⋆-quasi-pseudometrics on algebraic structures

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ABSTRACT

In this paper, we introduce some concepts of ⋆-(quasi)-pseudometric spaces, and give an example which shows that there is a ⋆-quasi-pseudometric space which is not a quasi-pseudometric space. We also study the conditions under which ⋆-quasi-pseudometric semitopological groups are paratopological groups or topological groups.

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1. Introduction

Finding a stronger topological structure is one of the central problems in topological algebra. In 1957, R. Ellis showed that every locally compact Hausdorff semitopological group is a topological group [3]. In 1960, W. Zelazko established that each completely metrizable semitopological group is a topological group [19]. Later, in 1982, N. Brand proved that every Čech-complete paratopological group is a topological group [2].

In 1975, Kramosil and Michalek introduced a notion of metric fuzziness [10] which quickly became an important issue (for example, [4, 5, 6, 7, 8]).

Definition 1.1. A fuzzy metric (in the sense of Kramosil and Michalek) on a set $X$ is a pair $(M, \ast)$ such that $M$ is a fuzzy set in $X \times X \times [0, \infty)$ and $\ast$ is a continuous $t$-norm satisfying for all $x, y, z \in X$:
1) \( M(x, y, 0) = 0; \)
2) \( M(x, y, t) = 1 \) for all \( t > 0 \) if and only if \( x = y; \)
3) \( M(x, y, t) = M(y, x, t); \)
4) \( M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \) for all \( t, s > 0; \)
5) \( M(x, y, -) : [0, +\infty) \rightarrow [0, 1] \) is a left continuous function.

Recently, fuzzy metric topological groups have been widely studied in fuzzy topological algebra (see, among others, [15, 18]).

In particular, I. Sánchez and M. Sanchis found that some special fuzzy metrics (such as left invariant fuzzy quasi-pseudometrics and invariant fuzzy pseudometrics) can improve some topological algebraic structures into stronger topological structures. The main results are: (1) If \((G, M, \ast)\) is a fuzzy quasi-pseudometric right topological group such that \((M, \ast)\) is left-invariant, then \((G, M, \ast)\) is a fuzzy paratopological group (see [16, Theorem 3.2]). (2) If \((G, M, \ast)\) is a fuzzy pseudometric right topological group such that \((M, \ast)\) is left-invariant, then \((G, M, \ast)\) is a fuzzy topological group (see [16, Theorem 3.3]). (3) Let \((M, \ast)\) be a fuzzy quasi-pseudometric on a semigroup \(S\). If \((M, \ast)\) is invariant, then \((S, M, \ast)\) is a fuzzy topological semigroup (see [16, Theorem 3.10]).

Given a function \(d : X \times X \rightarrow \mathbb{R}^+\) on a set \(X\), we consider the following conditions, for every \(x, y, z \in X\):

\[
\begin{align*}
(1) \quad & d(x, x) = 0; \\
(2) \quad & d(x, y) = d(y, x); \\
(3) \quad & d(x, y) \leq d(x, z) + d(z, y); \\
(4) \quad & \text{if } d(x, y) = 0, \text{ then } x = y; \\
(4') \quad & \text{if } d(x, y) = d(y, x) = 0, \text{ then } x = y,
\end{align*}
\]

for all \(x, y, z \in X\).

The function \(d\) is called a pseudometric if it satisfies (1), (2) and (3). A pseudometric that also satisfies (4) is called a metric. A quasi-pseudometric on an arbitrary set \(X\) is a function \(d : X \times X \rightarrow \mathbb{R}^+\) satisfying the conditions (1) and (3). If \(d\) satisfies further (4') then it is called a quasi-metric.

Recently, Khatami and Mirzavaziri (in [11]) generalized the concept of metric. They first gave a new operation called \(t\)-definer which is extended by \(t\)-conorm. It is defined as:

**Definition 1.2 ([11, Definition 2.1]).** A \(t\)-definer is a function \(\ast : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)\) satisfying the following conditions for each \(a, b, c \in [0, \infty)\):\n
\[
\begin{align*}
(T1) \quad & a \ast b = b \ast a; \\
(T2) \quad & a \ast (b \ast c) = (a \ast b) \ast c; \\
(T3) \quad & \text{if } a \leq b, \text{ then } a \ast c \leq b \ast c; \\
(T4) \quad & a \ast 0 = a; \\
(T5) \quad & \ast \text{ is continuous on its first component with respect to the Euclidean topology.}
\end{align*}
\]
The residuum of a \( t \)-definer plays a role such as the role of minus operator for addition operator. Let \( \star \) be a \( t \)-definer. The \textit{residuum of} \( \star \) is defined by
\[
 a \to b = \inf \{ c : c \star a \geq b \}.
\]
Then, by the residuation property of \( \star \) and \( \to \), we have
\[
 a \star (a \to b) = \max \{a, b\}.
\] (1.1)

Khatami and Mirzavaziri changed the condition (3) in the metric axiom into the \( \star \)-triangle inequality. Then the following definition of \( \star \)-metrics can be obtained.

