Criteria of strong nearest-cross points and strong best approximation pairs

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ABSTRACT. The concept of strong nearest-cross point (strong n.c. point) is introduced, which is the generalization of strong uniqueness of best approximation from a single point. The relation connecting to localization is discussed. Some criteria of strong n.c. points are given. The strong best approximation pairs are also studied.


Keywords: strong nearest-cross point, local strong nearest-cross point, strong best approximation pair.

1. Introduction

In [6], [5], [9] the first author of the paper studied the nearest cross points (in short, n.c. points) of two subsets of a normed space. More precisely, let G and F be two disjoint subsets of a normed space X. A point $y_0 \in G$ is called a n.c. point of G to F if $\rho(y_0, F) = \rho(G, F)$, where $\rho(G, F) = \inf_{y \in G, x \in F} \rho(y, x)$, $\rho(y, x) = \|x - y\|$ is the norm of $x - y$ in the space X. Moreover, if $x_0 \in F$ satisfies $\rho(x_0, y_0) = \rho(F, G)$, we say that $(x_0, y_0)$ is a best approximation pair of F and G. For details, one can see [4]. Obviously, if $(x_0, y_0)$ is a best approximation pair of F and G, then $y_0$ is a n.c. point of G to F, and $x_0$ is the best approximation of $y_0$ from F. The analogous result for $x_0$ also holds. However, the inverse is not true. If both n.c. points of F to G and G to F exist, a best approximation pair of F and G may not exist. But if n.c. points of G to F exist and F is a proximal set, then the best approximation pair of F and G exists. In [5], the author discussed the uniqueness of n.c. points (if it exists) and obtained that the n.c. point of G to F is unique if G is strict convex and F is convex. In this paper, we shall discuss a property which is stronger than being a n.c. point, which we will call a strong n.c. point. A strong n.c. point is

*The corresponding author Jingshi Xu was supported by the NNSF (No. 60474070) of China
the generalization of strong best approximation in a single best approximation problem. For strong best approximation, one can see [3], [7], [2] in detail.

The organization of the paper is as follows. In Section 2, we will give the definition of strong n.c. point. In Section 3, we shall discuss the criteria of strong nearest cross point. In Section 4, we shall discuss strong n.c. points and strong best approximation pairs by way of the concept of cusp. And we shall give more examples about the relation between strong best approximation pairs and strong n.c. points.

Finally, we declare that we will work in complex norm spaces in this paper and use the following notation. Let $X^*$ be the dual space of $X$. Let $u$ be a point of $u$. If $F$ denotes a subset of a normed space $X$, then $\|F\| = \sup_{x \in F} \|x\|$.

2. Definition of strong n.c. point

**Definition 2.1.** Let $F$ and $G$ be two disjoint sets, $y_0 \in G$ and a constant $r$, $0 < r < 1$. If the condition

$$\rho(y, F) - \rho(y_0, F) \geq r \rho(y, y_0)$$

holds for every $y \in G$, then $y_0$ is called a strong n.c. point of $G$ to $F$.

Notice that a strong nearest point is a n.c. point. In fact, since $\rho(y, F) - \rho(y_0, F) \geq 0$ for every $y \in G$, we have $\rho(G, F) = \rho(y_0, F)$. Then, for every $y \in G$, $y \neq y_0$, $\rho(y, F) > \rho(y_0, F)$, thus the n.c. point is unique. In Definition 2.1, the constant $r, r < 1$ holds automatically because $|\rho(y, F) - \rho(y', F)| \leq \rho(y, y')$ always holds. In fact, to say $y_0$ is a strong n.c. point, it suffices to remark that it exists a sufficiently small $r$, such that (1) holds. If $F$ is a singleton $x_0$, $y_0$ is a strong n.c. point of $G$ to $F$, then $y_0$ is the strong (unique) best approximation of $x_0$ from $G$; see [3], [7], [2].

**Definition 2.2.** Consider $F$, $G$, $r$, $y_0$ as in Definition 2.1. If (1) holds only for $y \in V_0 \cap G$, $V_0$ a neighborhood of $y_0$, then we say $y_0$ is a local strong n.c. point of $G$ to $F$.

