \subseteq U(F)$.

Proposition 2.8. If $(X, \mathcal{U})$ is a totally bounded uniform space, then the uniformity $\mathcal{K}(\mathcal{U})$ on $\mathcal{K}(X)$ is totally bounded.

Proof. Take $U \in \mathcal{U}$. Since $(X, \mathcal{U})$ is totally bounded, there exists a finite subset $A \subseteq X$ such that $X = U(A)$. Denote by $F$ the family of all non-empty finite subsets of $A$. Let us show that $\mathcal{K}(X) = \mathcal{K}[U](F)$. Fix $K \in \mathcal{K}(X)$. We can find $B \in F$ such that $K \subseteq U(B)$ and $K \cap U(b) \neq \emptyset$ for each $b \in B$. The choice of $B$ implies that $(B, K) \in \mathcal{K}[U]$. This completes the proof. \hfill \Box

Let $(X, \mathcal{U})$ be a uniform space. Denote by $\mathcal{F}^*(X)$ the family of fuzzy sets $u$ on $(X, \mathcal{U})$ satisfying the following conditions:

i) $u$ is upper semicontinuous.

ii) $[u]_{\alpha} \in \mathcal{K}(X)$ for every $\alpha \in (0, 1]$.

iii) $\mathcal{U}_0 = \bigcup\{[u]_{\alpha} : \alpha \in (0, 1]\}$.

Theorem 2.9 ([7, Proposition 4.9]). Let $X$ be a Hausdorff space and $u \in \mathcal{F}^*(X)$. If $L_u : [0, 1] \rightarrow (\mathcal{K}(X), \tau_V)$ is defined by $L_u(\alpha) = [u]_{\alpha}$ for all $\alpha \in (0, 1]$, then $L_u$ is left-continuous on $[0, 1]$.

Conversely, if $\{[u]_{\alpha} : \alpha \in (0, 1]\} \subseteq \mathcal{K}(X)$ is a decreasing family such that the function $L : (0, 1] \rightarrow (\mathcal{K}(X), \tau_V)$ defined by $L(\alpha) = [u]_{\alpha}$ is left-continuous, then there exists a unique $w \in \mathcal{F}^*(X)$ such that $[w]_{\alpha} = [u]_{\alpha}$ for every $\alpha \in (0, 1]$.

Remark 2.10. Let $X$ be a Hausdorff space and $u \in \mathcal{F}^*(X)$. If $L_u : (0, 1] \rightarrow (\mathcal{K}(X), \tau_V)$ is defined by $L_u(\alpha) = [u]_{\alpha}$ for all $\alpha \in (0, 1]$, then $\lim_{\alpha \rightarrow \beta^+} L_u(\alpha) = \bigcup_{\beta < \alpha} [u]_{\alpha}$ for each $\beta \in (0, 1)$ and we put $\lim_{\alpha \rightarrow \beta^+} L_u(\alpha) = u_{\beta^+}$. 
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Let \((X, \mathcal{U})\) be a uniform space. If \(u \in \mathcal{F}^*(X)\), then the endograph of \(u\) is defined as \(end(u) = \{(x, \alpha) \in X \times [0, 1] : u(x) \geq \alpha\}\). Notice that \(end(u) \in \mathcal{C}(X \times [0, 1])\). Consider the uniformity \(\mathcal{U}_\mathcal{E}\) defined on \(\mathcal{I} = [0, 1]\) by means of the base \(\{V_\epsilon : \epsilon > 0\}\), where \(V_\epsilon = \{(\alpha, \beta) \in \mathcal{I} \times \mathcal{I} : |\alpha - \beta| < \epsilon\}\). Then we can take the product uniformity \(\mathcal{U} \times \mathcal{U}_\mathcal{E}\) on \(X \times I\). We have that \(\{U \times V_\epsilon : U \in \mathcal{U}, \epsilon > 0\}\) is a base for \(U \times \mathcal{U}_\mathcal{E}\). Note that \(((a, \alpha), (b, \beta)) \in U \times V_\epsilon\) if and only if \((a, b) \in U\) and \(|\alpha - \beta| < \epsilon\). Let \((X, \mathcal{U})\) be a uniform space. Given \(U \in \mathcal{U}\) and \(\epsilon > 0\), we define the following sets:

\[E[U, \epsilon] = \{(u, v) \in \mathcal{F}^*(X) \times \mathcal{F}^*(X) : (end(u), end(v)) \in \mathcal{C}(U \times V_\epsilon)\}\]

It follows from Proposition 2.5 that the family \(\{E[U, \epsilon] : U \in \mathcal{U}, \epsilon > 0\}\) is base for a uniformity \(\mathcal{U}_E\) on \(\mathcal{F}^*(X)\). The uniformity \(\mathcal{U}_E\) is called the endograph uniformity.

We start this section with a characterization of totally bounded subsets in \(\mathcal{F}^*(X)\).

**Theorem 3.1.** Let \((X, \mathcal{U})\) be a uniform space and a non-empty subset \(A \subseteq \mathcal{F}^*(X)\). Then the following conditions are equivalent:

i) \(A\) is totally bounded in \((\mathcal{F}^*(X), \mathcal{U}_E)\).

ii) \(A(\alpha) = \bigcup\{[u]_\alpha : u \in A\}\) is totally bounded in \((X, \mathcal{U})\) for each \(\alpha \in (0, 1]\).

iii) \(A_\alpha = \{[u]_\alpha : u \in A\}\) is totally bounded in \((\mathcal{K}(X), \mathcal{K}(\mathcal{U}))\) for each \(\alpha \in (0, 1]\).