\textbf{Definition 1.3} ([11, Definition 2.2]). Let \( X \) be a non-empty set and \( \star \) a \( t \)-definer. If for every \( x, y, z \in X \), a function \( d^\star : X \times X \to [0, \infty) \) satisfies the following conditions:

(M1) \( d^\star(x, y) = 0 \) if and only if \( x = y \);
(M2) \( d^\star(x, y) = d^\star(y, x) \);
(M3) \( d^\star(x, y) \leq d^\star(x, z) \star d^\star(z, y) \),

then \( d^\star \) is called a \( \star \)-metric on \( X \). The set \( X \) with a \( \star \)-metric is called \( \star \)-metric space, denoted by \( (X, d^\star) \).

Assume that \( (X, d^\star) \) is a \( \star \)-metric space. For any \( a \in X \) and \( r > 0 \), denote by
\[
 B_d^\star(a, r) = \{ x \in X : d^\star(a, x) < r \}
\]
and
\[
 \mathcal{T}_d^\star = \{ U \subseteq X : \text{for each} \ a \in U \text{ there is} \ r > 0 \text{ such that} \ B_d^\star(a, r) \subseteq U \}.
\]

Khatami and Mirzavaziri proved the following result:

\textbf{Theorem 1.4} ([11, Theorems 3.2, 3.4, 3.5]). For every \( \star \)-metric space \( (X, d^\star) \), \( \mathcal{T}_d^\star \) forms a Hausdorff topology on \( X \) and the topological space \( (X, \mathcal{T}_d^\star) \) is first countable and satisfied the normal separation axiom.

Then, we have proved that

\textbf{Theorem 1.5} ([9, Theorem 2.4]). Every \( \star \)-metric space is metrizable.

In this paper, we extend some concepts of \( \star \)-metric spaces (in [11]) to \( \star \)-quasi-pseudometrics spaces, and give an example to show that \( \star \)-quasi-pseudometrics are not necessarily quasi-pseudometrics. Then, we will discuss the basic topological properties of \( \star \)-metric spaces. Further, we combine topological structure with algebraic structure. Our aim is to obtain conditions under which \( \star \)-quasi-pseudometric semitopological groups are paratopological groups or topological groups.

We show that: (1) if \( (G, d^\star) \) is a \( \star \)-quasi-pseudometric right topological group such that \( d^\star \) is left-invariant, then \( (G, d^\star) \) is a paratopological group (see Theorem 3.5); (2) if \( (G, d^\star) \) is a \( \star \)-quasi-pseudometric left topological group such that \( d^\star \) is right-invariant, then \( (G, d^\star) \) is a paratopological group. If in addition \( (G, d^\star) \) is a \( \star \)-pseudometric left topological group, then \( (G, d^\star) \) is a...
topological group (see Theorem 3.6); (3) let \( d^* \) be a left-invariant \( \ast \)-quasi-pseudometric on a monoid \( G \) such that for each \( x \in G \), \( \lambda_x \) is open and \( p_x \) is continuous at the identity \( e \) of \((G, d^*)\). Then \((G, d^*)\) is a topological semigroup (see Theorem 4.1).

2. Topology of \( \ast \)-quasi-metric

In this section, we extend some concepts of \( \ast \)-metric spaces to \( \ast \)-quasi-metric spaces and \( \ast \)-quasi-pseudometric spaces. Then we discussed the basic topological properties of \( \ast \)-quasi-metric spaces and \( \ast \)-quasi-pseudometric spaces.

**Definition 2.1.** Let \( X \) be a non-empty set and \( \ast \) a \( t \)-definer. A \( \ast \)-quasi-pseudometric on \( X \) is a function \( d^* : X \times X \to [0, \infty) \) satisfying the following conditions:

- (D1) \( d^*(x, x) = 0 \);
- (D2) \( d^*(x, y) \leq d^*(x, z) \ast d^*(z, y) \).

In this case \((X, d^*)\) is called a \( \ast \)-quasi-pseudometric space.

In addition, if \( d^* \) is a \( \ast \)-quasi-pseudometric and satisfies the condition:

- (D3) for every \( x, y \in X \), if \( d^*(x, y) = 0 \), then \( x = y \),

then \( d^* \) is called a \( \ast \)-quasi-metric on \( X \), and \((X, d^*)\) is called a \( \ast \)-quasi-metric space.

If \( d^* \) is a \( \ast \)-quasi-pseudometric and satisfies the condition:

- (D4) \( d^*(x, y) = d^*(y, x) \),

then \( d^* \) is called a \( \ast \)-pseudometric on \( X \), and \((X, d^*)\) is called a \( \ast \)-pseudometric space.

The following example shows that there are \( \ast \)-quasi-pseudometrics which are not quasi-pseudometrics.

