

## Topological groups: local versus global

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**ABSTRACT.** It is well known that locally compact groups are paracompact. We observe that this theorem can be generalized as follows: every locally paracompact group is paracompact. We prove a more general version of this statement using quotients. Similar ‘local implies global’ theorems hold also for many other properties, such as normality, metacompactness, stratifiability, etc.

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### 1. INTRODUCTION

It is well known that *every locally compact group is paracompact*<sup>1</sup>. Recall the easy proof. Let  $G$  be a locally compact group, and let  $V$  be a symmetric compact neighborhood of the neutral element. Then  $H = \cup_n V^n$  is an open  $\sigma$ -compact subgroup of  $G$ , and  $G$  is the disjoint union of closed-and-open  $\sigma$ -compact subsets – the left cosets with respect to  $H$ . It follows that  $G$  is strongly paracompact [3, Corollary 5.3.11]. Recall that a space is *strongly paracompact* if every open cover has a star-finite open refinement.

The aim of the present paper is to generalize the above assertion in several directions. In particular, we observe that every locally paracompact group is paracompact (Corollary 1.2), and every locally Lindelöf group has an open Lindelöf subgroup and hence is strongly paracompact.

Let  $\mathcal{P}$  be a class of topological spaces. We say that  $\mathcal{P}$  is *closed hereditary* if every closed subspace of every  $X \in \mathcal{P}$  is in  $\mathcal{P}$ . The notion of a class that is *closed under locally finite unions of closed sets* is self-explanatory. Note that every class which is closed under disjoint sums and (continuous) closed mappings is closed under locally finite unions of closed sets. Indeed, if  $X = \cup \gamma$

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<sup>1</sup>All spaces that we consider are assumed to be Tychonoff. For topological groups, Tychonoff is equivalent to Hausdorff.

and  $\gamma$  is a locally finite closed cover, then the natural map  $\oplus\gamma \rightarrow X$  is closed and onto.

A space  $X$  is *locally in  $\mathcal{P}$*  if every point in  $X$  has a (not necessarily open) neighborhood which belongs to  $\mathcal{P}$ .

**Theorem 1.1.** *Let  $\mathcal{P}$  be any class of topological spaces which is closed hereditary and closed under locally finite unions of closed sets. Then every topological group which is locally in  $\mathcal{P}$  is in  $\mathcal{P}$ .*

**Corollary 1.2.** *Every locally paracompact topological group is paracompact.*

Theorem 1.1 can be similarly applied in many other cases. For example, we have the following two results:

**Corollary 1.3.** *Every locally normal topological group is normal.*

**Corollary 1.4.** *Every locally stratifiable topological group is stratifiable.*

Similarly, every locally metacompact group is metacompact, every locally subparacompact group is subparacompact; etc. However, it is not clear if the analogue of Corollary 1.2 is true for strongly paracompact groups.

**Question 1.5.** *Is every locally strongly paracompact topological group strongly paracompact?*

The conclusion of Theorem 1.1 remains valid for some classes  $\mathcal{P}$  which are not closed under locally finite (or just finite) unions. For example, the union of two closed Dieudonné-complete subspaces need not be Dieudonné-complete [5] (recall that Dieudonné-complete is equivalent to realcompact for spaces of cardinality less than the first measurable cardinal). However, the counterpart of Theorem 1.1 holds for this class:

**Theorem 1.6.** *Let  $\mathcal{P}$  be the class of all Dieudonné-complete spaces. Then every topological group which is locally in  $\mathcal{P}$  is in  $\mathcal{P}$ .*

A locally Lindelöf group need not be Lindelöf (consider a discrete uncountable group). However, we have the following:

**Theorem 1.7.** *Every locally Lindelöf group contains an open Lindelöf subgroup and is strongly paracompact.*

Note that the easy argument that worked for locally compact groups does not work in the more general setting of locally Lindelöf groups, since the Lindelöf property is not preserved by products.

## 2. QUOTIENTS $G/H$ , WHERE $H$ IS LOCALLY COMPACT

We give some applications of the above results to quotients of topological groups.

A mapping  $f : X \rightarrow Y$  is *locally perfect* if every  $x \in X$  has a closed neighborhood  $N$  such that  $f|N : N \rightarrow Y$  is perfect (in particular,  $f(N)$  is closed in  $Y$ ). It was established in [2] that, for any locally compact subgroup  $H$  of

arbitrary topological group  $G$ , the natural quotient mapping  $\pi$  of  $G$  onto the quotient space  $G/H$  is locally perfect. This turns out to be a key result in the proof that a number of topological properties are transferred from the quotient space  $G/H$  to the topological group  $G$ , provided that  $H$  is locally compact. One of such statements is the classical result of J.-P. Serre that if  $H$  and  $G/H$  are locally compact, then the topological group  $G$  also is locally compact [7]. For example, it was shown in [2] in this way that if  $G/H$  is Čech-complete (or feathered), and the subgroup  $H$  is locally compact, then  $G$  is Čech-complete (respectively, feathered) as well. Now we have a tool to prove similar statements for paracompactness, normality, Dieudonné completeness, and for several other properties.