**Proof.** Let us show that i) implies ii). Suppose that \(A\) is a totally bounded subset in \((\mathcal{F}^*(X), \mathcal{U}_E)\). Fix \(\alpha \in (0, 1]\). Take \(U \in \mathcal{U}\). We can find a symmetric \(V \subseteq \mathcal{U}\) such that \(V^2 \subseteq U\). Put \(\epsilon = \frac{\alpha}{2} < \alpha\) and \(\delta = \alpha - \frac{\alpha}{4} > 0\). Since \(A\) is totally bounded in \((\mathcal{F}^*(X), \mathcal{U}_E)\), there exist \(u_1, ..., u_k \in A\) such that \(A \subseteq \bigcup_{i=1}^k E[V, \epsilon](u_i)\). We also put \(A_\alpha(k) = \bigcup_{i=1}^k [u_i]_\alpha\) and \(A_\epsilon(k) = \bigcup_{i=1}^k [u_i]_\epsilon\). Note that \(A_\alpha(k) \subseteq A_\epsilon(k)\). Clearly, \(A_\epsilon(k)\) is totally bounded in \((X, \mathcal{U})\). Hence, there exists a finite subset \(J \subseteq A_\epsilon(k)\) such that \(A_\epsilon(k) \subseteq V(J)\). Define \(J' = \{b \in J : V^2(b) \cap A(\alpha) \neq \emptyset\}\).

**Claim I:** \(A(\alpha) \subseteq U(J')\).

Take \(a \in A(\alpha)\). Then \(a \in [u]_\alpha\) for some \(u \in A\). So \((end(u), end(u_i)) \in \mathcal{C}[V \times V_\epsilon]\) for some \(i = 1, 2, ..., k\). Then there exists \((z_\alpha, \beta) \in end(u_i)\) with \(((a, \alpha), (z_\alpha, \beta)) \in V \times V_\epsilon\). So \((a, z_\alpha) \in V\) and \(\alpha - \beta < \epsilon = \frac{\alpha}{2}\). Hence \(\epsilon < \beta\). It follows that \(z_\alpha \in [u_i]_\beta \subseteq [u_i]_\epsilon \subseteq A_\epsilon(k)\).

By the choice of \(J\), we can find \(b \in J\) with \(z_\alpha \in V(b)\). Since \((a, z_\alpha) \in V\) and \((z_\alpha, b) \in V\), we have that \((a, b) \in V^2\). Hence \(a \in V^2(b) \cap A(\alpha)\). So \(b \in J'\) and \(a \in V^2(b) \subseteq U(b) \subseteq U(J')\), which proves Claim I. So Proposition 2.7 and Claim I imply that \(A(\alpha)\) is totally bounded in \((X, \mathcal{U})\).

Let us prove that ii) \(\Rightarrow\) iii). We now assume that \(A(\alpha)\) is totally bounded in \((X, \mathcal{U})\) for each \(\alpha \in (0, 1]\). Take \(\alpha \in (0, 1]\), we put \(X_\alpha = A(\alpha)\) and \(\mathcal{U}_\alpha = \mathcal{U}|_{X_\alpha}\).
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By Proposition 2.8, the uniform space $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$ is totally bounded. Note that $A_\alpha \subseteq \mathcal{K}(X)$. It follows from [1, Theorem 8.3.2] that $A_\alpha$ is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$. Given $U \in \mathcal{U}$, there exists a finite subset $J \subseteq A_\alpha$ such that $A_\alpha \subseteq \mathcal{K}[U \cap X^2](J) \subseteq \mathcal{K}[U](J)$. Therefore, $A_\alpha$ is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$.

In order to show that iii) implies i), assume that $A_\alpha = \{u_\alpha : u \in A\}$ is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$ for each $\alpha \in (0,1]$. Let us show that $A$ is totally bounded in $(\mathcal{F}^*(X), \mathcal{U}_E)$. Take $W \in \mathcal{U}$ and $\epsilon > 0$. We can assume that $\epsilon < 1$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Put $\alpha_i = \frac{n+1-i}{n}$ for each $i = 1, \ldots, n$ and $\alpha_{n+1} = 0$. Since $A_{\alpha_i}$ is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$ for each $i = 1, \ldots, n$, there exists a finite subset $I_i \subseteq A_{\alpha_i}$ such that $A_{\alpha_i} \subseteq \mathcal{K}[W](I_i)$ for each $i = 1, \ldots, n$. By Proposition 2.7, we can assume that $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ and every $I_i$ is closed under union. Let $V$ be the family of $v \in \mathcal{F}^*(X)$ such that $[v]_{\alpha_i} = K_i \in I_i$ for each $\alpha \in (\alpha_{i+1}, \alpha_i]$ and each $i = 1, 2, \ldots, n$. Clearly, $V$ is finite and non-empty. Let us prove the following:

$$A \subseteq E[W,e](V). \quad (3.1)$$

Take $u \in A$. Then there exists $K_i \in I_i$ such that $([u]_{\alpha_i}, K_i) \in \mathcal{K}(W)$ for each $i = 1, 2, \ldots, n$. By Lemma 2.6 and the fact that each $I_i$ is closed under union, we can suppose that $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$. Let $v \in V$ be such that $[v]_{\alpha_i} = K_i$ for each $\alpha \in (\alpha_{i+1}, \alpha_i]$ and each $i = 1, 2, \ldots, n$. Note that $v_0 = [v]_{\alpha_{n+1}} = K_n$. Pick $(x, \beta) \in \text{end}(u)$. If $\alpha_n \geq \beta \geq \alpha_{n+1}$, then

$$(x, \beta) \in [W \times V_e](x,0) \subseteq [W \times V_e](\text{end}(v)).$$

We now suppose that $\alpha_i \geq \beta > \alpha_{i+1}$ for some $i = 1, 2, \ldots, n - 1$. Since $([u]_{\alpha_i}, K_i) \in \mathcal{K}(W)$ and $x \in [u]_{\beta} \subseteq [u]_{\alpha_{i+1}}$ for each $i = 1, 2, \ldots, n - 1$, there exists $k \in K_{i+1}$ such that $(x, k) \in W$. So $((x, \beta), (k, \alpha_{i+1})) \in W \times V_e$. Therefore, $(x, \beta) \in [W \times V_e](\text{end}(v))$ for each $(x, \beta) \in \text{end}(u)$. We have thus proved that $\text{end}(u) \subseteq [W \times V_e](\text{end}(v))$.