Obviously, if $y_0$ is a strong n.c. point then $y_0$ is a local strong n.c. point, but the converse does not hold in general, as the following example shows.

**Example 2.3.** In the Euclidean space $\mathbb{R}^2$, let $F = \{(\xi, \eta) : (\xi - 2)^2 + \eta^2 = 1, \xi \geq 2\}$, $G = \{(\xi, \eta) : \xi^2 + \eta^2 = 1, \xi \leq 0\}$. If $y_0 = (0, 1)$, then $y_0$ is a local strong n.c. point of $G$ to $F$, since for every $y \in G$, $y$ be near $y_0$, $\rho(y, F)$ is equivalent to $\rho(y, x_0) = 2 + \rho(y, y_0)$. $x_0 = (2, 1)$. So $\rho(y, F) - \rho(y_0, F)$ is equivalent to $\rho(y, y_0)$. But $y_0$ is not a strong n.c. point. In fact, choose $y'$ suffice to $(0, -1)$, then $\rho(y_0, y')$ is to 2, $\rho(y_0, F) = 2$, thus $\rho(y', F) - \rho(y_0, F)$ converges to 0, so (1) does not hold. Moreover, (2,-1) is a n.c. point of $F$ to $G$, $(0,1)$ is a nearest cross point of $G$ to $F$, and (2,-1), (0,1) are not strong n.c. points.

In the above example, $F$ and $G$ are not convex sets, but under convexity, we shall have different results. To state our results, we need the following lemma, which is well known.
**Lemma 2.4.** Let \((X, \rho)\) be a metric space and let \(F \subset X\). Then the function \(\rho(\cdot, F)\) is uniformly continuous. Moreover if \(F\) is a convex set, then \(\rho(\cdot, F)\) is a convex function.

**Theorem 2.5.** Let \(F, G\) be convex sets, and let \(y_0 \in G, y_0\) is a strong nearest cross point of \(G\) to \(F\) if and only if \(y_0\) is a local strong n.c. point of \(G\) to \(F\).

**Proof.** As above stated, we only need to show that if \(y_0\) is a local strong n.c. point, then \(y_0\) is a strong n.c. point. Let \(V\) be a neighborhood of \(y_0\), where \(y_0\) is a strong n.c. point of \(G \cap V\) to \(F\). If \(y_0\) is not a strong n.c. point of \(G\) to \(F\), then for every \(r_n \to 0\), there exist \(y_n \in G, n = 1, 2, \ldots\), such that \(\rho(y_n, F) - \rho(y_0, F) < r_n\|y_n - y_0\|\). In the segment \([y_n, y_0]\), pick \(z_n = \lambda_n y_n + (1 - \lambda_n) y_0\), \(0 < \lambda_n < 1, \lambda_n \to 0\), such that for \(n\) sufficiently large, \(z_n \in G \cap V\). By Lemma 2.4, \(\rho(y, F)\) is a convex function. Thus,

\[
\rho(z_n, F) \leq \lambda_n \rho(y_n, F) + (1 - \lambda_n) \rho(y_0, F)
\]

\[
< \lambda_n r_n \|y_n - y_0\| + \lambda_n \rho(y_0, F) + (1 - \lambda_n) \rho(y_0, F)
\]

\[
= \lambda_n r_n \|y_n - y_0\| + \rho(y_0, F).
\]

So \(\rho(z_n, F) - \rho(y_0, F) < \lambda_n r_n \|y_n - y_0\|\). Since \(\|y_n - y_0\| = 1/\lambda_n\|z_n - y_0\|\), \(\rho(z_n, F) - \rho(y_0, F) \leq r_n\|z_n - y_0\|\), \(z_n \in G \cap V\). This contradicts the definition of a local strong n.c. point. This completes the proof. \(\square\)

In Theorem 2.5 we suppose \(F\) and \(G\) are convex sets. If one of them is not convex, then the result does not hold.

**Example 2.6.** In the Euclidean space \(\mathbb{R}^2\), let \(G = \{(\xi, \eta) : \xi^2 + \eta^2 = 1, \xi \leq 0\}\). Notice that \(G\) is not convex and \(F = \{(2, 0)\}\) is a singleton. Then \(y_0 = (0, 1)\) is a local strong n.c. point of \(G\) to \(F\), but \(y_0\) is not a strong n.c. point of \(G\) to \(F\). In fact, pick \(y = (0, -1)\). Then \(\rho(y, F) - \rho(y_0, F) = \sqrt{5} - \sqrt{5} = 0\), but \(\|y - y_0\| = 2\).