**Example 2.2.** Let \( X = [0, \infty) \). Clearly, \( x \ast y = (\sqrt{x} + \sqrt{y})^2 \) is a \( t \)-definer, for every \( x, y \in X \). The function

\[
d^*(x, y) = \begin{cases} 
(\sqrt{x} - \sqrt{y})^2, & x \geq y; \\
0, & x < y.
\end{cases}
\]

forms an \( \ast \)-quasi-pseudometric which is not a quasi-pseudometric.

Obviously, \( d^*(x, y) \) satisfies (D1) of Definition 2.1. Now, we show that also (D2) of Definition 2.1 holds.

Now, we need to prove the following 6 cases.
Therefore extend the meaningful. The following figure briefly describes the relationship between which is not a quasi-pseudometric. This shows that our promotion is very pseudometrics. In Example 2.2, we find that there is a metrics, and give an example that is not a quasi-pseudometric.

\[
\begin{align*}
\text{(1) When } x \geq z \geq y, \text{ we have } \\
d^*(x, y) &= (\sqrt{x} - \sqrt{y})^2 = (\sqrt{x} - \sqrt{z} + \sqrt{z} - \sqrt{y})^2 \\
&\leq \left[ \sqrt{(\sqrt{x} - \sqrt{z})^2 + (\sqrt{z} - \sqrt{y})^2} \right]^2 \\
&= \left[ \sqrt{d^*(x, z)} + \sqrt{d^*(z, y)} \right]^2 \\
&= d^*(x, z) \ast d^*(z, y).
\end{align*}
\]

\[
\begin{align*}
\text{(2) When } z \geq x \geq y, \text{ we have } d^*(x, y) &= (\sqrt{x} - \sqrt{y})^2, d^*(x, z) = 0, d^*(z, y) = (\sqrt{x} - \sqrt{z})^2. \text{ Therefore } d^*(x, y) &= (\sqrt{x} - \sqrt{y})^2 \leq (\sqrt{x} - \sqrt{z})^2 = 0 \ast (\sqrt{x} - \sqrt{y})^2 = d^*(x, z) \ast d^*(z, y).
\end{align*}
\]

\[
\begin{align*}
\text{(3) When } x \geq y \geq z, \text{ we have } d^*(x, y) &= (\sqrt{x} - \sqrt{y})^2, d^*(x, z) = (\sqrt{x} - \sqrt{z})^2, \text{ and } d^*(z, y) = 0. \text{ Therefore } d^*(x, y) &= (\sqrt{x} - \sqrt{y})^2 \leq (\sqrt{x} - \sqrt{z})^2 = (\sqrt{x} - \sqrt{z})^2 \ast 0 = d^*(x, z) \ast d^*(z, y).
\end{align*}
\]

\[
\begin{align*}
\text{(4) When } z \leq x < y, \text{ we have } d^*(x, z) &= (\sqrt{x} - \sqrt{z})^2, d^*(z, y) = 0, d^*(x, y) = 0. \text{ Therefore } d^*(x, y) &= 0 \leq (\sqrt{x} - \sqrt{z})^2 = (\sqrt{x} - \sqrt{z})^2 \ast 0 = d^*(x, z) \ast d^*(z, y).
\end{align*}
\]

\[
\begin{align*}
\text{(5) When } x \leq z \leq y, \text{ we have } d^*(x, z) = 0, d^*(z, y) = 0, d^*(x, y) = 0. \text{ Therefore } d^*(x, y) &= 0 = 0 \ast 0 = d^*(x, z) \ast d^*(z, y).
\end{align*}
\]

\[
\begin{align*}
\text{(6) When } x < y \leq z, \text{ we have } d^*(x, z) = 0, d^*(z, y) = (\sqrt{x} - \sqrt{z})^2, d^*(x, y) = 0. \text{ Therefore } d^*(x, y) &= 0 \leq (\sqrt{x} - \sqrt{y})^2 = 0 \ast (\sqrt{x} - \sqrt{y})^2 = d^*(x, z) \ast d^*(z, y).
\end{align*}
\]

Thus, (D2) holds. However, \(d^*(1, 25) = 16 \neq d^*(1, 16) + d^*(16, 25) = 10\), which means \(d^*(x, y)\) is not a quasi-pseudometric.

