\(\top\)-quasi-Cauchy spaces — a non-symmetric theory of completeness and completion

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ABSTRACT

Based on the concept of Cauchy pair \(\top\)-filters, we develop an axiomatic theory of completeness for non-symmetric spaces, such as \(\top\)-quasi-uniform (limit) spaces or \(L\)-metric spaces. We show that the category of \(\top\)-quasi-Cauchy spaces is topological and Cartesian closed and we construct a finest completion for a non-complete \(\top\)-quasi-Cauchy space. In the special case of symmetry, \(\top\)-quasi-Cauchy spaces can be identified with \(\top\)-Cauchy spaces.

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KEYWORDS: fuzzy topology; pair \(\top\)-filter; Cauchy pair \(\top\)-filter; \(\top\)-quasi-Cauchy space; \(\top\)-quasi-uniform space; \(\top\)-quasi-uniform limit space; \(L\)-metric space.

1. INTRODUCTION

For studying completeness and completion, apart from uniform (limit) spaces, Cauchy spaces have been considered an appropriate framework, see for example Reed’s fundamental paper [20]. A Cauchy space is a set, together with a set of filters satisfying axioms that are derived from the properties of Cauchy filters of a uniform limit space [16]. As uniform limit spaces, uniform spaces and metric spaces, are symmetric, Cauchy spaces are from their very nature designed to describe completeness in symmetric spaces.
For many years it has been observed that non-symmetric spaces are of interest. Non-symmetric metric spaces, as so-called L-categories, are at the basis of monoidal topology, and allow a “categorical” treatment of completeness and completion [18, 12]. Non-symmetric uniform spaces, bearing the name quasi-uniform spaces, have been studied since the middle of the last century, see e.g. [17]. Completeness and completion in these spaces were defined and studied, among others, by Lindgren and Fletcher [6] with the help of Cauchy pair filters. Recently, this approach has been used – in the realm of many-valued topology – in $\top$-quasi-uniform spaces [25] using $\top$-Cauchy pair filters. In [15], we have stated the properties of the $\top$-Cauchy pair filters in a $\top$-quasi-uniform space (with a small error that we will correct in this paper). These properties can be used as an axiomatic definition of $\top$-quasi-Cauchy spaces, thus allowing a non-symmetric theory of completeness and completion that encompasses the corresponding approaches in $\top$-quasi-uniform spaces or in L-metric spaces. We use commutative and integral quantales L as basis for treating “many-valuedness” and would like to point out that for the two-point chain $L = \{0, 1\}$, $\top$-filters can be identified with ordinary filters and the theory developed in this paper can be used also in the classical case of quasi-uniform (limit) spaces.

The paper is organized as follows. In the next section, we collect the necessary facts and concepts about quantales, L-sets and $\top$-filters. In the third section we define our category of $\top$-quasi-Cauchy spaces and give important examples, and the fourth section is devoted to the categorical properties. Completeness and completion are studied in section 5 and section 6 shows that if a symmetry axiom is satisfied, then our spaces can be identified with the $\top$-Cauchy spaces of [21]. Finally we draw some conclusions.

2. Preliminaries

The triple $L = (L; \leq, \ast)$, where $(L, \leq)$ is a complete lattice with order relation $\leq$ and with distinct top and bottom elements $\top \neq \bot$, and $(L, \ast)$ is a commutative semigroup for which the top element of $L$ acts as the unit, i.e. $\alpha \ast \top = \alpha$ for all $\alpha \in L$, and $\ast$ is distributive over arbitrary joins, i.e. $(\bigvee_{i \in I} \alpha_i) \ast \beta = \bigvee_{i \in I} (\alpha_i \ast \beta)$, is called a commutative and integral quantale, see e.g. [12]. Important examples of such quantales are e.g. the unit interval $[0, 1]$ with a left-continuous t-norm [22], or Lawvere’s quantale, the interval $[0, \infty]$ with the opposite order and addition $\alpha \ast \beta = \alpha + \beta$, extended by $\alpha + \infty = \infty + a = \infty$, see e.g. [5]. A further important example is the quantale $(\Delta^+, \leq, \ast)$ of distance distribution functions [5]. This quantale finds applications, for example, in the theory of probabilistic metric spaces [22].