Using a similar argument, we can show that $\text{end}(v) \subseteq [W \times V_e](\text{end}(u))$. Hence $u \in E[W,e](v)$. Therefore, $A \subseteq E[W,e](V)$. By (3.1) and Proposition 2.7, we have that $A$ is totally bounded in $(\mathcal{F}^*(X), \mathcal{U}_E)$.

Corollary 3.2. Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{D} \subseteq \mathcal{K}(X)$. Then the following conditions are equivalent:

i) $\mathcal{D} = \bigcup\{C \in \mathcal{D}\}$ is totally bounded in $(X, \mathcal{U})$.

ii) $\mathcal{D}$ is totally bounded in $(\mathcal{K}(X), \mathcal{K}(\mathcal{U}))$.

Proof. We put $A = \{X_K : K \in \mathcal{D}\} \subseteq \mathcal{F}^*(X)$ and apply Theorem 3.1.

We need the following three results in order to prove Theorem 3.6.

Lemma 3.3. Consider a uniform space $(X, \mathcal{U})$ and $\mathcal{D} \subseteq \mathcal{K}(X)$. If $(\mathcal{D}, \mathcal{K}(\mathcal{U})|_D)$ is compact, then $\mathcal{D} = \bigcup\{C \in \mathcal{D}\}$ is compact with respect to the uniformity $\mathcal{U}|_D$.

Proof. We can assume that $(X, \mathcal{U})$ is complete, otherwise we can take its completion. Let $\{x_\sigma\}_{\sigma \in \Sigma}$ be a net in $\mathcal{D}$. Pick $C_\sigma \in \mathcal{D}$ such that $x_\sigma \in C_\sigma$. Since
(\mathcal{D}, \mathcal{K}(\mathcal{U})|_{\mathcal{D}}) is compact, the net \{C_{\sigma}\}_{\sigma \in \Sigma} has a finer net \{C_{\sigma'}\}_{\sigma' \in \Sigma'} which converges to \(C \in \mathcal{D}\). The set \(\mathcal{E} = \{C\} \cup \{C_{\sigma'} : \sigma' \in \Sigma'\} \subseteq \mathcal{D}\) is totally bounded, since \(\mathcal{D}\) is compact. By Corollary 3.2, \(\mathcal{E} = \bigcup\{E \in \mathcal{E}\}\) is totally bounded in \((X, \mathcal{U})\). Then \(\mathcal{E}\) is totally bounded in \((X, \mathcal{U})\). So \(\mathcal{E}\) is compact, since \((X, \mathcal{U})\) is complete. We know that \(x_{\sigma'} \in \mathcal{E}\) for each \(\sigma' \in \Sigma'\). Hence there exists a net \(\{x_{\sigma'}\}_{\sigma' \in \Sigma'}\) finer than \(\{x_{\sigma'}\}_{\sigma' \in \Sigma'}\) which converges to \(x \in \mathcal{E}\). It is straightforward to show that \(x \in C\). We have thus proved that \(\{x_{\sigma}\}_{\sigma \in \Sigma}\) has a finer net which converges to \(x \in \mathcal{D}\). Therefore, \(\mathcal{D}\) is compact. □

**Lemma 3.4.** Consider a uniform space \((X, \mathcal{U})\) and \(\mathcal{D} \subseteq \mathcal{K}(X)\). If \(\mathcal{D} = \bigcup\{C \in \mathcal{D}\}\) is complete with respect to the uniformity \(\mathcal{U}|_{\mathcal{D}}\) and \(\mathcal{D}\) is closed in \(\mathcal{K}(X)\), then \((\mathcal{D}, \mathcal{K}(\mathcal{U})|_{\mathcal{D}})\) is complete.

**Proof.** If \(\mathcal{D}\) is complete with respect to the uniformity \(\mathcal{U}|_{\mathcal{D}}\), then \((\mathcal{K}(\mathcal{D}), \mathcal{K}(\mathcal{U})|_{\mathcal{K}(\mathcal{D})})\) is complete by [12]. Since \(\mathcal{D}\) is closed in \(\mathcal{K}(X)\), we have that \(\mathcal{D}\) is closed in \(\mathcal{K}(\mathcal{D})\). The completeness of \((\mathcal{K}(\mathcal{D}), \mathcal{K}(\mathcal{U})|_{\mathcal{K}(\mathcal{D})})\) implies that \((\mathcal{D}, \mathcal{K}(\mathcal{U})|_{\mathcal{D}})\) is complete. □

**Proposition 3.5.** Consider a uniform space \((X, \mathcal{U})\) and \(\mathcal{D} \subseteq \mathcal{K}(X)\). Then the following conditions are equivalent:

i) \(\mathcal{D}\) is compact in \((\mathcal{K}(X), \mathcal{K}(\mathcal{U}))\).

ii) \(\mathcal{D} = \bigcup\{C \in \mathcal{D}\}\) is compact in \((X, \mathcal{U})\) and \(\mathcal{D}\) is closed in \((\mathcal{K}(X), \mathcal{K}(\mathcal{U}))\).