Khatami and Mirzavaziri gave a generalization of metrics, put forward *-metrics, and give an example that *-metrics are not metrics. Further, we extend the *-metrics to obtain *-quasi-metrics, *-pseudometrics, and *-quasi-pseudometrics. In Example 2.2, we find that there is a *-quasi-pseudometric, which is not a quasi-pseudometric. This shows that our promotion is very meaningful. The following figure briefly describes the relationship between them.

\[
\text{pseudometrics} \quad \ast\ast\text{-pseudometrics} \\
\ast\ast\text{-metrics} \quad \ast\ast\text{-quasi-metrics} \\
\text{metrics} \quad \ast\text{-metrics} \\
\text{quasi-metrics} \quad \ast\text{-quasi-metrics} \\
\text{pseudo-metrics} \quad \ast\text{quasi-metrics}
\]

Similar to metric spaces, we will give the definition of open balls in *-quasi-pseudometric spaces below.
Definition 2.3. Let \((X,d^*)\) be a \(\ast\)-quasi-pseudometric space. We define open ball \(B_{d^*}(x,r)\) with \(x \in X\) and radius \(r > 0\) as
\[
B_{d^*}(x,r) = \{ y \in X : d^*(x,y) < r \}.
\]

Theorem 2.4. Let \((X,d^*)\) be a \(\ast\)-quasi-pseudometric space. Define
\[
\mathcal{T}_{d^*} = \{ U \subseteq X : \text{ for each } x \in U \text{ there is } r > 0 \text{ such that } B_{d^*}(x,r) \subseteq U \}.
\]
Then \(\mathcal{T}_{d^*}\) is a topology on \(X\).

Lemma 2.5. In \(\ast\)-quasi-pseudometric space \((X,d^*)\) every open ball is an open set.

Proof. Let \(\ast\) be a \(t\)-definer, \(\rightarrow\) be the residuum of \(\ast\). For every \(x \in X\) and \(r > 0\), we claim that there exist \(\epsilon > 0\), such that for every \(y \in B_{d^*}(x,r)\), we have
\[
B_{d^*}(y,\epsilon) \subseteq B_{d^*}(x,r).
\]
In fact, take \(\epsilon = d^*(x,y) \rightarrow r\) and for every \(z \in B_{d^*}(y,\epsilon)\), then \(d^*(y,z) < d^*(x,y) \rightarrow r\). By formula (1.1), we have
\[
d^*(x,y) \ast d^*(y,z) < d^*(x,y) \ast (d^*(x,y) \rightarrow r) = r.
\]
Therefore, we have \(d^*(x,z) \leq d^*(x,y) \ast d^*(y,z) < r\) which shows that \(z \in B_{d^*}(x,r)\).

Now, by Definition 2.3 and Lemma 2.5, for a \(\ast\)-quasi-(pseudo)metric space \((X,d^*)\), the set \(\mathcal{B} = \{ B_{d^*}(x,\epsilon) | x \in X, \epsilon > 0 \}\) is a base for the topology induced by \(d^*\) on \(X\).

Definition 2.6. Let \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence of a \(\ast\)-quasi-pseudometric space \((X,d^*)\), and \(x \in X\). If for every \(\epsilon > 0\), there exists \(k \in \mathbb{N}\) such that \(d^*(x,x_n) < \epsilon\) whenever \(n \geq k\), then the sequence \(\{x_n\}_{n \in \mathbb{N}}\) converges \(x\) under \(d^*\).

The following propositions are easy to prove.

Proposition 2.7. Let \((X,d^*)\) be a \(\ast\)-quasi-pseudometric space. Then the following statements are equivalent:

1. \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(x_0\) under \(\mathcal{T}_{d^*}\);
2. \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(x_0\) under \(d^*\).

Remark 2.8. The Proposition 2.7 illustrates that for a \(\ast\)-quasi-pseudometric space, \(x_n \to x\) if and only if \(d^*(x,x_n) \to 0\).

Proposition 2.9. Let \((X,d^*)\) be a \(\ast\)-quasi-pseudometric space. Then the set \(X\) with the topology induced by \(d^*\) is first countable.

In Proposition 2.9, we get that, for every \(x \in X\), \(\mathcal{B}_x = \{ B_{d^*}(x,\frac{1}{n}) : n \in \mathbb{N} \}\) is a neighborhood base at \(x\) in the \(\ast\)-quasi-pseudometric space \((X,d^*)\).

Proposition 2.10. Every \(\ast\)-quasi-metric space \((X,d^*)\) is a Hausdorff space.
Proof. Choose two distinct points \( x, y \in X \). We shall show that, there exists \( r > 0 \) such that \( B_{d^*}(x, r) \cap B_{d^*}(y, r) = \emptyset \). Since the \( \ast \) is continuous and \( d^*(x, y) > 0 \), we have \( d^*(x, y) > r \ast r \). Now, we assume that there exists \( z \in B_{d^*}(x, r) \cap B_{d^*}(y, r) \) then we get the following contradiction:

\[
d^*(x, y) \leq d^*(x, z) \ast d^*(z, y) < r \ast r < d^*(x, y).
\]

Hence, \( B_{d^*}(x, r) \cap B_{d^*}(y, r) = \emptyset \). □

The notions and concepts of topological spaces are defined as usual (e.g. see [1] or [13]). Unless otherwise stated, \( \ast \)-quasi-metric spaces and \( \ast \)-quasi-pseudometric spaces do not satisfy any separation axiom.