In a quantale, we can define an implication by $\alpha \to \beta = \bigvee \{\delta \in L : \alpha \ast \delta \leq \beta\}$. The implication is characterized by $\delta \leq \alpha \to \beta$ if, and only if, $\delta \ast \alpha \leq \beta$.

We list some of the properties of the implication, that we will use later on. We omit the straightforward proofs.

**Lemma 2.1.** Let $\alpha, \beta, \gamma, \delta, \alpha_j, \beta_j \in L$ for $j \in J$. The following assertions hold.

1. If $\alpha \leq \beta$, then $\alpha \to \gamma \geq \beta \to \gamma$ and $\gamma \to \alpha \leq \gamma \to \beta$. 
(2) $(\alpha \to \beta) \ast (\gamma \to \delta) \leq (\alpha \ast \gamma) \to (\beta \ast \delta)$.

(3) $\bigwedge_{j \in J} (\alpha_j \to \beta_j) = (\bigvee_{j \in J} \alpha_j) \to \beta$.

(4) $\bigwedge_{j \in J} (\alpha_j \to \beta_j) \leq (\bigvee_{j \in J} \alpha_j) \to (\bigwedge_{j \in J} \beta_j)$.

We denote the set of $L$-sets in $X$, or, more precisely, $L$-subsets of $X$, $a, b, c, \ldots$ by $L^X = \{a : X \to L\}$. In particular, we denote for $A \subseteq X$, the characteristic function $\top_A \in L^X$ by $\top_A(x) = \top$ if $x \in A$ and $\top_A(x) = \bot$ otherwise. The lattice operations are extended pointwisely from $L$ to $L^X$. If $a \in L^X$, $b \in L_Y$ and $\varphi : X \to Y$ is a mapping, then we define $\varphi(a) \in L^Y$ by $\varphi(a)(y) = \bigvee_{x : \varphi(x) = y} a(x)$ for $y \in Y$ and $\varphi^{-1}(b) = b \circ \varphi$.

For $a \in L^X$ and $b \in L^Y$ we define the monoidal product $a \otimes b \in L^X \times L^Y$ by $(a \otimes b)(x, y) = a(x) \ast b(y)$ for all $(x, y) \in X \times Y$.

For $b, d \in L^X$ we denote the fuzzy inclusion order $[2]$ by $[b, d] = \bigwedge_{x \in X} (b(x) \to d(x))$. We collect some of the properties that we will need later.

**Lemma 2.2.** Let $a, a', b, b', c \in L^X$, $d \in L^Y$ and let $\varphi : X \to Y$ be a mapping. Then

(i) $a \leq b$ if and only if $[a, b] = \top$;

(ii) $a \leq a'$ implies $[a', b] \subseteq [a, b]$ and $b \leq b'$ implies $[a, b] \leq [a, b']$;

(iii) $[a, c] \wedge [b, c] = [a \vee b, c]$;

(iv) $[\varphi(a), d] = [a, \varphi^{-1}(d)]$.

For $L$-sets $b, c \in L^X \times X$ we define the composition, $b \circ c \in L^X \times X$, by $b \circ c(x, y) = \bigvee_{z \in X} c(x, z) \ast b(z, y)$ for all $x, y \in X$ and the inverse, $b^{-1} \in L^X \times X$, by $b^{-1}(x, y) = b(y, x)$ for all $x, y \in X$.

**Definition 2.3 ([24, 9, 7]).** A subset $F \subseteq L^X$ is called a $\top$-filter on $X$ if

(\text{T-F1}) $\bigvee_{x \in X} b(x) = \top$ for all $b \in F$;

(\text{T-F2}) $a, b \in F$ implies $a \wedge b \in F$;

(\text{T-F3}) $\bigvee_{b \in F} [b, d] = \top$ implies $d \in F$.