**Example 2.7.** In the Euclidean space \(\mathbb{R}^2\), let \(G = \{(\xi, \eta) : -3 \leq \xi \leq 2, \eta = 0\}\) be a convex set, indeed, a segment, and \(F = \{(\xi, \eta) : \xi^2 + \eta^2 = 25\}\) a non-convex set. Then \(y_0 = (2, 0)\) is a local strong n.c. point, but \(y_0\) is not a strong n.c. point, even if it is not a n.c. point. In fact, \((-3, 0)\) is a strong n.c. point.

3. **Kolmogorov type and differential type criteria of strong n.c. points**

In [6], the author gave sufficient and necessary conditions for a point to be a n.c. point by means of linear functions and differentials. Following the same idea we first obtain a sufficient condition for strong n.c. points.

**Theorem 3.1.** Let \(F, G\) be disjoint sets, \(y_0 \in G\), and \(r\) a constant, \(0 < r < 1\). If \(y_0\) satisfies one of the following conditions, then \(y_0\) is a strong n.c. point of \(G\) to \(F\).

(i) For every \(\epsilon > 0\), there exists \(f^* \in X^*, \|f^*\| = 1\), such that \(\inf_{x \in F} Re f^*(x - y) = \rho(y_0, F)\) and for every \(y \in G\), \(Re f^*(y_0 - y) + \epsilon \geq r\|y_0 - y\|\) holds.
Theorem 3.2. Let $F, G$ be two disjoint convex sets of $X$. If $y_0 \in G$ is a strong n.c. point of $G$ to $F$, then

$$ (D_s) \quad \rho_{x}^{+}(y_0, y - y_0, F) \geq r\|y - y_0\| $$

for all $y \in G$, $r > 0$.

Proof. From Definition 2.1, $\rho(y, F) - \rho(y_0, F) \geq r\|y - y_0\|$. Put $h = y - y_0$,

$$ \lim_{t \to 0^+} \frac{\rho(y_0 + th, F) - \rho(y_0, F)}{t} \geq \frac{r\|th\|}{t} = r\|h\|.$$ 

Set $t$ towards to 0 from right of 0, then $\rho_{x}^{+}(y_0, h, F) \geq r\|y - y_0\|$. Note that $t > 0$, $t$ is sufficiently small and $y_0 + th \in G$, since $G$ is convex. This completes the proof. \qed

We shall consider whether condition $(D_s)$ is sufficient. Some lemmas are required. To state them we give first some notation.

$$ \Gamma = \{ \varphi \in X^* : \|\varphi\| = 1, \ \text{Re} \varphi(u) \leq \varphi(y_0), \text{for } u \in H \}, \text{here, } H = \{ u : \rho(u, F) \leq \rho(F, G) \}.$$ 

$$ N_F = \{ f \in X^* : \inf_{x \in F} \text{Re} f(x) = \inf_{u \in F} \|u\| \}.$$ 

The following lemma is Lemma 2.3 and Lemma 2.4 in [6].

**Lemma 3.3.** For every $h \neq 0 \in X$, $\sup_{\varphi \in \Gamma} \frac{\text{Re} \varphi(h)}{\varphi(y_0)} = \frac{\rho_{x}^{+}(y_0, h, F)}{\rho(y_0, F)}$, and $-\Gamma = N_{F-y_0}.$
Theorem 3.4. Let $F$ be a subspace of $X$, $G$ a convex subset of $X$, $F \cap G = \emptyset$ and $0 < r < 1$. If $(D_s)$ holds, then the following condition holds

\((K_s)\) for every $\epsilon > 0$, and every $y \in G$, there exists $f_0$ (depend on $\epsilon$, $y$) $\in X^*$ with $\|f_0\| = 1$, such that $\text{Re } f_0(y_0 - y) + \epsilon \geq r \|y_0 - y\|$.