3. \( \ast \)-quasi-pseudometric topological groups

We now move on to notions from topological algebra. Let \( G \) be an algebraic group. For a fixed element \( x \in G \). The function \( \lambda_x \colon G \to G \) defined by \( \lambda_x(g) = xg \) is called the left translation of \( x \) on \( G \). Similarly, \( \rho_x \colon G \to G \) defined as \( \rho_x(g) = gx \) is known as the right translation of \( x \) on \( G \).

A topological semigroup \((G, \tau)\) is an algebraic semigroup \( G \) with a topology \( \tau \) that makes the multiplication in \( G \) jointly continuous. A paratopological group \( G \) is a topological semigroup such that \( G \) is an algebraic group. A topological group \( G \) is a paratopological group \( G \) such that the inverse mapping is continuous.

\((G, \tau)\) is said to be a left (respectively, right) topological group if the translations \( \lambda_x \) (respectively, \( \rho_x \)) are continuous in \( G \) for all \( x \in G \), and a semitopological group is a left topological group which is also a right topological group.

Next, we will give the definitions related to \( \ast \)-quasi-pseudometric topological groups.

**Definition 3.1.** By a \( \ast \)-(quasi)-pseudometric semigroup we mean a pair \((G, d^*)\) such that \((G, d^*)\) is a \( \ast \)-(quasi)-pseudometric space and \((G, T_{d^*})\) is a topological semigroup.

A \( \ast \)-(quasi)-pseudometric paratopological group is a \( \ast \)-(quasi)-pseudometric semigroup \((G, d^*)\) such that \( G \) is an algebraic group.

**Definition 3.2.** By a \( \ast \)-(quasi)-pseudometric right (left) topological group we mean a pair \((G, d^*)\) such that \((G, d^*)\) is a \( \ast \)-(quasi)-pseudometric space and \((G, T_{d^*})\) is a right (left) topological group.

We give the definition of left (right) invariance in \( \ast \)-(quasi)-pseudometric topological groups. This notion plays an important role in our results.

**Definition 3.3.** A \( \ast \)-(quasi)-pseudometric \( d^* \) on a group \( G \) is left-invariant (respectively, right-invariant) if \( d^*(x, y) = d^*(ax, ay) \) (respectively, \( d^*(x, y) = d^*(xa, ya) \)) whenever \( a, x, y \in G \). We say that \( d^* \) is invariant if it is both left-invariant and right-invariant.
Now, we give a well known result which is an internal characterization of a (para)topological group.

**Proposition 3.4** ([1, Theorem 1.2.12]). Let $G$ be a group with identity $e$ and $\mathcal{U}$ a family of subsets of $G$ containing $e$. If $\mathcal{U}$ satisfies the following conditions:

(i) for every $U, V \in \mathcal{U}$, there exists an $W \in \mathcal{U}$ such that $W \subseteq U \cap V$;

(ii) for every $U \in \mathcal{U}$ and $x \in U$, there exists an $V \in \mathcal{U}$ such that $Vx \subseteq U$;

(iii) for every $U \in \mathcal{U}$ and $x \in G$, there exists an $V \in \mathcal{U}$ such that $xVx^{-1} \subseteq U$;

(iv) for every $U \in \mathcal{U}$, there exists an $V \in \mathcal{U}$ such that $V^2 \subseteq U$;

then the family $\{Ux : x \in G, U \in \mathcal{U}\}$ is a base for a topology $\tau_\mathcal{U}$ on $G$. With this topology, $G$ is a paratopological group, and the family $\{xU : x \in G, U \in \mathcal{U}\}$ is a base for the same topology on $G$. In addition, if $\mathcal{U}$ satisfies

(v) for every $U \in \mathcal{U}$, there exists an $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.

Then $(G, \tau_\mathcal{U})$ is a topological group.

**Theorem 3.5.** If $(G, d^*)$ is a $\ast$-quasi-pseudometric right topological group such that $d^*$ is left-invariant, then $(G, d^*)$ is a paratopological group.

**Proof.** Let $e$ be the identity of $G$. According to Proposition 2.9, $\mathcal{B}_e = \{B_{d^*}(e, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at $e$. Let us show that $\mathcal{B}_e = \{B_{d^*}(e, \frac{1}{n}) : n \in \mathbb{N}\}$ satisfies conditions (i) – (iv) in Theorem 3.4, that is, the topology $\mathcal{T}_{\mathcal{B}_e}$ associated to the family $\mathcal{B}_e$ makes $G$ into a paratopological group.

(i). It follows from the fact that $\mathcal{B}_e$ is a local base at $e$ in $(G, \mathcal{T}_{d^*})$. So, $\mathcal{B}_e$ satisfies (i).

(ii). Take $n \in \mathbb{N}$ and $x \in B_{d^*}(e, \frac{1}{n})$. Since $\rho_x$ is continuous at $e$ and $\rho_x(e) = ex = x \in B_{d^*}(e, \frac{1}{n})$, there exists $m \in \mathbb{N}$ such that

$$\rho_x(B_{d^*}(e, \frac{1}{m})) = B_{d^*}(e, \frac{1}{m})x \subseteq B_{d^*}(e, \frac{1}{n}).$$

Thus, (ii) holds.