We denote the set of all $\top$-filters on $X$ by $\mathcal{F}_L(X)$.

**Example 2.4.** For $x \in X$, $[x] = \{a \in L^X : a(x) = \top\}$ is a $\top$-filter. More general, if $a(x) = \top$ for some $x \in X$, then $[a] = \{b \in L^X : a \leq b\}$ is a $\top$-filter.

**Definition 2.5 ([24, 9, 7]).** A subset $B \subseteq L^X$ is called a $\top$-filter basis if

(\text{T-B1}) $\bigvee_{x \in X} b(x) = \top$ for all $b \in B$;

(\text{T-B2}) $a, b \in B$ implies $\bigvee_{c \in B} [c, a \wedge b] = \top$.

For a $\top$-filter base $B$, $[B] = \{a \in L^X : \bigvee_{b \in B} [b, a] = \top\}$ is a $\top$-filter, the $\top$-filter generated by $B$. For an ordinary filter $F$ on $X$, the set $\{\top_F : F \in F\}$ is a $\top$-filter basis and we denote the generated $\top$-filter by $\top_F$.

For the following definitions and properties, we refer to [9, 24, 14]. The set $\mathcal{F}_L(X)$ is ordered by $F \leq G$ if $F \subseteq G$. The meet of a non-empty family $(F_j)_{j \in J}$ of $\top$-filters on $X$ is given by $\bigwedge_{j \in J} F_j = \bigcap_{j \in J} F_j$ and a $\top$-filter base for $\bigwedge_{j \in J} F_j$ is given by $\{\bigvee_{j \in J} f_j : f_j \in F_j \forall j \in J\}$. 

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For a $\top$-filter $F \in F_\top^I(X)$ and a mapping $\varphi : X \to Y$, the set $\{\varphi(f) : f \in F\}$ is a $\top$-filter basis and we call the generated $\top$-filter, $\varphi(F) \in F_\top^I(Y)$, the \emph{image of $F$ under $\varphi$}. We then have $\varphi([x]) = [\varphi(x)]$ and for two mappings $\varphi : X \to Y$ and $\psi : Y \to Z$ we have $\psi(\varphi(F)) = (\psi \circ \varphi)(F)$. For a $\top$-filter $G \in L_\top^I(Y)$, the set $\{\varphi^\leftarrow(g) : g \in G\}$ is a $\top$-filter basis if and only if $\bigvee_{y \in \varphi(X)} g(y) = \top$ for all $g \in G$. In this case, we denote the generated $\top$-filter by $\varphi^\leftarrow(G) \in F_\top^I(X)$ and call it the \emph{inverse image of $G$ under $\varphi$}. We also say that $\varphi^\leftarrow(G)$ \emph{exists}. In the special case of a subset $A \subseteq X$ and the embedding $i_A : A \to X$, we denote for $G \in F_\top^I(X)$, in case of existence, $i_A^\leftarrow(G) = G_A$ and call it the \emph{trace of $G$ on $A$}.

For $F \in F_\top^I(X), G \in F_\top^I(U)$ we define $F \otimes G$ as the $\top$-filter on $X \times U$ generated by the $\top$-filter basis $\{f \otimes g : f \in F, g \in G\}$. For mappings $\varphi : X \to Y, \psi : U \to V$ we have, with the product mapping $(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y))$, $(\varphi \times \psi)(F \otimes G) = \varphi(F) \otimes \psi(G)$.

Finally, for $\top$-filters $\Phi, \Psi$ on $X \times X$, we define the \emph{inverse} $\Phi^{-1} = \{a^{-1} : a \in \Phi\}$ and the \emph{composition} $\Phi \circ \Psi$ as the $\top$-filter generated by the $\top$-filter basis $\{b \circ c : b \in \Phi, c \in \Psi\}$.

