Tightness of function spaces

SHOU LIN

ABSTRACT. The purpose of this paper is to give higher cardinality versions of countable fan tightness of function spaces obtained by A. Arhangel’ski. Let vet(X), ωH(X) and H(X) denote respectively the fan tightness, ω-Hurewicz number and Hurewicz number of a space X, then vet(Cp(X)) = ωH(X) = sup{H(X^n) : n ∈ N}.

2000 AMS Classification: 54C35; 54A25; 54D20; 54D99.

Keywords: Function spaces; fan tightness; Hurewicz spaces; cardinal functions.

The general question in the theory of function spaces is to characterize topological properties of the space, C(X), of continuous real-valued functions on a topological space X. A study of some convergence properties in function spaces is an important task of general topology. It have been obtained interested results on some higher cardinal properties of first-countability, Fréchet properties, tightness[2, 4, 6, 9]. Arhangel’ski-Pykeev theorem[2] is a nice result about tightness of function spaces: t(Cp(X)) = sup{L(X^n) : n ∈ N} for any Tychonoff space X. The following result on countable fan tightness of function spaces is shown by A. Arhangel’ski[1]: Cp(X) has countable fan tightness if and only if X^n is a Hurewicz space for each n ∈ N for an arbitrary space X. In this paper the higher cardinality versions of countable fan tightness of Cp(X) are obtained.

In this paper all spaces will be Tychonoff spaces. Let α be a network of compact subsets of a space X, which is closed under finite unions and closed subsets. Then the space Ca(X) is the set C(X) with the set-open topology as follows[9]: The subbasic open sets of the form [A, V] = {f ∈ C(X) : f(A) ⊂ V}, where A ∈ α and V is open in R. Then Ca(X) is a topological vector space[9]. The family of all compact subsets of X generates the compact-open topology.

*Supported by the NNSF of China (10271026).
denoted by $C_k(X)$. Also the family of all finite subsets of $X$ generates the topology of pointwise convergence, denoted by $C_p(X)$). For each $f \in C(X)$, a basic neighborhood of $f$ in $C_p(X)$ can be expressed as $W(f, K, \varepsilon)$ for each finite subset $K$ of $X$ and $\varepsilon > 0$, here $W(f, K, \varepsilon) = \{g \in C(X) : |f(x) - g(x)| < \varepsilon \text{ for each } x \in K\}$. In this paper the alphabet $\lambda$ is an infinite cardinal number, $\gamma$ is an ordinal number, and $i, m, n, j, k$ are natural numbers.

The fan tightness of a space $X$ is defined by $\text{vet}(X) = \sup \{\text{vet}(X, x) : x \in X\}$, here $\text{vet}(X, x) = \omega + \min\{\lambda : \text{ for each family } \{A_\gamma\}_{\gamma < \lambda} \text{ of subsets of } X \text{ with } x \in \bigcap_{\gamma < \lambda} A_\gamma, \text{ there is a subset } B_\gamma \subset A_\gamma \text{ with } |B_\gamma| < \lambda \text{ for each } \gamma < \lambda \text{ such that } x \in \bigcup_{\gamma < \lambda} B_\gamma\}$. A space $X$ has countable fan tightness if and only if $\text{vet}(X) = \omega$. An $\alpha$-cover of a space $X$ is a family of subsets of $X$ such that every member of $\alpha$ is contained in some member of this family. An $\alpha$-cover is called a $k$-cover if $\alpha$ is the set of all compact subsets of $X$. Also an $\alpha$-cover is called an $\omega$-cover if $\alpha$ is the set of all finite subsets of $X$. The $\alpha$-Hurewicz number of $X$ is defined by $\alpha H(X) = \omega + \min\{\lambda : \text{ for each family } \{U_\gamma\}_{\gamma < \lambda} \text{ of open } \alpha \text{-covers of } X \text{ there is a subset } B_\gamma \subset U_\gamma \text{ with } |B_\gamma| < \lambda \text{ for each } \gamma < \lambda \text{ such that } \bigcup_{\gamma < \lambda} B_\gamma \text{ is an } \alpha \text{-cover of } X\}$. The $\alpha$-Hurewicz number of $X$ is called the Hurewicz number of $X$ and written $H(X)$ if $\alpha$ consists of the singleton of $X$. A space $X$ is Hurewicz space if and only if $H(X) = \omega$.