Proof. By the condition $(D_s)$, Theorem 3.1 and Lemma 3.3, we have

\[
\sup_{f \in N_{F-y_0}} \frac{\text{Re } f(y_0 - y)}{\|f(y_0)\|} \geq \frac{r \|y_0 - y\|}{\rho(y_0, F)}
\]

So for every $\epsilon$, $y \in G$, there exists $f_0 \in N_{F-y_0}$, such that $\frac{\text{Re } f_0(y_0 - y)}{\|f_0(y_0)\|} + \epsilon \geq r \|y_0 - y\|/\rho(y_0, F)$. Since $0 \in F$, the definition of $N_{F-y_0}$, $\text{Re } f_0(y_0) \geq \rho(y_0)$. This implies that $\text{Re } f_0(y_0 - y) + \epsilon \geq r \|y_0 - y\|$, so $(K_s)$ holds. This completes the proof. \(\square\)

Finally, we give a condition $(B_s)$ which is equivalent to $(K_s)$ for general disjoint sets $F, G$. Before stating it, we require a notation. Let $F$ be a subset of the space $X$. Denote

\[Q_F = \{u : \text{Re } \phi(u) \leq \|F\|, \text{ for all } \phi \in N_F\}.\]

Notice that $Q_F$ is a cone type set including $F$. For if $z \in Q_F$, $x \in F, z' = x + t(z - x), t > 0$, then $z' \in Q_F$. Because for every $\phi \in N_F$, $\text{Re } \phi(z) \leq \|F\|$, $\text{Re } \phi(z') = \text{Re } \phi(x) + t \text{Re } \phi(z - x) = (1 - t) \text{Re } \phi(x) + 2t \text{Re } \phi(z) \leq (1 - t) \|F\| + t \|F\| = \|F\|$. Specially, if $F$ is a singleton $x_0$, then $Q_F$ is a cone including ball $B(0, \|x_0\|)$; see [3].

Theorem 3.5. Let $F, G$ be two disjoint sets, then $(K_s)$ is equivalent to

\[(B_s) \quad Q_{F-y_0} \cap \text{cone}(y_0 - G) \text{ is bounded},\]

$\text{cone}(E)$ denotes the cone closure of $E$.

Proof. Suppose $(K_s)$ holds, for every $y \in G$. Then

\[
\sup_{y \in N_{F-y_0}} \text{Re } f(y_0 - y) \geq r \|y_0 - y\|, \quad 0 < r < 1.
\]

We should conclude that $Q_{F-y_0} \cap \text{cone}(y_0 - G) \subset B(0, \|F - y_0\|/r)$. If this is not true, there exists $t > 0$, and some $y \in G$ such that $t(y_0 - y) \in Q_{F-y_0}$, $\|t(y_0 - y)\| > 1/r \|F - y_0\|$. From the definition of $Q_{F-y_0}$, for every $f \in N_{F-y_0}$, $\|f(y_0 - y)\| \leq \|F - y_0\|$ and $\|F - y_0\| \geq r \|y_0 - y\| > \|F - y_0\|$, which leads us to a contradiction.

Now if $(B_s)$ holds, from the above statement, there exists a sufficient large number $\alpha$, such that $Q_{F-y_0} \cap \text{cone}(y_0 - G) \subset \text{int}B(0, \alpha)$, $\alpha > 0$. So for every $y \in G, y_0 - y \in y_0 - G$, and $\frac{y_0 - y}{\|y_0 - y\|} \in \text{cone}(y_0 - G)$. But since $\alpha \frac{y_0 - y}{\|y_0 - y\|} \notin \text{int}B(0, \alpha)$, then $\alpha \frac{y_0 - y}{\|y_0 - y\|} \notin Q_{F-y_0}$. There exists $f_0 \in N_{F-y_0}$, such that $\text{Re } f_0(\frac{y_0 - y}{\|y_0 - y\|}) > \|F - y_0\|$. It means that $\text{Re } f_0(y_0 - y) > \frac{1}{\alpha} \|F - y_0\| \|y_0 - y\|$. If we put $r = \frac{\|F - y_0\|}{\alpha}$ and we take $\alpha$ large enough such that $0 < r < 1$, then $(K_s)$ holds. \(\square\)
4. CUSP AND STRONG BEST APPROXIMATION PAIRS

In this section, we shall discuss the case when either strong nearest cross point or strong best approximation pair involve a cusp. In the end of this section, we shall give three examples of strong best approximation pairs. Let us begin with the definition of a cusp.