**Proof.** i) ⇒ ii) by Lemma 3.3. Let us show that ii) ⇒ i). If \(\mathcal{D}\) is compact, then \(\mathcal{D}\) is totally bounded in \((\mathcal{K}(X), \mathcal{K}(\mathcal{U}))\) by Corollary 3.2. On the other hand, \(\mathcal{D}\) is complete by Lemma 3.4. Therefore, \(\mathcal{D}\) is compact in \((\mathcal{K}(X), \mathcal{K}(\mathcal{U}))\). □

**Theorem 3.6.** Let \((X, \mathcal{U})\) be a uniform space and a non-empty subset \(A \subseteq \mathcal{F}^*(X)\). Then the following conditions are equivalent:

i) \(A\) is compact in \((\mathcal{F}^*(X), \mathcal{U}_E)\).

ii) \(A\) is closed in \((\mathcal{F}^*(X), \mathcal{U}_E)\) and \(A(\alpha) = \bigcup\{[u]_\alpha : u \in A\}\) is compact in \((X, \mathcal{U})\) for each \(\alpha \in (0, 1]\).

**Proof.** Let \((\hat{X}, \hat{\mathcal{U}})\) the completion of \((X, \mathcal{U})\). Then \(\mathcal{F}^*(X) \subseteq \mathcal{F}^*(\hat{X})\). Let us show that i) implies ii). Clearly, \(A\) is compact in \((\mathcal{F}^*(\hat{X}), \mathcal{U}_E)\). By Theorem 3.1, \(A(\alpha)\) is totally bounded in \((\hat{X}, \hat{\mathcal{U}})\) for each \(\alpha \in (0, 1]\). Let us show that \(A(\alpha)\) is closed in \((\hat{X}, \hat{\mathcal{U}})\) for each \(\alpha \in (0, 1]\). Take \(\alpha \in (0, 1]\) and \(x \in A(\alpha)\). Then there exists a net \(\{x_{\sigma}\}_{\sigma \in \Sigma}\) in \(A(\alpha)\) which converges to \(x\). For every \(\sigma \in \Sigma\), we can choose \(u_{\sigma} \in A\) such that \(x_{\sigma} \in [u_{\sigma}]_\alpha\). Since \(A\) is compact \(\{u_{\sigma}\}_{\sigma \in \Sigma}\) has a finer net \(\{u_{\sigma}\}_{\sigma \in \Sigma'}\) which converges to \(u \in A\). We define \(v \in \mathcal{F}^*(\hat{X})\) as follows:

\[
[v]_\beta = \begin{cases} [u]_\beta, & \text{if } \beta \in (\alpha, 1], \\ \{x\} \cup [u]_\beta, & \text{if } \beta \in (0, \alpha]. 
\end{cases}
\]

Let us show that \(\{u_{\sigma}\}_{\sigma \in \Sigma'}\) converges to \(v\). Given \(U \in \hat{\mathcal{U}}\) and \(\epsilon > 0\), there exists \(\sigma_0 \in \Sigma'\) such that \((x, x_{\sigma}) \in U\) and \((u, u_{\sigma}) \in E[U, \epsilon]\) for every \(\sigma \geq \sigma_0\). Take \(\sigma \geq \sigma_0\). Clearly, \(end(u_{\sigma}) \subseteq [U \times V_{\sigma}](end(v)) \subseteq [U \times V_{\sigma}](end(v))\). We now pick \((y, \beta) \in end(v)\). If \(y \neq x\), then \((y, \beta) \in end(u) \subseteq [U \times V_{\sigma}](end(u_{\sigma}))\). On the
other hand, if \( y = x \), the definition of \( v \) implies that \( \beta \leq \alpha \). Then \( x_\sigma \in [u_\sigma]_\alpha \subseteq [u_\sigma]_\beta \). So \( (x_\sigma, \beta) \in \text{end}(u_\sigma) \) and \( (x, \beta) \in [U \times V_\epsilon](x_\sigma, \beta) \subseteq [U \times V_\epsilon](\text{end}(u_\sigma)) \). Hence, \( \text{end}(v) \subseteq [U \times V_\epsilon](\text{end}(u_\sigma)) \). We have thus proved that \((v, u_\sigma) \in E[U, \epsilon] \) for every \( \sigma \geq \sigma_0 \). Therefore, \( u = v \) and \( x \in [u]_\alpha \subseteq A(\alpha) \). So \( A(\alpha) \) is closed and totally bounded in \( (\tilde{X}, \tilde{U}) \). It follows that \( A(\alpha) \) is compact.

In order to show that ii) \( \Rightarrow i) \), assume that \( A \) is closed in \((\mathcal{F}^*(X), \mathcal{U}_E)\) and \( A(\alpha) = \bigcup\{[u]_\alpha : u \in A\} \) is compact in \((X, \mathcal{U})\) for each \( \alpha \in (0, 1) \). By Theorem 3.1, \( A \) is totally bounded in \((\mathcal{F}^*(\tilde{X}), \tilde{U}_E)\). We put \( X_\alpha = A(\alpha) \) for each \( \alpha \in (0, 1) \). Given \( u \in \mathcal{F}^*(X) \) and \( \alpha \in (0, 1) \), we put \( \text{end}_\alpha(u) = [u_\alpha+ \times \{\alpha\}] \cup [\text{end}(u) \cap (X \times (\alpha, 1))] \), see Remark 2.10 for the symbol \( u_\alpha+ \). Note that \( \text{end}_\alpha(u) \in \mathcal{K}(X_\alpha \times [0, 1]) \). Since \( X_\alpha \) is compact, we can conclude that \( \mathcal{K}(X_\alpha \times [0, 1]) \) is compact for every \( \alpha \in (0, 1) \). We can argue as in the proof of [6, Theorem 5.3] to show that \( E_\alpha = \{\text{end}_\alpha(u) : u \in A\} \) is closed in \( \mathcal{K}(X_\alpha \times [0, 1]) \). Hence \( E_\alpha \) is compact for each \( \alpha \in (0, 1) \).