Indeed, the following general statement holds [2]:

**Lemma 2.1.** *Let  $\mathcal{P}$  be a topological property preserved by preimages of spaces under perfect mappings (in the class of Tychonoff spaces) and also inherited by closed sets. Suppose further that  $G$  is a topological group and  $H$  is a locally compact subgroup of  $G$  such that the quotient space  $G/H$  has the property  $\mathcal{P}$ . Then there exists an open neighbourhood  $U$  of the neutral element  $e$  such that  $\overline{U}$  has the property  $\mathcal{P}$ .*

The next theorem immediately follows from Corollary 1.2 and from Lemma 2.1.

**Theorem 2.2.** *Suppose that  $G$  is a topological group, and  $H$  is a locally compact subgroup of  $G$  such that the quotient space  $G/H$  is paracompact. Then the space  $G$  is paracompact.*

Similarly, the results obtained above imply the following statements:

**Theorem 2.3.** *Suppose that  $G$  is a topological group, and  $H$  is a locally compact subgroup of  $G$  such that the quotient space  $G/H$  is Dieudonné complete. Then the space  $G$  is Dieudonné complete.*

**Theorem 2.4.** *Suppose that  $G$  is a topological group, and  $H$  is a locally compact subgroup of  $G$  such that the quotient space  $G/H$  is metacompact (subparacompact). Then the space  $G$  is metacompact (subparacompact).*

**Question 2.5.** *Suppose that  $G$  is a topological group, and that  $H$  is a locally compact subgroup of  $G$  such that the quotient space  $G/H$  is strongly paracompact. Is then the space  $G$  strongly paracompact?*

Clearly, this question is closely related to Question 1.5. Note that if  $G/H$  in Question 2.5 is assumed to be Lindelöf, then  $G$  is locally Lindelöf (Lemma 2.1) and hence strongly paracompact (Theorem 1.7). Note that for locally connected groups the answer for Questions 1.5 and 2.5 is in the affirmative, since connected strongly paracompact spaces are Lindelöf.

**Question 2.6.** *Suppose that  $G$  is a topological group, and that  $H$  is a locally compact metrizable subgroup of  $G$  such that the quotient space  $G/H$  is stratifiable. Is then the space  $G$  stratifiable?*

## 3. PROOFS

**Proposition 3.1.** *Let  $G$  be a topological group with the neutral element  $e$ , and let  $U$  be a neighborhood of  $e$ . There exist a metric space  $(M, d)$  on which  $G$  acts continuously and transitively by isometries and a neighborhood  $O$  of  $a = p(e)$  in  $M$  such that  $p^{-1}(O) \subset U$ , where  $p : G \rightarrow M$  is the map defined by  $p(g) = ga$ .*

*Proof.* This follows from the fundamental fact that every topological group can be embedded in the group of isometries of a metric space, see e.g. [8]. See [6] for the history of this assertion that was rediscovered many times by various authors.

There also is an easy direct argument for the proposition. There exists a continuous left-invariant pseudometric  $\rho$  on  $G$  such that  $\{x \in G : \rho(x, e) < 1\} \subset U$ . This follows from the fact that every topological group has natural uniform structures [3, Example 8.1.17]. Let  $(M, d)$  be the metric space associated with the pseudometric space  $(G, \rho)$ .  $\square$

*Proof of Theorem 1.1.* Let  $\mathcal{P}$  be a class which is closed hereditary and closed under locally finite unions of closed sets. Let  $G$  be a topological group which is locally in  $\mathcal{P}$ . We must prove that  $G$  is in  $\mathcal{P}$ .

Let  $U$  be a neighborhood of  $e$  such that  $U \in \mathcal{P}$ . In virtue of Proposition 3.1, there exists a metric space  $(M, d)$  on which  $G$  acts continuously and transitively by isometries, a point  $a \in M$  and an open neighborhood  $O$  of  $a$  in  $M$  such that  $p^{-1}(O) \subset U$ , where  $p : G \rightarrow M$  is the map defined by  $p(g) = ga$ . Since  $M$  is paracompact, there exists a locally finite closed cover  $\mathcal{F}$  of  $M$  which refines the open cover  $\{gO : g \in G\}$ . Consider the closed cover  $\mathcal{H} = p^{-1}(\mathcal{F}) = \{p^{-1}(F) : F \in \mathcal{F}\}$  of  $G$ . Each element of this cover is a closed subset of the set of the form  $gU$ . Since  $gU$  is homeomorphic to  $U$ ,  $U \in \mathcal{P}$ , and  $\mathcal{P}$  is closed hereditary, it follows that  $\mathcal{H} \subset \mathcal{P}$ . Since  $\mathcal{H}$  is locally finite and  $\mathcal{P}$  is closed under locally finite unions of closed sets, we have  $G \in \mathcal{P}$ .  $\square$

To prove Theorem 1.6, we need some preparations. For a space  $X$  we denote by  $\mathcal{U}_X$  the *fine uniformity* on  $X$ , that is, the finest compatible uniformity. A space  $X$  is Dieudonné-complete iff the uniform space  $(X, \mathcal{U}_X)$  is complete.