**Definition 4.1.** Let $G$ be a nonempty subset of $X$, and let $\partial G$ be the boundary of $G$. Given $y_0 \in G \cap \partial G$, a point $y_0$ is called a cusp of $G$ if there exists a hyperplane $P$ supporting $G$ at $y_0$, and $\frac{\rho(y, P)}{\rho(y_0, y)} > \sigma > 0$ holds for every $y \in G$, where $\sigma$ is a constant.

Obviously, every cusp is a strongly exposed point. We say that $y_0$ is a strongly exposed point of $G$ if there exists a hyperplane $P$ supporting $G$ at $y_0$, $x \in P, f(x) = c$ and such that if for every arbitrary $\epsilon > 0$, there exists $\delta > 0$, such that $|f(y) - f(y_0)| < \delta$ for $y \in G$, then $\rho(y, y_0) < \epsilon$. In fact, $\rho(y, P) = \frac{|f(y) - c|}{\|f\|}$. Without loss of generality, we assume $\|f\| = 1$, then $\rho(y, P) = |f(y) - f(y_0)|$. Since $y_0$ is a cusp of $G$, $\frac{\rho(y, P)}{\rho(y_0, y)} > \sigma, \rho(y, y_0) < |f(y) - f(y_0)|/\sigma$ for $y \in G$. For exposed points and strongly exposed points, one can see [1], [8] and the references there in.

Let $F, G$ be two nonempty sets with $\rho(F, G) > 0$. We say that two hyperplanes $P, Q$ regular separate $F$ and $G$, if $P, Q$ are parallel, $F$ and $G$ are in two outer sides of $P$ and $Q$, and $\rho(P, Q) = \rho(F, G) = \rho(F, P) = \rho(G, Q)$. Furthermore, if $y_0 \in G$ is such that $\rho(y_0, F) = \rho(F, G)$, and $\frac{\rho(y, P)}{\rho(y_0, y)} > \delta > 0$, we say that $y_0$ is a cusp of $G$ to $F$. Obviously, if $y_0$ is a cusp of $G$ to $F$, then $y_0$ is a cusp of $G$.

**Theorem 4.2.** Let $F, G$ be two disjoint convex sets and let $y_0 \in G$. If $y_0$ is a cusp of $G$ to $F$, then $y_0$ is a strong n.c. point of $G$ to $F$.

**Proof.** From the definition of cusp of $G$ to $F$, there exist hyperplanes $P, Q$ separating $F, G$ such that $\frac{\rho(y, P)}{\rho(y_0, y)} > \delta$ for every $y \in G$. Then, $\rho(y, F) - \rho(y_0, F) \geq \rho(y, Q) - \rho(y_0, F)$. Note that since $P$ is parallel to $Q$, then $\rho(y, Q) = \rho(y, P) + \rho(P, Q)$, and $\rho(P, Q) = \rho(y_0, F)$. Thus $\rho(y, F) - \rho(y_0, F) = \rho(y, P) > \delta \rho(y, y_0)$. This means that $y_0$ is a strong n.c. point. □

Let $h$ be the Hausdorff metric $h(F_0, F_1) = \max\{\Delta(F_0, F_1), \Delta(F_1, F_0)\}$, where $\Delta(F_0, F_1) = \sup_{x \in F_0} \inf_{x' \in F_1} \|x - x'\|$. We have

**Theorem 4.3.** (Fred type proposition) Suppose $y_0$ is a strong n.c. point of $G$ to $F$. If $y_1$ is a strong n.c. point of $G$ to $F_1$, then $\|y_0 - y_1\| < 2/rh(F_0, F_1)$.