(iii). First we show that, for each $n \in \mathbb{N}$ and $x \in G$, we have

$$xB_{d^*}(e, \frac{1}{n}) = B_{d^*}(x, \frac{1}{n}).$$

In fact, take $y \in B_{d^*}(e, \frac{1}{n})$, namely $xy \in xB_{d^*}(e, \frac{1}{n})$. Since $d^*$ is left-invariant, we have

$$d^*(x, xy) = d^*(e, y) < \frac{1}{n}.$$ 

By the foregoing, $xB_{d^*}(e, \frac{1}{n}) \subseteq B_{d^*}(x, \frac{1}{n})$.

On the other hand, take $z \in B_{d^*}(x, \frac{1}{n})$. Because $d^*$ is left-invariant, we have

$$d^*(e, x^{-1}z) = d^*(x, z) < \frac{1}{n}.$$ 

This proves that $x^{-1}z \in B_{d^*}(e, \frac{1}{n})$, and from this it follows further that $z \in xB_{d^*}(e, \frac{1}{n})$, which shows (1).

Now, we shall show (iii). Take $n \in \mathbb{N}$ and $x \in G$. Note that every right translation is a homeomorphism and $x \in B_{d^*}(e, \frac{1}{n})$. So $B_{d^*}(e, \frac{1}{n})x$ is an open
neighborhood of \( x \). Hence there is \( m \in \mathbb{N} \) such that \( B_{d^*}(x, \frac{1}{m}) \subseteq B_{d^*}(e, \frac{1}{n})x \). From this and (1) it follows that

\[
x B_{d^*}(e, \frac{1}{m})x^{-1} = B_{d^*}(x, \frac{1}{m})x^{-1} \subseteq B_{d^*}(e, \frac{1}{n}).
\]

So, \( \mathcal{B}_e \) satisfies (iii).

(iv). For every \( n \in \mathbb{N} \), since the \( * \) is continuous, there is \( m \in \mathbb{N} \) such that \( \frac{1}{m} \cdot \frac{1}{m} \leq \frac{1}{n} \). Then for each \( y, z \in B_{d^*}(e, \frac{1}{m}) \), the following inequalities hold

\[
d^*(e, yz) \leq d^*(e, y) \star d^*(y, yz) = d^*(e, y) \star d^*(e, z) < \frac{1}{m} \star \frac{1}{m} < \frac{1}{n}.
\]

Therefore, \( B_{d^*}(e, \frac{1}{m})B_{d^*}(e, \frac{1}{n}) \subseteq B_{d^*}(e, \frac{1}{n}) \), \( \mathcal{B}_e \) satisfies (iv).

By Proposition 3.4, \((G, \mathcal{T}_{\mathcal{B}_e})\) is a paratopological group and \( \{xB_{d^*}(e, \frac{1}{m}) : x \in G, n \in \mathbb{N}\} \) is a base for \( \mathcal{T}_{\mathcal{B}_e} \). Notice that equation (1) implies that \( \{xB_{d^*}(e, \frac{1}{n}) : x \in G, n \in \mathbb{N}\} \) also is a base for \( \mathcal{T}_{d^*} \), so that \( \mathcal{T}_{\mathcal{B}_e} = \mathcal{T}_{d^*} \). This shows that \((G, d^*)\) is a paratopological group.

**Theorem 3.6.** If \((G, d^*)\) is a \( * \)-quasi-pseudometric right topological group such that \( d^* \) is left-invariant, then \((G, d^*)\) is a topological group.

**Proof.** Since, \( * \)-quasi-pseudometrics are \( * \)-pseudometrics, according to Theorem 3.5, \((G, d^*)\) is a paratopological group. To complete the proof, it is enough to show that the family \( \mathcal{B} = \{B_{d^*}(e, \frac{1}{n}) : n \in \mathbb{N}\} \) satisfies (v) of Proposition 3.4. For every \( n \in \mathbb{N} \). Take \( x \in B_{d^*}(e, \frac{1}{n}) \). As a consequence of left-invariance of \( d^* \), we have

\[
d^*(e, x^{-1}) = d^*(x, e) = d^*(e, x) < \frac{1}{n}.
\]

We conclude that \( x^{-1} \in B_{d^*}(e, \frac{1}{n}) \). So, \((G, d^*)\) is a topological group. \( \Box \)

Similar to the proof of Theorems 3.5 and 3.6, we can obtain the following Theorem.

**Theorem 3.7.** If \((G, d^*)\) is a \( * \)-quasi-pseudometric left topological group such that \( d^* \) is right-invariant, then \((G, d^*)\) is a paratopological group. If furthermore \((G, d^*)\) is a \( * \)-pseudometric left topological group, then \((G, d^*)\) is a topological group.