**Lemma 3.2.** *Let  $A \subset B$  be subsets of a topological space  $X$  such that  $A$  and  $X \setminus B$  are functionally separated. Then the uniformities  $\mathcal{U}_X$  and  $\mathcal{U}_B$  induce the same uniformity on  $A$ .*

*Proof.* It suffices to prove that for every continuous pseudometric  $d$  on  $B$  there exists a continuous pseudometric  $d_1$  on  $X$  such that  $d(x, y) = d_1(x, y)$  for all  $x, y \in A$ . There exists a normed linear space  $E$  and a map  $p : B \rightarrow E$  such that  $d(x, y) = \|p(x) - p(y)\|$ . Let  $f : X \rightarrow [0, 1]$  be a function such that  $f = 1$  on  $A$  and  $f = 0$  on a neighborhood of  $X \setminus B$ . Define  $q : X \rightarrow E$  as follows:  $q(x) = f(x)p(x)$  if  $x \in B$ ;  $q(x) = 0$  if  $x \in X \setminus B$ . Then  $q$  is continuous and  $q = p$  on  $A$ . The pseudometric  $d_1$  on  $X$  defined by  $d_1(x, y) = \|q(x) - q(y)\|$  is as required.  $\square$

**Proposition 3.3.** *Let  $f : X \rightarrow Y$  be an onto map. Suppose that  $Y$  is Dieudonné-complete, and every point in  $Y$  has a neighbourhood  $N$  such that  $f^{-1}(N)$  is Dieudonné-complete. Then  $X$  is Dieudonné-complete.*

*Proof.* Let  $\mathcal{F}$  be a Cauchy filter on  $(X, \mathcal{U}_X)$ . We must prove that  $\mathcal{F}$  converges. Since  $f(\mathcal{F})$  is a Cauchy filter on the complete space  $(Y, \mathcal{U}_Y)$ , it has a limit  $y \in Y$ . Let  $N$  be a neighborhood of  $y$  such that  $B = f^{-1}(N)$  is Dieudonné-complete. Find a closed neighborhood  $K$  of  $y$  such that  $K \subset N$  and  $K$  and  $Y \setminus N$  are functionally separated. Let  $A = f^{-1}(K)$ . According to Lemma 3.2,  $\mathcal{U}_X|A = \mathcal{U}_B|A$ . Since  $A$  is closed in the complete space  $(B, \mathcal{U}_B)$ , the uniform space  $(A, \mathcal{U}_X|A) = (A, \mathcal{U}_B|A)$  is complete. Every member of  $\mathcal{F}$  meets  $A$ , because every member of  $f(\mathcal{F})$  meets  $K$ . It follows that  $\mathcal{F}$  induces a Cauchy filter on the complete space  $(A, \mathcal{U}_X|A)$  and hence converges.  $\square$

*Proof of Theorem 1.6.* Let  $G$  be a topological group such that some neighborhood  $U$  of the neutral element  $e$  is Dieudonné-complete. We prove that  $G$  is Dieudonné-complete.

In virtue of Proposition 3.1, there exists a metric space  $M$  on which  $G$  transitively acts and a closed neighborhood  $N$  of  $a = p(e)$  in  $M$  such that  $p^{-1}(N) \subset U$ , where  $p : G \rightarrow M$  is the map defined by  $p(g) = ga$ . Every point  $x \in M$  has a neighbourhood  $K$  such that  $p^{-1}(K)$  is Dieudonné-complete. Indeed, write  $x = ga$  for some  $g \in G$ , and put  $K = gN$ . Then  $p^{-1}(K) = gp^{-1}(N)$  is Dieudonné-complete, since  $p^{-1}(N)$  is closed in  $U$ . Proposition 3.3 implies that  $G$  is Dieudonné-complete.  $\square$

*Proof of Theorem 1.7.* It suffices to note that every topological group  $G$  generated by a Lindelöf neighborhood of its neutral element is Lindelöf. This follows from Guran's theorem [4]: if  $G$  is generated by a Lindelöf set, then for every neighborhood  $U$  of  $e$  the cover  $\{gU : g \in G\}$  has a countable subcover. See e.g. [8] for the proof.  $\square$

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