**Proof.** According to Definition 2.1, there exists $0 < r < 1$, such that $r\|y_1 - y_0\| \leq \Delta(F, G) + \rho(y_1, F) - \rho(y_0, F)$. It is easy to see that $\rho(y, B) - \rho(y, A) \leq (A, B)$ holds for every $y$. Thus

$$r\|y_1 - y_0\| \leq \Delta(F_1, F_0) + \rho(y_1, F_1) - \rho(y_0, F_0) \leq \Delta(F_1, F_0) + \rho(y_0, F_1) - \rho(y_0, F_0) \leq \Delta(F_1, F_0) + \Delta(F_0, F_1) \leq 2h(F_0, F_1),$$
(to obtain the second inequality, we used that \( y_1 \) is a strong n.c. point of \( G \) to \( F \)). This completes the proof. \( \square \)

In smooth normed spaces, \( F \) is a singleton and \( G \) is a normed subspace, then Theorem 4.3 is the result of Wulbert [3, page 95].

**Theorem 4.4.** Let \( F, G \) be two disjoint sets with \( F \) convex and \( G \) a linear subspace and \( y_0 \in G \). If \( \rho(y, F) \) is Gateaux differential at \( y_0 \), then \( y_0 \) is not a strong n.c. point of \( G \) to \( F \).

**Proof.** Suppose \( y_0 \) is a strong n.c. point. By the Gateaux differentiable of \( \rho(y, F) \), we have \( \rho'(y_0, h, F) + \rho'(y_0, -h, F) = 0 \), for \( h \neq 0 \). By Theorem 3.2, \( \rho'_{c}(y_0, y - y_0, F) \geq r\|y - y_0\| \) holds for all \( y \neq y_0 \), \( y \in G \), where \( 0 < r < 1 \). If \( y - y_0 \) is either \( h \) or \( -h \), we have \( 0 \geq 2r\|y - y_0\| \), which is a contradiction. The proof is complete. \( \square \)

**Definition 4.5.** Let \( F, G \) be two disjoint sets, \( x_0 \in F \), and \( y_0 \in G \). We say that \((x_0, y_0)\) is a strong best approximation pair of \( F \) and \( G \) if there exist positive constants \( r, r' \) such that \( \rho(y, y) - \rho(x, y_0) \geq r'\|x - x_0\| + r''\|y - y_0\| \) for all \( x \in F \), \( y \in G \).

Obviously, a strong best approximation pair of \( F \) and \( G \) is a best approximation pair of \( F \) and \( G \); for best approximation pairs one can see [5] in detail. In the following, we shall discuss the connection between strong best approximation pairs and strong n.c. points.

**Theorem 4.6.** If \( F, G \) are two disjoint sets, then \((x_0, y_0)\) is the strong best approximation pair of \( F \) and \( G \), if and only if, \( y_0 \) is a strong n.c. point of \( F \) to \( G \), \( x_0 \) is the strong n.c. point of \( F \) to \( G \), and \((x_0, y_0)\) is a best approximation pair of \( F \) and \( G \). In this case, it is unique.

**Proof.** If \((x_0, y_0)\) is the strong best approximation pair of \( F \) and \( G \), by Definition 4.5, \( \rho(x_0, y_0) = \rho(F, G) = \rho(y_0, G) \) and \( \rho(x, y) - \rho(y_0, F) \leq r''\|y - y_0\| \) for all \( x \in F \). If we take the infimum over all \( x \in F \), we have \( \rho(y, F) - \rho(y, F) \geq r''\|y - y_0\| \). Thus \((x_0, y_0)\) is the strong n.c. point of \( F \) to \( G \). Similarly, \( x_0 \) is a strong n.c. point of \( F \) to \( G \).

Conversely, since \((x_0, y_0)\) is a best approximation pair of \( F \) and \( G \), then \( \rho(x_0, y_0) = \rho(F, G) = \rho(y_0, F) = \rho(x_0, G) \). Note that \( \rho(x, y) \geq \rho(\rho, F) \) and \( y_0 \) is a strong n.c. point of \( G \) from \( F \). So, \( \rho(x, y) - \rho(x_0, y_0) \geq \rho(F, F) - \rho(y_0, F) \geq r''\|y - y_0\| \). Similarly, \( \rho(x, y) - \rho(x_0, y_0) \geq r''\|x - x_0\| \). Thus, \( \rho(x, y) - \rho(x_0, y_0) \geq r''/2\|x - x_0\| + r''/2\|y - y_0\| \). Therefore \((x_0, y_0)\) is a strong best approximation pair. This completes the proof. \( \square \)

**Theorem 4.7.** Let \( F, G \) be two disjoint sets, \( \rho(F, G) > 0 \), and \((x_0, y_0)\) a best approximation pair of \( F \) and \( G \). If \( y_0 \) is a cusp of \( G \) to \( F \), and \( x_0 \) is a cusp of \( F \) to \( G \), then \((x_0, y_0)\) is a strong best approximation pair.