Since a semitopological group is both a left and right topological group. According to the result of Theorems 3.5, 3.6 and 3.7 we can get the following corollary.

**Corollary 3.8.** Suppose that \((G, \mathcal{T}_{d^*})\) is a semitopological group whose topology \( \mathcal{T}_{d^*} \) is induced by a right-(or left-)invariant \( * \)-quasi-pseudometric \( d^* \). Then \((G, \mathcal{T}_{d^*})\) is a paratopological group.

**Corollary 3.9.** Suppose that \((G, \mathcal{T}_{d^*})\) is a semitopological group whose topology \( \mathcal{T}_{d^*} \) is induced by a right-(or left-)invariant \( * \)-pseudometric \( d^* \). Then \((G, \mathcal{T}_{d^*})\) is a topological group.
It is known that a (quasi-)pseudometric is a ∗-(quasi-)pseudometric. So, it is easy to draw the following conclusions:

**Corollary 3.10 ([16, Corollary 3.4]).** Suppose that \((G, \tau)\) is a left (right) topological group whose topology \(\tau\) is induced by a right-(left-)invariant quasi-pseudometric. Then \((G, \tau)\) is a paratopological group.

**Corollary 3.11 ([16, Corollary 3.8]).** Suppose that \(G\) is a left (right) topological group whose topology is induced by a right-(left-)invariant pseudometric. Then \(G\) is a topological group.

We said that a topological space \(X\) is said to be ∗-(quasi-)metrizable if there exists a ∗-(quasi-)metric \(d^\ast\) on the set \(X\) that induces the topology of \(X\). A ∗-quasi-metric \(d^\ast(x, y)\) is called left-continuous if \(d^\ast(x, _y)\) is continuous.

Recall that a topological space \(X\) is called a sequential space if a set \(A \subset X\) is closed if and only if together with any sequence it contains all its limits.

**Theorem 3.12.** Suppose that \(G\) is a ∗-quasi-metrizable paratopological group with respect to a left continuous, left-invariant ∗-quasi-metric. Then \(G\) is a ∗-metrizable topological group.

**Proof.** First we prove that the ∗-quasi-metrizable paratopological group \(G\) is a topological group. It is sufficient to prove that the inverse operation is continuous.

Let \(G\) be a paratopological group with respect to a left continuous, left-invariant ∗-quasi-metric \(d^\ast\) and \(e\) be the neutral element. First we prove that if \(x_n \to x\), then \(x_n^{-1} \to x^{-1}\). Since \(x_n \to x\) and \(d^\ast\) is left continuous, then \(d^\ast(x_n, x) \to d^\ast(x, x) = 0\). As a consequence of the left invariance of \(d^\ast\), we have

\[
d^\ast(e, x_n^{-1}x) = d^\ast(x_ne, x_nx_n^{-1}x) = d^\ast(x_n, x) \to 0.
\]

Then \(x_n^{-1}x \to e\) by Proposition 2.7. By the foregoing, \(x_n^{-1} \to x^{-1}\). Let \(U\) be open. We shall prove that \(U^{-1}\) is open. Since \(G\) is a sequential space, it is sufficient to prove \(U^{-1}\) is sequential open. Let \(y_n \to y \in U^{-1}\), then \(y_n^{-1} \to y^{-1} \in U\). Since \(U\) is open, \(\{y_n^{-1} : n \in \mathbb{N}\}\) is eventually in \(U\). Hence \(\{y_n : n \in \mathbb{N}\}\) is eventually in \(U^{-1}\). Therefore, \(U^{-1}\) is open.

The inverse operation on \(G\) is continuous, hence \(G\) is a topological group. According to [1, Theorem 3.3.12], A Hausdorff topological group satisfying the first-countable axiom is metrizable. By Propositions 2.9 and 2.10, \(G\) is a Hausdorff topological group satisfying the first-countable axiom and from this it follows by the foregoing that \(G\) is metrizable. Therefore \(G\) is ∗-metrizable by Theorem 1.5. □

From Theorem 3.12, we can easily get Liu’s conclusion in [12]

**Corollary 3.13 ([12, Theorem 2.1]).** Suppose that \(G\) is a quasi-metrizable paratopological group with respect to a left continuous, left-invariant quasi-metric. Then \(G\) is a metrizable topological group.
4. *-QUASI-PSEUDOMETRIC TOPOLOGICAL SEMIGROUPS

We now move on to *-quasi-pseudometric semigroups.

Theorem 4.1. Suppose that $d^*$ be a *-quasi-pseudometric on a semigroup $S$. If $d^*$ is invariant, then $(S, d^*)$ is a topological semigroup.