### 3. $\top$-quasi-Cauchy spaces

Following [6] and [25], we call, for $F, G \in F_\top^I(X), (F, G)$ a \emph{pair} $\top$-filter if for all $f \in F, g \in G$ we have $\bigvee_{x \in X} f(x) \ast g(x) = \top$. If this condition is satisfied, we also say that $F$ and $G$ are \emph{linked}. The set of all pair $\top$-filters on $X$ is denoted by $PF_\top^I(X)$. For $(F, G), (F', G') \in PF_\top^I(X)$ we write $(F, G) \leq (F', G')$ if $F \leq F'$ and $G \leq G'$.

**Lemma 3.1.** Let $\varphi : X \to X'$ be a mapping and let $(F, G) \in PF_\top^I(X)$. Then $(\varphi(F), \varphi(G)) \in PF_\top^I(X')$.

**Proof:** This follows as for $f \in F$ and $g \in G$ we have
\[
\bigvee_{x' \in X'} \varphi(f)(x') \ast \varphi(g)(x') \geq \bigvee_{x' \in X'} \bigvee_{\varphi(x') = x'} f(x) \ast g(x) = \bigvee_{x \in X} f(x) \ast g(x) = \top.
\]
\[\square\]

**Definition 3.2.** A set $C \subseteq PF_\top^I(X)$ is called a $\top$-\emph{quasi-Cauchy structure} if
- (TQC1) $([x], [x]) \in C$ for all $x \in X$;
- (TQC2) $(F', G') \in C$ whenever $(F, G) \in C$ and $(F, G) \leq (F', G') \in PF_\top^I(X)$;
- (TQC3) $(F \land F', G \land G') \in C$ whenever $(F, G), (F', G') \in C$ and $\bigvee_{x \in X} f(x) \ast g'(x) = \top = \bigvee_{x \in X} f'(x) \ast g(x)$ for all $f \in F, f' \in F', g \in G, g' \in G'$,

that is, if $F$ and $G$ as well as $F'$ and $G'$ are linked.

The pair $(X, C)$ is called a $\top$-\emph{quasi-Cauchy space}. A mapping $\varphi : (X, C) \to (X', C')$ is called \emph{Cauchy continuous} if $(\varphi(F), \varphi(G)) \in C'$ whenever $(F, G) \in C$.

We note that a $\top$-quasi-Cauchy structure is not just a pair of $\top$-Cauchy structures as defined in [21], see Section 7 below, one for each component of a
Example 3.3 \((T\text{-}Q\text{Unif and } L\text{-Met})\). A \(T\text{-quasi-uniform space}(\text{fuzzy }L\text{-quasi-uniform space}[9], \text{probabilistic quasi-uniform space}[25])\) is a pair \((X, \mathcal{U})\) of a set \(X\) and a \(T\)-filter \(\mathcal{U} \in F^u_T(X \times X)\) with the properties \((TU1) \mathcal{U} \subseteq [(x, x)]\) for all \(x \in X\). \((TU2) \mathcal{U} \subseteq \mathcal{U} \circ \mathcal{U}\). A mapping \(\varphi : (X, \mathcal{U}) \rightarrow (X', \mathcal{U}')\) between two \(T\)-quasi-uniform spaces \((X, \mathcal{U}), (X', \mathcal{U}')\) is called uniformly continuous if \(\mathcal{U}' \leq (\varphi \times \varphi)(\mathcal{U})\).

A pair \(T\)-filter \((F, G)\) is called a Cauchy pair \(T\)-filter in \((X, \mathcal{U})\) if \(\mathcal{U} \leq G \otimes F\), \([25]\). We denote the set of Cauchy pair \(T\)-filters in \((X, \mathcal{U})\) by \(\mathcal{C}P^u\). Then \((X, \mathcal{C}P^u)\) is a \(T\)-quasi-Cauchy space. The axiom \((TQC1)\) follows with \([x] \otimes [x] = [(x, x)]\). The axiom \((TQC2)\) is obvious. We show \((TQC3)\). Let \((F, G), (F', G') \in \mathcal{C}P^u\) and let \(\bigvee_{x \in X} f(x) * g(x) = \top\) and \(\bigvee_{x \in X} f'(x) * g(x) = \top\) for all \(f \in F, f' \in F', g \in G, g' \in G'\). Then \(\mathcal{U} \leq G \otimes F\) and \(\mathcal{U} \leq G' \otimes F'\) and hence, by \((TU2)\), \(\mathcal{U} \leq \mathcal{U} \circ \mathcal{U} \subseteq (G \otimes F) \circ (G' \otimes F')\). For \(g \in G, f \in F, g' \in G', f' \in F'\) we have