**Theorem 1.** $\text{vet}(C_\alpha(X)) = \alpha H(X)$ for any space $X$.

**Proof.** Let $\lambda = \text{vet}(C_\alpha(X))$, and let $\{U_\gamma\}_{\gamma < \lambda}$ be any family of open $\alpha$-covers of $X$. For each $\gamma < \lambda$, put $A_\gamma = \{f \in C_\alpha(X) : \text{ there is } U \in U_\gamma \text{ such that } f(X \setminus U) \subset \{0\}\}$. Then $A_\gamma$ is dense in $C_\alpha(X)$. In fact, let $\bigcap_{i \leq m}[K_i, V_i]$ be a non-empty basic open set of $C_\alpha(X)$, fix $f \in \bigcap_{i \leq m}[K_i, V_i]$. There is $U \in U_\gamma$ such that $\bigcup_{i \leq m} K_i \subset U$ because $U_\gamma$ is an $\alpha$-cover on $X$. Since $\bigcup_{i \leq m} K_i$ is compact in Tychonoff space $X$, there is $g \in C_\alpha(X)$ such that $g|_{\bigcup_{i \leq m} K_i} = f|_{\bigcup_{i \leq m} K_i}$ and $g(X \setminus U) \subset \{0\}$. Then $g \in A_\gamma \cap (\bigcap_{i \leq m}[K_i, V_i])$, and $\bigcap_{\gamma < \lambda} A_\gamma = C_\alpha(X)$.

Take $f_1 \in C(X)$ with $f_1(X) = \{1\}$, then $f_1 \in \bigcap_{\gamma < \lambda} A_\gamma$. For each $\gamma < \lambda$ there is a subset $B_\gamma \subset A_\gamma$ with $|B_\gamma| < \lambda$ such that $f_1 \in \bigcup_{\gamma < \lambda} B_\gamma$ by $\lambda = \text{vet}(C_\alpha(X))$. Denote $B_\gamma = \{f_\kappa\}_{\kappa \in \Phi_\gamma}$, here $|\Phi_\gamma| < \lambda$. There is $U_\kappa \in U_\gamma$ such that $f_\kappa(X \setminus U_\kappa) \subset \{0\}$ for each $\kappa \in \Phi_\gamma$. Put $U_\gamma = \{U_\kappa\}_{\kappa \in \Phi_\gamma}$. Then $\bigcup_{\gamma < \lambda} U_\gamma$ is an $\alpha$-cover of $X$. In fact, for each $A \in \alpha$, since $f_1 \in [A, (0, 2)]$, there are $\gamma$ and $\kappa \in \Phi_\gamma$ such that $f_\kappa \in [A, (0, 2)]$, then $A \subset U_\kappa$, so $\bigcup_{\gamma < \lambda} U_\gamma$ is an $\alpha$-cover of $X$. This shows that $\alpha H(X) \leq \text{vet}(C_\alpha(X))$.

To show the reverse inequality, let $\lambda = \alpha H(X)$. Since $C_\alpha(X)$ is a topological vector space, it is homogeneous. It suffices to show that $\text{vet}(C_\alpha(X), f_0) \leq \lambda$, where $f_0 \in C(X)$ with $f_0(X) = \{0\}$. Suppose that $f_0 \in \bigcap_{\gamma < \lambda} A_\gamma$ with each $A_\gamma \subset C_\alpha(X)$. For each $\gamma < \lambda$ and $n \in \mathbb{N}$, put $U_{\gamma, n} = \{f^{-1}(O_n) : f \in A_\gamma\}$, here $\{O_n\}_{n \in \mathbb{N}}$ is a decreasing local base of $0$ in $\mathbb{R}$. Then $U_{\gamma, n}$ is an open $\alpha$-cover of $X$. In fact, for each $A \in \alpha, f_0 \in [A, O_n]$, there is $f \in [A, O_n] \cap A_\gamma$, thus $A \subset f^{-1}(O_n) \in U_{\gamma, n}$.
Case 1. $\lambda > \omega$. For each $n \in \mathbb{N}$, since $\{U_{\gamma,n}\}_{\gamma<\lambda}$ is a family of open covers of $X$, there is a subset $U'_{\gamma,n} \subset U_{\gamma,n}$ with $|U'_{\gamma,n}| < \lambda$ for each $\gamma < \lambda$ such that $\bigcup_{\gamma<\lambda} U'_{\gamma,n}$ is an open $\alpha$-cover of $X$. Denote $U'_{\gamma,n} = \{U_\tau : \tau \in \Phi_{\gamma,n}\}$. There is $f_\tau \in A_\gamma$ such that $U_\tau = f_\tau^{-1}(O_{\tau})$ for each $\tau \in \Phi_{\gamma,n}$. Let $B_{\gamma} = \{f_\tau : \tau \in \Phi_{\gamma,n}, n \in \mathbb{N}\}$. Then $B_\gamma \subset A_\gamma$ and $|B_\gamma| < \lambda$. We show that $f_0 \in \bigcup_{\gamma<\lambda} B_\gamma$.