Claim 1: Take \( 0 < \beta < \alpha < 1 \). Suppose that \( \{\text{end}_\alpha(u_\sigma)\}_{\sigma \in \Sigma} \) and \( \{\text{end}_\beta(u_\sigma)\}_{\sigma \in \Sigma} \) have a finer net \( \{\text{end}_\alpha(u_\sigma')\}_{\sigma \in \Sigma'} \) and \( \{\text{end}_\beta(u_\sigma')\}_{\sigma \in \Sigma'} \) which converge to \( \text{end}_\alpha(u) \) and \( \text{end}_\beta(v) \), respectively. Then \([u]_\gamma = [v]_\gamma \) for each \( \gamma \in (\alpha, 1) \).

Pick \( \gamma \in (\alpha, 1) \). Let us show that \((\{u\}_\gamma, \{v\}_\gamma) \in \mathcal{K}[\mathcal{W}] \) for every \( W \in \mathcal{U} \). Take a symmetric \( U \in \mathcal{U} \) such that \( U^4 \subseteq W \). Put \( d = \gamma - \alpha \) and \( \alpha_n = \gamma - \frac{d}{2n} \) for each \( n \in \mathbb{N} \). Then the sequence \( \{\alpha_n\}_n \subseteq (\alpha, \gamma) \) is increasing and converges to \( \gamma \). Since \( \{\text{end}_\alpha(u_\sigma)\}_{\sigma \in \Sigma'} \) and \( \{\text{end}_\beta(u_\sigma')\}_{\sigma \in \Sigma'} \) converge to \( \text{end}_\alpha(u) \) and \( \text{end}_\beta(v) \), respectively; then for every \( n \in \mathbb{N} \), there exists \( \sigma_n \in \Sigma' \) such that

\[
\text{end}_\alpha(u_{\sigma_n}) \subseteq [U \times V_{\frac{d}{2n}}](\text{end}_\alpha(u)) \quad \text{and} \quad \text{end}_\alpha(u) \subseteq [U \times V_{\frac{d}{2n}}](\text{end}_\alpha(u_{\sigma_n})).
\]  

\[\text{(3.2)}\]

\[
\text{end}_\beta(u_{\sigma_n}) \subseteq [U \times V_{\frac{d}{2n}}](\text{end}_\beta(v)) \quad \text{and} \quad \text{end}_\beta(v) \subseteq [U \times V_{\frac{d}{2n}}](\text{end}_\beta(u_{\sigma_n})).
\]  

\[\text{(3.3)}\]

From (3.2) and (3.3), we have that \( \text{end}_\alpha(u) \subseteq [U^2 \times V_{\frac{d}{2n}}](\text{end}_\beta(v)) \) for each \( n \in \mathbb{N} \). Fix \( x \in [u]_\gamma \). Since \((x, \alpha_n) \in \text{end}_\alpha(u) \), we can take \((y_n, \beta_n) \in \text{end}_\beta(v) \) such that \((x, \alpha_n, (y_n, \beta_n)) \in U^2 \times V_{\frac{d}{2n}} \). Since \( |\alpha_n - \beta_n| < \frac{d}{2n} \) and \( \{\alpha_n\}_n \) converges to \( \gamma \), we can conclude that \( \{\beta_n\}_n \) converges to \( \gamma \). Note that the sequence \( \{(y_n, \beta_n)\}_n \) is in the compact set \( \text{end}_\beta(v) \). Therefore, we can suppose that \( \{(y_n, \beta_n)\}_n \) converges to \((y, \gamma) \). Hence \( y \in [v]_\gamma \). On the other hand, \((x, y_n) \in U^2 \) for each \( n \in \mathbb{N} \). The latter fact implies that \( (x, y) \in \mathcal{U}^2 \subseteq U^3 \). So \( x \in U^3(y) \subseteq W(y) \). Hence \([u]_\gamma \subseteq W([v]_\gamma) \).

Fix \( x \in [v]_\gamma \). By (3.3) and \((x, \alpha_n) \in \text{end}_\beta(v) \), we can take \((y_n, \beta_n) \in \text{end}_\beta(u_{\sigma_n}) \) such that \((x, \alpha_n, (y_n, \beta_n)) \in U \times V_{\frac{d}{2n}} \). Since \( |\alpha_n - \beta_n| < \frac{d}{4n} \) for every \( n \in \mathbb{N} \), we have the following:

\[
\alpha = \frac{(2n - 1)\alpha + \alpha}{2n} < \frac{(2n - 1)\gamma + \alpha}{2n} = \gamma - \frac{d}{2n} = \alpha_n - \frac{d}{4n} < \beta_n < \alpha_n + \frac{d}{4n} = \gamma.
\]
Compactness in the endograph uniformity

It follows that \( \beta_n \in (\alpha, \gamma) \) for all \( n \in \mathbb{N} \). So \( (y_n, \beta_n) \in \text{end}_\alpha(u_{\sigma_n}) \). By (3.2), we can take \( (z_n, \delta_n) \in \text{end}_\alpha(u) \) such that \( ((y_n, \beta_n), (z_n, \delta_n)) \in U \times V_{\frac{d}{2n}} \). For each \( n \in \mathbb{N} \), we have that

\[
|\alpha_n - \delta_n| \leq |\alpha_n - \beta_n| + |\beta_n - \delta_n| < \frac{d}{2n}.
\]

Since \( \{\alpha_n\}_n \) converges to \( \gamma \), we can conclude that \( \{\delta_n\}_n \) converges to \( \gamma \). Note that the sequence \( \{(z_n, \delta_n)\}_n \) is in the compact set \( \text{end}_\alpha(u) \). Therefore, we can suppose that \( \{(z_n, \delta_n)\}_n \) converges to \( (z, \gamma) \). Hence \( z \in [u]_\gamma \). On the other hand, \( (x, z_n) \in U^2 \) for each \( n \in \mathbb{N} \). The latter fact implies that \( (x, z) \in U^2 \subseteq U^3 \subseteq W \). So \( x \in W(z) \) and \( [v]_\gamma \subseteq W([u]_\gamma) \). Hence \( ([u]_\gamma, [v]_\gamma) \in K[W] \) for every \( W \in \mathcal{U} \), whence \( [u]_\gamma = [v]_\gamma \) for each \( \gamma \in (\alpha, 1] \). This completes the proof of Claim 1.