\[
(g \otimes f) \circ (g' \otimes f')(s, t) = \bigvee_{x \in X} g'(s) * f'(x) * g(x) * f(t)
= g'(s) * f(t) * \bigvee_{x \in X} f'(x) * g(x) = g' \circ f(s, t).
\]

Therefore, we have \(\mathcal{U} \leq (G \otimes F) \circ (G' \otimes F') = G' \otimes F\). Similarly, we can show that \(\mathcal{U} \leq G \otimes F'\) and we conclude

\[
\mathcal{U} \leq (G \otimes F) \wedge (G' \otimes F') \wedge (G' \otimes F') = (G \wedge G') \otimes (F \wedge F').
\]

Hence \((F \wedge F', G \wedge G') \in \mathcal{C}P^u\).

A uniformly continuous mapping \(\varphi : (X, \mathcal{U}) \rightarrow (X', \mathcal{U}')\) is Cauchy continuous as a mapping \(\varphi : (X, \mathcal{C}P^u) \rightarrow (X', \mathcal{C}P'^u)\): For \((F, G) \in \mathcal{C}P^u\) we have \(\mathcal{U} \leq G \otimes F\) and hence \(\mathcal{U}' \leq (\varphi \times \varphi)(\mathcal{U}) \leq (\varphi \times \varphi)(G \otimes F) = \varphi(G) \otimes \varphi(F)\), that is, we have \((\varphi(G), \varphi(F)) \in \mathcal{C}P'^u\).

This example encompasses \(L\)-metric spaces (also called continuity spaces \([5]\), \(L\)-categories \([18, 12]\) or \(L\)-preordered sets \([26]\)) \((X, d)\) with \((LM1)\) \(d(x, x) = \top\) for all \(x \in X\) and \((LM2)\) \(d(x, y) * d(y, z) \leq d(x, z)\) for all \(x, y, z \in X\). Then \([d] = \{u \in L^X \times X : d \leq u\}\) is a \(T\)-quasi-uniformity \([13]\) and we call a pair \(T\)-filter \((F, G)\) a Cauchy pair \(T\)-filter if \([d] \leq G \otimes F\), i.e., if \(\top = \bigvee_{g \in G, f \in F} [g \otimes f, d]\).

If \(F = T\mathcal{F}\) and \(G = T\mathcal{G}\) with filters \(F, G\) on \(X\), then \((T\mathcal{F}, T\mathcal{G})\) is a Cauchy pair \(T\)-filter if and only if \(F \vee G\) exists and \(\top = \bigvee_{F \otimes G} F \otimes G = [y, x]\).

For Lawvere’s quantale \(\mathcal{L} = ([0, \infty), \geq, +]\), this establishes a connection with the definition of Cauchy sequences in quasi-metric spaces given by Doitchinov \([4]\). A sequence \((x_n)\) in a quasi-metric space \((X, d)\) is called a Cauchy sequence if there is a co-sequence \((y_n)\) such that for all \(\epsilon > 0\) there is \(N \in \mathbb{N}\) such that for all \(m, n \geq N\) we have \(d(y_m, x_n) \leq \epsilon\). If we denote the generated filters by \(F = \langle(x_n)\rangle\) and \(G = \langle(y_n)\rangle\), then \((T\mathcal{F}, T\mathcal{G})\) being a Cauchy pair implies that
$(x_n)$ is a Cauchy sequence with $(y_n)$ as co-sequence. We note that Doitchinov does not demand that $\top_X$ and $\top_G$ are linked.