For arbitrary basic neighborhood $[A,V]$ of $f_0$ in $C_\alpha(X)$, there is $n \in \mathbb{N}$ such that $O_n \subset V$. Since $\bigcup_{\gamma<\lambda} U'_{\gamma,n}$ is an open $\alpha$-cover of $X$, there are $\gamma < \lambda$ and $\tau \in \Phi_{\gamma,n}$ such that $A \subset U_\tau = f_\tau^{-1}(O_{\tau})$, hence $f_\tau(A) \subset V$, i.e., $f_\tau \in [A,V]$, so $f_0 \in \bigcup_{\gamma<\lambda} B_\gamma$.

Case 2. $\lambda = \omega$. Put $M = \{n \in \mathbb{N} : X \in U_{n,n}\}$. If $M$ is infinite, there is $m \in M$ such that $m \subset V$ for arbitrary basic neighborhood $[A,V]$ of $f_0$ in $C_\alpha(X)$. By the definition of $U_{m,m}$, there is $g_m \in A_m$ such that $X = g_m^{-1}(O_m)$, then $g_m(X) \subset V$, so $g_m \in [A,V]$, thus the sequence $\{g_m\}_{m \in M}$ converges to $f_0$. If $M$ is finite, there is $n_0 \in \mathbb{N}$ such that for each $m \geq n_0$ and $g \in A_m$, $g^{-1}(O_m) \neq X$. Since $\{U_{m,m}\}_{m \geq n_0}$ is a sequence of open $\alpha$-covers of $X$, there is a finite subset $U''_{m,m}$ of $U_{m,m}$ for each $m \geq n_0$ such that $\bigcup_{m \geq n_0} U''_{m,m}$ is an open $\alpha$-cover of $X$. Denote $U''_{m,m} = \{U_{m,j}\}_{j \leq i(m)}$. There is $f_{m,j} \in A_m$ such that $U_{m,j} = f_{m,j}^{-1}(O_{m,j})$ for each $m \geq n_0, j \leq i(m)$. Next, we shall show that $f_0 \in \{f_{m,j} : m \geq n_0, j \leq i(m)\}$. For arbitrary basic neighborhood $[A,V]$ of $f_0$ in $C_\alpha(X)$, let $F = \{(m,j) \in \mathbb{N}^2 : m \geq n_0, j \leq i(m)\}$ and $A \subset U_{m,j}$. Obviously, $F \neq \emptyset$. If $F$ is finite, take $x_{m,j} \in X \setminus U_{m,j}$ for each $(m,j) \in F$ because $U_{m,j} \neq X$. There is $K \subset \omega$ with $A \cup \{x_{m,j} : (m,j) \in F\} \subset K$. Then $K$ is not contained by any element of $\bigcup_{m \geq n_0} U''_{m,m}$, so $\bigcup_{m \geq n_0} U''_{m,m}$ is not an $\alpha$-cover of $X$, a contradiction. Hence $F$ is infinite, and there are $m \geq n_0$ and $j \leq i(m)$ such that $A \subset U_{m,j} = f_{m,j}^{-1}(O_{m,j})$ and $O_m \subset V$, so $f_{m,j}(A) \subset V$, i.e., $f_{m,j} \in [A,V]$.

Thus $f_0 \in \{f_{m,j} : m \geq n_0, j \leq i(m)\}$.

This shows that $\text{vet}(C_\alpha(X)) \leq \alpha H(X)$. \hfill \Box

By Theorem 1, $C_p(X)$ has countable fan tightness if and only if for each sequence $\{U_n\}$ of open $\omega$-covers of $X$ there is a finite subset $U'_n \subset U_n$ for each $n \in \mathbb{N}$ such that $\bigcup_{n \in \mathbb{N}} U'_n$ is an $\omega$-cover of $X$.

Theorem 2. $\text{vet}(C_p(X)) = \sup \{H(X^n) : n \in \mathbb{N}\}$ for any space $X$.