Take a net \( \{u_\sigma\}_{\sigma \in \Sigma} \) in \( A \). Since \( E_\alpha \) is compact for each \( (0, 1) \), the net \( \{\text{end}_{\frac{1}{n}}(u_\sigma)\}_{\sigma \in \Sigma_1} \) has a finer net \( \{\text{end}_{\frac{1}{n}}(u_\sigma)\}_{\sigma \in \Sigma_2} \) which converges to \( \text{end}_{\frac{1}{n}}(v_2) \) with \( v_2 \in A \). By induction, for every \( n \in \mathbb{N} \), we can obtain a net \( \{\text{end}_{\frac{1}{n}}(u_\sigma)\}_{\sigma \in \Sigma_{n+1}} \) which is finer than \( \{\text{end}_{\frac{1}{n}}(u_\sigma)\}_{\sigma \in \Sigma_n} \) and \( \{\text{end}_{\frac{1}{n}}(u_\sigma)\}_{\sigma \in \Sigma_{n+1}} \) converges to \( \text{end}_{\frac{1}{n+1}}(v_{n+1}) \) with \( v_{n+1} \in A \).

By Claim 1, the set \( (X \times \{0\}) \cup \bigcup_{n \geq 2} \text{end}_{\frac{1}{n}}(v_n) \) is the endograph of a fuzzy set \( v \in F^*(X) \). Let us show that \( v \) is an accumulation point of \( \{u_\sigma\}_{\sigma \in \Sigma} \). Take \( U \in \mathcal{U} \) and \( \epsilon > 0 \). We can choose \( n \geq 2 \) such that \( \frac{1}{n} < \epsilon \). Fix \( \sigma_0 \in \Sigma \). Since \( \{\text{end}_{\frac{1}{n}}(u_\sigma)\}_{\sigma \in \Sigma_n} \) converges to \( \text{end}_{\frac{1}{n}}(v_n) \), we can find \( \sigma \geq \sigma_0 \) such that

\[
\text{end}_{\frac{1}{n}}(u_\sigma) \subseteq [U \times V_\frac{1}{n}](\text{end}_{\frac{1}{n}}(v_n)) \quad \text{and} \quad \text{end}_{\frac{1}{n}}(v_n) \subseteq [U \times V_\frac{1}{n}](\text{end}_{\frac{1}{n}}(u_\sigma)).
\]

(3.4)

Take \( (x, \alpha) \in \text{end}(v) \) with \( \alpha \in [0, \frac{1}{n}] \). Then \( (x, x) \in U \) and \( (\alpha, 0) \in V_\epsilon \). So \( (x, \alpha) \in [U \times V_\epsilon](\text{end}(u_\sigma)) \). If \( \alpha > \frac{1}{n} \), (3.4) implies the following:

\[
(x, \alpha) \in \text{end}_{\frac{1}{n}}(v_n) \subseteq [U \times V_{\frac{1}{n}}](\text{end}_{\frac{1}{n}}(u_\sigma)) \subseteq [U \times V_\epsilon](\text{end}(u_\sigma)).
\]

We have thus proved that \( \text{end}(v) \subseteq [U \times V_\epsilon](\text{end}(u_\sigma)) \). Similarly, we can show that \( \text{end}(u_\sigma) \subseteq [U \times V_\epsilon](\text{end}(v)) \). Therefore, \( v \) is an accumulation point of \( \{u_\sigma\}_{\sigma \in \Sigma} \). Finally, we know that \( A \) is closed in \( F^*(X) \), so \( v \in A \). We can conclude that every net in \( A \) has an accumulation point in \( A \), i.e., \( A \) is compact. \( \square \)

Consider now a metric space \( (X, d) \). Define the metric \( d^* \) on \( X \times [0, 1] \) as follows:

\[
d^*((x, a), (y, b)) = \max\{d(x, y), |a - b|\}.
\]

The endograph metric \( d_E \) on \( F^*(X) \) is the Hausdorff distance \( d_H \) (with respect to \( X \times [0, 1] \)) between \( \text{end}(u) \) and \( \text{end}(v) \) for each \( u, v \in F^*(X) \). Recall that a metric space \( (X, d) \) has a natural uniformity \( U_d \) determined by the base \( \{U_\epsilon : \epsilon > 0\} \), where \( U_\epsilon = \{ (x, y) \in X \times X : d(x, y) < \epsilon \} \).

**Corollary 3.7** ([4]). Let \( (X, d) \) be a metric space and a non-empty subset \( A \subseteq F^*(X) \). Then the following conditions are equivalent:

\[\square\]

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is compact (closed) in $(\mathcal{F}^*(X), d_E)$.

ii) $A$ is closed in $(\mathcal{F}^*(X), d_E)$ and $A(\alpha) = \bigcup\{[u]_\alpha : u \in A\}$ is compact in $(X, d)$ for each $\alpha \in (0, 1]$.