**Remark 3.4.** In [25], completeness for $\top$-quasi-uniform spaces was studied using Cauchy pair $\top$-filters and also using the concept of adjoint promodules. It was shown that both approaches are equivalent. The keypoint is, that adjoint promodules can be identified with (minimal) Cauchy pair $\top$-filters, see also [15]. The condition $\forall x \in X \ f(x) \ast g(x) = \top$ for all $f \in \mathbb{F}, g \in \mathbb{G}$ for a Cauchy pair $\top$-filter $(\mathbb{F}, \mathbb{G})$ is the one half of the adjointness. We would like to mention in this respect also the work of Preuß [19] who studied completeness in so-called preuniform convergence spaces by means of pre-Cauchy filters $\mathcal{F}$ on $X$. Also here the existence of a “co-filter” $\mathcal{G}$ on $X$ is required such that $\mathcal{G} \times \mathcal{F}$ belongs to the preuniform convergence structure, without demanding the existence of the join of $\mathcal{F}$ and $\mathcal{G}$. It seems that both Preuß’ and Doitchinov’s theory do not fully fit into a lax algebraic setting as it was developed e.g. in [3].

**Example 3.5 ($\top$-QULim).** Relaxing one axiom of a definition given in [21], we call a pair $(X, \Lambda)$ a $\top$-quasi-uniform limit space if $\Lambda \subseteq F^L_\top(X \times X)$ satisfies

1. ($\text{TULS1}$) $\{(x, x)\} \in \Lambda$ for all $x \in X$;
2. ($\text{TULS2}$) If $\Phi \leq \Psi$ and $\Phi \in \Lambda$, then $\Psi \in \Lambda$;
3. ($\text{TULS3}$) $\Phi \land \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$;
4. ($\text{TULS4}$) $\Phi \lor \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$ and $\Phi \lor \Psi$ exists.

A mapping $\varphi : (X, \Lambda) \rightarrow (X', \Lambda')$ between the $\top$-quasi-uniform limit spaces $(X, \Lambda)$ and $(X', \Lambda')$ is called **uniformly continuous** if $(\varphi \times \varphi)(\Phi) \in \Lambda'$ whenever $\Phi \in \Lambda$.

We call a pair $\top$-filter $(\mathbb{F}, \mathbb{G}) \in PF^L_\top(X)$ a **Cauchy pair $\top$-filter** if $\mathbb{G} \otimes \mathbb{F} \in \Lambda$ and we denote the set of all Cauchy pair $\top$-filters on $(X, \Lambda)$ by $\mathcal{CP}^A$. It is then not difficult to see that $(X, \mathcal{CP}^A)$ is a $\top$-quasi-Cauchy space and that a uniformly continuous mapping $\varphi : (X, \Lambda) \rightarrow (X', \Lambda')$ is Cauchy continuous as a mapping $\varphi : (X, \mathcal{CP}^A) \rightarrow (X', \mathcal{CP}^A)$.

**Example 3.6 ($\top$-PUConv).** Generalizing a definition of Preuß [19] to the lattice-valued case, we define a $\top$-preuniform convergence space $(X, \Lambda)$ with $\Lambda \subseteq F^L_\top(X \times X)$ satisfying ($\text{TULS1}$) and ($\text{TULS2}$). We further call $\mathbb{F} \in F^L_\top(X)$ a **Cauchy $\top$-filter** if $\mathbb{F} \otimes \mathbb{F} \in \Lambda$ and we call a pair $\top$-filter $(\mathbb{F}, \mathbb{G})$ a **Cauchy pair $\top$-filter** if $\mathbb{G} \otimes \mathbb{F} \in \Lambda$. We denote the set of Cauchy filters by $\mathcal{C}^A$ and the set of Cauchy pair $\top$-filters by $\mathcal{CP}^A$. It is then not difficult to show that with $\tilde{\Lambda}$ defined by

$$\tilde{\Lambda} = \{ \Phi \in F^L_\top(X \times X) : \exists \mathbb{F} \in \mathcal{C}^A \text{ s.t. } \mathbb{F} \otimes \mathbb{F} \leq \Phi \},$$