Proof. Take $y, z \in S$. Since the * is continuous, for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $\frac{1}{m} \leq \frac{1}{n}$. We can claim that $B_{d^*}(y, \frac{1}{m})B_{d^*}(z, \frac{1}{m}) \subseteq B_{d^*}(y, \frac{1}{n})$. Choose $a \in B_{d^*}(y, \frac{1}{m})$ and $b \in B_{d^*}(z, \frac{1}{m})$, then $ab \in B_{d^*}(y, \frac{1}{m})B_{d^*}(z, \frac{1}{m})$. Since $d^*$ is invariant, we have

$$d^*(yz, ab) \leq d^*(yz, yb) \ast d^*(yb, ab) = d^*(z, b) \ast d^*(y, a) < \frac{1}{m} \ast \frac{1}{m} < \frac{1}{n}.$$  

We have proved that multiplication is continuous in $(S, \mathcal{T}_{d^*})$. As a consequence, $(S, d^*)$ is a topological semigroup. □

Let us recall that a monoid is a semigroup with a neutral element.

Theorem 4.2. Let $d^*$ be a left-invariant *-quasi-pseudometric on a monoid $G$ such that for each $x \in G$, $\lambda_x$ is open and $\rho_x$ is continuous at the identity $e$ of $(G, d^*)$. Then $(G, d^*)$ is a topological semigroup.

Proof. Let $e$ be the identity of $G$. We claim that for each $n \in \mathbb{N}$ and $x \in G$ we have

$$xB_{d^*}(e, \frac{1}{n}) \subseteq B_{d^*}(x, \frac{1}{n}). \quad (2)$$

Indeed, take $y \in B_{d^*}(e, \frac{1}{n})$. Since $d^*$ is left-invariant, we have

$$d^*(x, xy) = d^*(e, y) < \frac{1}{n}.$$  

This proves (2). As a consequence of (2), we have that left translations are continuous at $e$.

Now, we shall show that for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ satisfying

$$B_{d^*}(e, \frac{1}{m})B_{d^*}(e, \frac{1}{m}) \subseteq B_{d^*}(e, \frac{1}{n}). \quad (3)$$

Since the * is continuous, for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $\frac{1}{m} \ast \frac{1}{m} < \frac{1}{n}$. Then, for each $y, z \in B_{d^*}(e, \frac{1}{m})$, the following inequalities hold:

$$d^*(e, yz) \leq d^*(e, y) \ast d^*(y, z) = d^*(e, y) \ast d^*(e, z) < \frac{1}{m} \ast \frac{1}{m} < \frac{1}{n}.$$  

Now, we will prove that the multiplication is continuous in $(G, \mathcal{T}_{d^*})$. Take $x, y \in G$ and $n \in \mathbb{N}$. By (2), we have $xyB_{d^*}(e, \frac{1}{n}) \subseteq B_{d^*}(xy, \frac{1}{n})$. By (3), $B_{d^*}(e, \frac{1}{m})B_{d^*}(e, \frac{1}{m}) \subseteq B_{d^*}(e, \frac{1}{n})$ for some $m \in \mathbb{N}$. Therefore

$$xyB_{d^*}(e, \frac{1}{m})B_{d^*}(e, \frac{1}{m}) \subseteq xyB_{d^*}(e, \frac{1}{n}) \subseteq B_{d^*}(xy, \frac{1}{n}). \quad (4)$$
It follows from the hypothesis that left translations are open. Hence \( yB_{d^*}(e, \frac{1}{m}) \) is an open set in \((G, d^*)\) which contains \( y \). According to assumptions \( \rho_x \) is continuous at \( e \). Hence there is \( k \in \mathbb{N} \) satisfying

\[
\rho_y(B_{d^*}(e, \frac{1}{k})) = B_{d^*}(e, \frac{1}{k}) \subseteq yB_{d^*}(e, \frac{1}{m}).
\]  

(5)

According to (4)-(5), we have

\[
xB_{d^*}(e, \frac{1}{k})yB_{d^*}(e, \frac{1}{m}) \subseteq xyB_{d^*}(e, \frac{1}{m})B_{d^*}(e, \frac{1}{m}) \subseteq B_{d^*}(xy, \frac{1}{n}).
\]

Since left translations are open, \( xB_{d^*}(e, \frac{1}{k}) \) and \( yB_{d^*}(e, \frac{1}{m}) \) are open neighborhoods of \( x \) and \( y \), respectively. Hence multiplication in \((G, d^*)\) is continuous. \( \square \)

Applying the previous results, we get the following results in semigroups and topological monoids.

**Corollary 4.3** ([16, Corollary 3.12]). Suppose that \( d \) is a invariant quasi-pseudometric on a semigroup \( S \). Then \((S, d)\) is a topological semigroup.

**Corollary 4.4** ([16, Corollary 3.14]). Let \( d \) be a left-invariant quasi-pseudometric on a monoid \( G \) such that for each \( x \in G \), \( \lambda_x \) is open and \( \rho_x \) is continuous at the identity \( e \) of \((G, d)\). Then \((G, d)\) is a topological semigroup.

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References

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