**Proof.** By a result of [6], we have that $\mathcal{U}_{d_E} = (\mathcal{U}_d)_E$. It is easy to see that $A$ is compact (closed) in $(\mathcal{F}^*(X), \mathcal{U}_{d_E})$ if and only if $A$ is compact (closed) in $(\mathcal{F}^*(X), d_E)$ if and only if $A$ is compact (closed) in $(\mathcal{F}^*(X), (\mathcal{U}_d)_E)$. We also have that $A(\alpha)$ is compact in $(X, \mathcal{U}_d)$ if and only if $A(\alpha)$ is compact in $(X, d)$ for each $\alpha \in (0, 1]$. It remains to apply Theorem 3.6 to the uniform space $(X, \mathcal{U}_d)$. $\square$

4. COMPACTNESS IN THE SENDOGRAPH UNIFORMITY

Given a uniform space $(X, \mathcal{U})$, we denote by $\mathcal{F}(X)$ the elements of $\mathcal{F}^*(X)$ with compact support. If $u \in \mathcal{F}(X)$, the sendograph of $u$ is defined by $\text{send}(u) = \text{end}(u) \cap (u_0 \times [0, 1])$. Observe that $\text{send}(u) \in \mathcal{K}(X \times [0, 1])$. Given $U \in \mathcal{U}$ and $\epsilon > 0$, we define the following sets:

$$S[U, \epsilon] = \{(u, v) \in \mathcal{F}(X) \times \mathcal{F}(X) : (\text{send}(u), \text{send}(v)) \in \mathcal{K}[U \times V_{\epsilon}]\}.$$  

By Proposition 2.3, the family $\{S[U, \epsilon] : U \in \mathcal{U}, \epsilon > 0\}$ is base for a uniformity $\mathcal{U}_S$ on $\mathcal{F}(X)$. The uniformity $\mathcal{U}_S$ is called the sendograph uniformity.

Consider now a metric space $(X, d)$. Define the metric $d^*$ on $X \times [0, 1]$ as follows:

$$d^*((x, a), (y, b)) = \max\{d(x, y), |a - b|\}.$$  

The sendograph metric $d_E$ on $\mathcal{F}(X)$ is the Hausdorff metric $d_H^*$ (on $\mathcal{K}(X \times [0, 1])$) between the non-empty compact subsets $\text{send}(u)$ and $\text{send}(v)$ for every $u, v \in \mathcal{F}(X)$ (see [10]).

**Theorem 4.1.** Let $A$ be a non-empty subset of a uniform space $(X, \mathcal{U})$. Then $A$ is totally bounded in $(\mathcal{F}(X), \mathcal{U}_S)$ if and only if $A(0) = \bigcup_{u \in A} u_0$ is totally bounded in $(X, \mathcal{U})$.

**Proof.** Suppose that $A$ is a totally bounded subset in $(\mathcal{F}(X), \mathcal{U}_S)$. Take $U \in \mathcal{U}$. We can find a symmetric $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Since $A$ is totally bounded in $(\mathcal{F}(X), \mathcal{U}_S)$, there exist $u_1, \ldots, u_k \in A$ such that $A \subseteq \bigcup_{i=1}^k S[V, 1](u_i)$. We also put $A(k) = \bigcup_{i=1}^k [u_i]_0$. Clearly, $A(k)$ is totally bounded in $(X, \mathcal{U})$. Hence, there exists a finite subset $J \subseteq A(k)$ such that $A(k) \subseteq V(J)$. Define $J' = \{b \in J : V^2(b) \cap A(0) \neq \emptyset\}$.

**Claim II:** $A(0) \subseteq U(J')$.

Take $a \in A(0)$. Then $a \in [u_i]_0$ for some $u \in A$. So $(\text{send}(u), \text{send}(u_i)) \in \mathcal{K}[V \times V_1]$ for some $i = 1, 2, \ldots, k$. Then there exists $(z_a, \beta) \in \text{send}(u_i)$ with $((a, 0), (z_a, \beta)) \in V \times V_1$. So $(a, z_a) \in V$ and $\beta < 1$. It follows that $z_a \in [u_i]_\beta \subseteq [u_i]_0 \subseteq A(k)$.

By the choice of $J$, we can find $b \in J$ with $z_a \in V(b)$. Then $(a, z_a), (z_a, b) \in V$. So $(a, b) \in V^2$. Hence $a \in V^2(b) \cap A(0)$. So $b \in J'$ and $a \in V^2(b) \subseteq U(b) \subseteq \mathcal{U}(b)$.
U(J'). This completes the proof of Claim II. Proposition 2.7 and Claim II imply that A(0) is totally bounded in (X, U).

For the converse, we assume that A(0) is totally bounded in (X, U). Hence A(α) is totally bounded in (X, U) for every α ∈ [0, 1]. For each α ∈ [0, 1], we put X_α = A(α) and U_α = U|_{X_α}. By Proposition 2.8, the uniform space (K(X_α), K(U_α)) is totally bounded. Let us show that A is totally bounded in (F(X), U_S). Take W ∈ U and ε > 0. We can assume that ε < 1. Choose n ∈ N such that 1/n < ε. Put α_i = \frac{n+1}{n+1} for each i = 1, ..., n and α_{n+1} = 0. Since (K(X_α), K(U_α)) is totally bounded for each i = 1, ..., n, there exists a finite subset I_i ⊆ K(X_α) such that K(X_α) = \bigcap_{i=1}^{n+1} [K(X)\cap X_\alpha]\{I_i\} for each i = 1, ..., n. By Proposition 2.7, we can assume that I_1 ⊆ I_2 ⊆ ... ⊆ I_n and every I_i is closed under union. Let V be the family of v ∈ F(X) such that [v]_α = K_i for each α ∈ (α_{i+1}, α_i] and each i = 1, 2, ..., n. Clearly, V is finite and non-empty. Let us prove the following:

$$A \subseteq S[\epsilon, \alpha](V).$$

(4.1)