**Proof.** By the definition, \( y_0 \) is a cusp of \( G \) to \( F \), and there exist parallel hyperplanes separating \( P, Q \), such that \( F \) and \( G \) are in the outer side of \( P \) and \( Q \), \( y_0 \in P \), and \( \rho(P, Q) = \rho(F, G) = \rho(P, F) = \rho(Q, G) = \rho(y_0, F) \). Obviously,
\( \rho(x, y) \geq \rho(y, Q) \) for all \( x \in F, y \in G \). Since \((x_0, y_0)\) is a best approximation pair of \( F \) and \( G \), \( \rho(x_0, y_0) = \rho(y_0, F) \). So \( \rho(x, y) - \rho(x_0, y_0) \geq \rho(y, Q) - \rho(y_0, F) \). Note that since \( \rho(y_0, F) = \rho(P, Q) \), and \( \rho(y, Q) = \rho(y, P) + \rho(P, Q), \) then \( \rho(x, y) - \rho(x_0, y_0) > \rho(y, P) > \sigma' ||y - y_0|| \). Thus \( \rho(x, y) - \rho(x_0, y_0) > \sigma'/2 ||x - x_0|| + \sigma'/2 ||y - y_0|| \). This completes the proof. \( \square \)

Finally, we shall give three examples.

**Example 4.8.** Denote \( C[0,1] \) be all continuous function \( f \) on \([0,1]\) with norm \( ||f|| = \max_{t \in [0,1]} |f(t)| \). In \( C[0,1] \), let \( F = \{ \mu t : -\infty < \mu < \infty \}, G = \{ \lambda t^2 : \sqrt{2} + 1 \leq \lambda \leq 5 \} \). We consider n.c. points, best approximation pairs, strong n.c. points and strong best approximation pairs between \( F \) and \( G \). Denote \( x(t) = \mu t, y(t) = \lambda t^2 \). Then

\[
\rho(x, y) = \| \mu t - \lambda t^2 \| = \begin{cases} 
\lambda - \mu, & \text{for } \mu/\lambda \leq \sqrt{2} - 2 \\
\mu^2/4\lambda, & \text{for } \mu/\lambda \geq \sqrt{2} - 2.
\end{cases}
\]

First we compute \( \inf_{\mu \leq \infty, \lambda < \infty} \| \mu t - \lambda t^2 \| \). For fixed \( \lambda \), \( \| \mu t - \lambda t^2 \| \) takes its infimum at \( \mu = \mu_\lambda \). By the representation of \( \| \mu t - \lambda t^2 \| \), \( \mu_\lambda \) satisfies \( \lambda - \mu_\lambda = \mu_\lambda^2/4\lambda \), so \( \mu_\lambda = (\sqrt{2} - 2)\lambda \). Thus

\[
\rho(F, G) = \inf_{\sqrt{2} + 1 \leq \lambda \leq 5} \| \lambda t^2 - (\sqrt{2} - 2)\lambda t \|
= (\sqrt{2} + 1)\|t^2 - (\sqrt{2} - 2)t\|
= (\sqrt{2} + 1)(3 - \sqrt{2}) = \sqrt{2} - 1.
\]

From above we obtain that \( \|x_0 - y_0\| = \rho(F, G) \), when \( x_0(t) = \mu_0 t, y_0(t) = \lambda_0, \lambda_0 = \sqrt{2} + 1, \mu_0 = 2 \). We declare that \( y_0(t) \) is a strong n.c. point of \( G \) to \( F \), since for every \( y(t) = \lambda t^2 \in F \),

\[
\rho(y, F) = \inf_{-\infty < \mu < \infty} \| \lambda t^2 - \mu t^2 \| = \| \lambda t^2 - \mu t^2 \| = \lambda \| t^2 - (\sqrt{2} - 2)t \| = (3 - \sqrt{2})\lambda.
\]