Proof. Let $\lambda = \text{vet}(C_p(X))$ and $n \in \mathbb{N}$. Suppose that $\{U_{\gamma}\}_{\gamma<\lambda}$ is a family of open covers of the space $X^n$. For each $\gamma < \lambda$, a family $V$ of subsets of $X$ is called having a property $P_{n,\gamma}$ if for each $\{V_i\}_{i \leq \gamma} \subset V$ there is $U \in U_\gamma$ such that $\prod_{i \leq \gamma} V_i \subset U$. Denote by $\Gamma_{n,\gamma}$ the family of all finite sets, which has the property $P_{n,\gamma}$, of open sets in $X$. For each $\gamma \in \Gamma_{n,\gamma}$, let $F_{\gamma} = \{f \in C_p(X) : f(X \setminus \bigcup V) \subset \{0\}\}$. We show that the set $A_{\gamma} = \bigcup F_{\gamma} \in \Gamma_{n,\gamma}$ is dense in $C_p(X)$.

Let $W(f,K,\varepsilon)$ be any basic neighborhood of $f$ in $C_p(X)$. Since $K$ is finite, there is a finite family $W$ of open subsets in $X$ such that for any $(x_1,x_2,\ldots,x_n) \in K^n$ there are $U \in \mathcal{U}_n$ and a finite subset $\{W_i\}_{i \leq n} \subset W$ such that $(x_1,x_2,\ldots,x_n) \in \prod_{i \leq n} W_i \subset U$. Then $K \subset \bigcup W$. For each $x \in K$,
put $V_x = \bigcap\{W \in \mathcal{W}_i : x \in W\}$, and $V = \{V_x : x \in K\}$. Then $K \subset \bigcup V$ and the family $V$ has the property $P_{n,\gamma}$. In fact, take an arbitrary $(x_1, x_2, ..., x_n) \in K^n$, there are $\{W_i\}_{i \leq n} \subset \mathcal{W}$ and $U \in \mathcal{U}$ such that $(x_1, x_2, ..., x_n) \in \bigcap_{i \leq n} W_i \subset U$. Since each $V_{x_i} \subset W_i$, $\prod_{i \leq n} V_{x_i} \subset U$. Now, take $g \in C_p(X)$ such that $f_{IK} = g|_K$ and $g(X \setminus \bigcup V) = \{0\}$, then $g \subset V \subset A_\gamma$, so $W(f, K, \varepsilon) \cap A_\gamma \neq \emptyset$. Thus $\mathcal{A}_\gamma = C_p(X)$.

Let $f_1 \in C(X)$ with $f_1(X) = \{1\}$. Then $f_1 \in \bigcap_{\gamma \leq \lambda} \mathcal{A}_\gamma$. There is a subset $B_\gamma \subset A_\nu$ with $|B_\gamma| < \lambda$ for each $\gamma < \lambda$ such that $f_1 \in \bigcup_{\gamma \leq \lambda} B_\gamma$. Then there is a subset $\Delta_{\gamma, \nu} \subset \Gamma_{\gamma, \nu}$ with $|\Delta_{\gamma, \nu}| < \lambda$ such that $B_\gamma \subset \bigcup \{\{f\} : \gamma \in \Delta_{\gamma, \nu}\}$. Let $\mathcal{V} \in \Delta_{\gamma, \nu}$. For each $\xi = (V_1, V_2, ..., V_n) \in \mathcal{V}$, take $G_\xi \in \mathcal{U}_\xi$ such that $\bigcap_{i \leq n} V_i \subset G_\xi$. Put $G_\nu = \{G_\xi : \xi \in \mathcal{V}, \mathcal{V} \in \Delta_{\gamma, \nu}\}$. Clearly, $|G_\nu| < \lambda$ and $G_\nu \subset \mathcal{U}_\xi$. We show that $\bigcup_{\gamma \leq \lambda} G_\gamma$ covers $X$.

For an arbitrary $(x_1, x_2, ..., x_n) \in X^n$, let $F = \{f \in C_p(X) : f(x_i) > 0$ for each $i \leq n\}$. Then $F$ is an open neighborhood of $f_1$ in $C_p(X)$. Since $f_1 \in \bigcup_{\gamma \leq \lambda} B_\gamma$, there is $\gamma < \lambda$ such that $F \cap B_\gamma \neq \emptyset$. Then $F \cap B_\gamma \neq \emptyset$ for some $\mathcal{V} \in \Delta_{\gamma, \nu}$. Let $g \in F \cap B_\gamma$. Then $g(X \setminus \bigcup \mathcal{V}) = 0$ and $g(x_i) > 0$ for each $i \leq n$. Take $V_i \in \mathcal{V}$ such that $x_i \in V_i$ for each $i \leq n$, then there is $G_\xi \in G_\nu$ such that $(x_1, x_2, ..., x_n) \in \bigcap_{i \leq n} V_i \subset G_\xi$. So $(x_1, x_2, ..., x_n) \in \bigcup_{\gamma \leq \lambda} G_\gamma$. Hence $H(X^n) \leq \text{vet}(C_p(X))$.