Take u ∈ A. Then there exists K_i ∈ I_i such that ([u]_α, K_i) ∈ K[W \cap X_\alpha] for each i = 1, 2, ..., n. By Lemma 2.6 and the fact that each I_i is closed under union, we can suppose that K_1 ⊆ K_2 ⊆ ... ⊆ K_n. Let v ∈ V be such that [v]_α = K_i for each α ∈ (α_{i+1}, α_i] and each i = 1, 2, ..., n. Note that v_0 = K_n. Pick (x, β) ∈ send(u). Suppose that α_i ≥ β > α_{i+1} for some i = 1, 2, ..., n − 1. Since ([u]_α, K_i) ∈ K[W \cap X_\alpha] and x ∈ [u]_β ⊆ [u]_{α_{i+1}} for each i = 1, 2, ..., n − 1, there exists k ∈ K_{i+1} such that (x, k) ∈ W. So ((x, β), (k, α_{i+1})) ∈ W × V. Therefore, (x, β) ∈ [W × V](send(v)). Now if (x, β) ∈ send(u) and 0 ≤ β ≤ \frac{1}{n}, then x ∈ u_0 = \bigcup_{\alpha > 0} [u]_\alpha. Hence u_0 ∩ W(x) ≠ ∅. So we can find y ∈ [u]_α for some α > 0 such that (x, y) ∈ W. We can assume that α ∈ \{0, \frac{1}{n}\}. Therefore, (x, β) ∈ [W × V](y, α) ⊆ [W × V](send(v)). We have thus proved that send(u) ⊆ [W × V](send(v)).

Using a similar argument, we can show that send(v) ⊆ [W × V](send(u)). Hence u ∈ S[\epsilon, \alpha](v). Therefore, A \subseteq S[\epsilon, \alpha](V). By (4.1) and Proposition 2.7, we have that A is totally bounded in (F(X), U_S).

\[\square\]

**Theorem 4.2.** Let A be a non-empty subset of a uniform space (X, U). Then A is compact in (F(X), U_S) if and only if A is closed in (F(X), U_S) and A(0) is compact in (X, U).

**Proof.** Assume that A is compact in (F(X), U_S). Let \(\hat{X}, \hat{U}\) be the completion of (X, U). Then F(X) ⊆ F(\hat{X}). Clearly, A is compact in (F(\hat{X}), \hat{U}_S). By Theorem 4.1, A(0) is totally bounded in (\hat{X}, \hat{U}). Let us show that A(0) is closed in (\hat{X}, \hat{U}). Take x ∈ A(0) and a net \(\{x_\sigma\}_{\sigma \in \Sigma}\) in A(0) which converges to x. For every σ ∈ Σ, we take u_σ ∈ A such that x_σ ∈ [u_σ]_0. Since A is compact, the net \(\{u_\sigma\}_{\sigma \in \Sigma}\) in A has a finer net \(\{u_\sigma\}_{\sigma \in \Sigma'}\) which converges to u ∈ A. Let us show that x ∈ u_0. Suppose the contrary, then there exists W ∈ \hat{U} such that W(x) ∩ u_0 = ∅. Pick V ∈ \hat{U} such that V^2 ⊆ U. On the other hand, there exists σ_0 ∈ Σ' such that (u, u_σ) ∈ S[V, 1] and (x, x_σ) ∈ V for
each $\sigma \geq \sigma_0$. Hence $(x_{\sigma_0}, 0) \in \text{send}(u_{\sigma_0}) \subseteq [V \times V_1](\text{send}(u))$. So there exists $(y, \beta) \in \text{send}(u)$ with $(x_{\sigma_0}, y) \in V$ and $\beta < 1$. Then $y \in [u]_{\beta} \subseteq u_0$. Since $(x, x_{\sigma_0}) \in V$ and $(x_{\sigma_0}, y) \in V$, we have that $(x, y) \in W$. So $y \in W(x)$, which contradicts that $W(x) \cap u_0 = \emptyset$. Therefore, $A(0)$ is compact in $(X, U)$.

We now suppose that $A$ is closed in $(\mathcal{F}(X), U_S)$ and $A(0)$ is compact in $(X, U)$. Put $Y = A(0)$ and $V = U|_Y$. We can assume that $A \subseteq \mathcal{F}(Y) \subseteq \mathcal{F}(X)$. Since $(Y, V)$ is compact, $(\mathcal{F}(Y), V_S)$ is complete by a result of [6]. Hence $A$ is complete, since $A$ is closed in $(\mathcal{F}(Y), V_S)$. On the other hand, $A$ is totally bounded in $(\mathcal{F}(Y), V_S)$ by Theorem 4.1. Therefore, $A$ is compact in $(\mathcal{F}(X), U_S)$.

Corollary 4.3. [4] Let $A$ be a non-empty subset of a metric space $(X, d)$. Then $A$ is compact in $(\mathcal{F}(X), d_S)$ if and only if $A$ is closed in $(\mathcal{F}(X), d_S)$ and $A(0)$ is compact in $(X, d)$.

Proof. It is easy to see that $A$ is compact (closed) in $(\mathcal{F}(X), d_S)$ if and only if $A$ is compact (closed) in $(\mathcal{F}(X), U_{d_S})$. Since $U_{d_S} = (U_d)_S$, we have that $A$ is compact (closed) in $(\mathcal{F}(X), d_S)$ if and only if $A$ is compact (closed) in $(\mathcal{F}(X), (U_d)_S)$. On the other hand, $A(0)$ is compact in $(X, d)$ if and only if $A(0)$ is compact in $(X, U_d)$. If we apply Theorem 4.2 to the uniform space $(X, U_d)$, we obtain the required conclusion.

5. ACKNOWLEDGEMENTS

This research was supported by Conacyt, grant: Ciencia de Frontera 64356. The author thank to the anonymous referee for the careful reading of the original manuscript and her/his comments, which helped to improve this work.

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