Therefore, \( \rho(y_0, F) = \rho_0 = (3 - \sqrt{2})\lambda_0, \rho(y, F) - \rho(y_0, F) = (3 - \sqrt{2})(\lambda - \lambda_0) \leq r' ||y - y_0||, r' = 3 - \sqrt{2} \). Similarly, we have that \( x_0 \) is a strong n.c. point of \( F \) to \( G \), since for every \( x(t) = \mu t, \)

\[
\rho(x, G) = \inf_{\sqrt{2} + 1 \leq \lambda \leq 5} \| \lambda t^2 - \mu t^2 \| = \inf_{\sqrt{2} + 1 \leq \lambda \leq 5} \| \lambda - \mu, \text{ for } \lambda \geq \mu/\sqrt{2} - 2 \\
\| \mu^2/4\lambda, \text{ for } \lambda \leq \mu/\sqrt{2} - 2.
\]

So, \( \rho(x, G) \) takes the infimum at \( \lambda_\mu = \sqrt{2} + 1/2\mu \), and \( \rho(x, G) = \lambda_\mu - \mu = \sqrt{2}/2\mu \). Thus, \( \rho(x, G) = \rho(x_0, G) = \sqrt{2} + 1/2\mu - \sqrt{2} - 1 = r ||x - x_0|| \), where \( r = \sqrt{2} - 1/2 \). By Theorem 3.4, we obtain that \( (x_0, y_0) \) is the strong best approximation pair of \( F \) and \( G \).

The following example shows that a n.c. point always exists, but strong n.c. points can fail to exist.

**Example 4.9.** Denote \( l^1_2 = \{ (\xi, \eta) : \xi, \eta \in \mathbb{R}, ||(\xi, \eta)|| = ||\xi| + |\eta|| \} \). In \( l^2_3 \) space, let \( F = \{ (\xi, \eta) : \xi = \eta \}, G = \{ (\xi, \eta) : \eta = 0, 2 \leq \xi \leq 3 \} \). It is easy to see that \( y_0 = (2, 0) \) is a n.c. point of \( G \) to \( F \). But the best approximation of \((2,0)\)
from $F$ is not unique. Thus the nearest cross points of $G$ to $F$ is not unique. Therefore a strong n.c. point of $G$ to $F$ does not exist.

At the end, we shall give an example, which shows that it is possible to find best approximation pairs which are not strong best approximation pairs.

Example 4.10. In the Euclidean $\mathbb{R}^2$ space, set $F = \{(\xi, \eta) : \xi^2 + \eta^2 \leq 1\}$, $G = \{(\xi, \eta) : \xi \geq 2, -\xi + 2 \geq \eta \geq \xi - 2\}$. It is easy to see that $x_0 = (1, 0), y_0 = (2, 0)$ is the unique best approximation pair of $F$ and $G$. Denote an arbitrary point of $F$ as $x = (\cos \theta, \sin \theta), 0 \leq \theta < 2\pi$. Put $y = (2 + \delta, \delta) \in G, \delta > 0$. Then $\rho_0 = \rho(x_0, y_0) = 1, \rho^2 = \rho^2(x, y) = (2 + \delta - \cos \theta)^2 + (\delta - \sin \theta)^2$. Setting $\theta \to 0, \delta \to 0$, then $\rho^2 - \rho_0^2$ is asymptotic to $2(\rho - \rho_0)$. $\|x - x_0\| = |\sin \theta|, \|y - y_0\| = \sqrt{2}\delta$. Since $\frac{\delta}{\rho^2 - \rho_0^2}$ and $\frac{\theta}{\rho^2 - \rho_0^2}$ are not bounded, then $\frac{\|y - y_0\|}{\rho - \rho_0}$ and $\frac{\|x - x_0\|}{\rho - \rho_0}$ are also not bounded. This means that $(x_0, y_0)$ is not a strong best approximation pair. However, $y_0$ is a strong n.c. point of $G$ to $F$.

Acknowledgements. The authors would like to give their deep gratitude to the referee for his careful reading of the manuscript and his suggestions which made this article more readable.

References

Received March 2004
Accepted December 2005

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