Conversely, suppose $\lambda = \sup\{H(X^n) : n \in \mathbb{N}\}$. Fix $f \in C_p(X)$ and a family $\{A_\gamma\}_{\gamma \leq \lambda}$ of subsets in $C_p(X)$ such that $f \in \bigcap_{\gamma \leq \lambda} A_\gamma$. For each $n \in \mathbb{N}$, $\gamma < \lambda$ and $x = (x_1, x_2, ..., x_n) \in X^n$, there is $g_{x, \gamma} \in W(f, \{x_1, x_2, ..., x_n\}, 1/n) \bigcap A_\gamma$. For each $i \leq n$, since $|g_{x, \gamma}(x_i) - f(x_i)| < 1/n$, by the continuity of $f$ and $g_{x, \gamma}$, there is an open neighborhood $O_i$ of $x_i$ in $X$ such that $|g_{x, \gamma}(y_i) - f(y_i)| < 1/n$ if $y_i \in O_i$. The set $U_{x, \gamma} = \bigcap_{i \leq n} O_i$ is a neighborhood of $x$ in $X^n$. Thus $U_{x, \gamma} = \{U_{x, \gamma} : x \in X^n\}$ covers $X^n$, and $|g_{x, \gamma}(y_i) - f(y_i)| < 1/n$ for each $(y_1, y_2, ..., y_n) \in U_{x, \gamma}$.

**Case 1.** $\lambda > \omega$. Since $H(X^n) \leq \lambda$, there is a family $\{S_{n, \gamma}\}_{\gamma \leq \lambda}$ of subsets in $\mathcal{S}_n$ with $|S_{n, \gamma}| < \lambda$ for each $\gamma < \lambda$ such that $\bigcup_{\gamma \leq \lambda} S_{n, \gamma}$ covers $X^n$, here each $S_{n, \gamma} = \{U_{x, \gamma} : x \in S_{n, \gamma}\}$. For each $\gamma < \lambda$, let $B_{n, \gamma} = \{g_{x, \gamma} : x \in S_{n, \gamma}\}$, and $B_\gamma = \bigcup_{n \in \mathbb{N}} B_{n, \gamma}$. Then $B_\gamma \subset A_\gamma$, $|B_\gamma| < \lambda$, and $f \in \bigcup_{\gamma \leq \lambda} B_\gamma$.

In fact, let $W(f, \{y_1, y_2, ..., y_n\}, \varepsilon)$ be a basic neighborhood of $f$ in $C_p(X)$ with $1/n < \varepsilon$. There is $\gamma < \lambda$ such that $(y_1, y_2, ..., y_n) \in \bigcup S_{n, \gamma}$, thus there is $x \in S_{n, \gamma}$ such that $(y_1, y_2, ..., y_n) \in U_{x, \gamma}$, so $g_{x, \gamma}(y_i) - f(y_i) < 1/n < \varepsilon$ for each $i \leq n$, hence $g_{x, \gamma} \in W(f, \{y_1, y_2, ..., y_n\}, \varepsilon) \cap B_\gamma$. This shows that $f \in \bigcup_{\gamma \leq \lambda} B_\gamma$.

**Case 2.** $\lambda = \omega$. Replace $\gamma < \lambda$ by $k \geq n$, and let $B_k = \bigcup_{n \leq k} B_{n, k}$ in the proof of Case 1, then $B_k$ is finite subset of $A_k$ and $f \in \bigcup_{k \in \mathbb{N}} B_k$.

In a word, vet($C_p(X)$) $\leq \sup\{H(X^n) : n \in \mathbb{N}\}$.

The following result obtained by A. Arhangel’skii[1] is generalized: $C_p(X)$ has countable fan tightness if and only if $X^n$ is a Hurewicz space for each $n \in \mathbb{N}$. \qed
REFERENCES


RECEIVED AUGUST 2004

ACCEPTED MAY 2005

SHOU LIN (linshou@public.ndptt.fj.cn)
Department of Mathematics, Zhangzhou Teachers’ College, Fujian 363000, P. R. China

Department of Mathematics, Ningde Teachers’ College, Fujian 352100, P. R. China