$(X, \tilde{\Lambda})$ is the finest $\top$-preuniform convergence space such that $\mathcal{C}^A = \mathcal{C}^{\Lambda}$, i.e. for any $\top$-preuniform convergence space $(X, \Lambda^*)$ with $\mathcal{C}^{\Lambda^*} = \mathcal{C}^A$ we have $\tilde{\Lambda} \subseteq \Lambda^*$. We note that $\tilde{\Lambda}$ is **symmetric** in the sense that $\Phi \in \tilde{\Lambda}$ implies $\Phi^{-1} \in \tilde{\Lambda}$. If we are looking for the finest non-symmetric $\top$-preuniform convergence space
(X, L) such that CP^L = CP^\overline{L}, then analogously we get
\overline{L} = \{ \Phi \in F_1^+ (X \times X) : \exists (F, G) \in CP^L \text{ s.t. } G \otimes F \leq \Phi \}.

As F \in C^L implies (F, G) \in CP^L we see that \overline{L} \subseteq L.

4. CATEGORICAL PROPERTIES

It is clear that for \( \tau \)-quasi-Cauchy spaces \((X, CP), (X', CP')\) and \((X'', CP'')\), the identity mapping \(id_X : (X, CP) \rightarrow (X, CP)\) is Cauchy continuous and that for two Cauchy continuous mappings \(\varphi : (X, CP) \rightarrow (X', CP')\), \(\psi : (X', CP') \rightarrow (X'', CP'')\) the composition \(\psi \circ \varphi : (X, CP) \rightarrow (X'', CP'')\) is Cauchy continuous. Therefore, we can form a category which has as objects the \( \tau \)-quasi-Cauchy spaces and as morphisms the Cauchy continuous mappings. We denote this category by \( \tau \)-QChy.

**Proposition 4.1.** \( \tau \)-QChy is a well-fibred and topological category.

**Proof.** The class of all \( \tau \)-quasi-Cauchy structures on a fixed set \( X \) is a subset of \( \{0, 1\}^{PF^+_1(X)} \), that is, it is a set and hence \( \tau \)-QChy is fibre-small. Furthermore, on a one-point set \( X = \{x\} \), there is exactly one \( \tau \)-quasi-Cauchy structure, namely \( CP = \{([x], [x])\} \). Hence \( \tau \)-QChy is well-fibred.

We show the existence of initial constructions. Consider a source \((\varphi_j : X \rightarrow (X_j, CP_j))_{j \in J}\). We define the initial \( \tau \)-quasi-Cauchy structure on \( X \) as follows. For \((F, G) \in PF^+_1(X)\), we define \((F, G) \in CP\) if for all \( j \in J \) we have \((\varphi_j(F), \varphi_j(G)) \in CP_j\).

Then \((X, CP)\) is a \( \tau \)-quasi-Cauchy space. As for each \( j \in J \) and each \( x \in X \) we have \((\varphi_j([x]), \varphi_j([x])) = ([\varphi_j(x)], [\varphi_j(x)]) \in CP_j\), we see that \(([x], [x]) \in CP\) for all \( x \in X \) and (TQC1) is valid. For (TQC2) consider \((F', G') \geq (F, G) \in CP\). Then for all \( j \in J \) we have \((\varphi_j(F'), \varphi_j(G')) \in CP_j\) and \(\varphi_j(F') \geq \varphi_j(F)\) and \(\varphi_j(G') \geq \varphi_j(G)\). Hence \((\varphi_j(F'), \varphi_j(G')) \in CP_j\) for all \( j \in J \) which implies \((F', G') \in CP\). To show (TQC3), let \((F, G), (F', G') \in CP\) and \(\bigvee_{x \in X} f(x) \ast g'(x) = \top = \bigvee_{x \in X} f'(x) \ast g(x)\) for all \( f \in F, f' \in F', g \in G, g' \in G'\). Then \(\bigvee_{x \in X} \varphi_j(f)(x) \ast \varphi_j(g')(x) = \top = \bigvee_{x \in X} \varphi_j(f')(x) \ast \varphi_j(g)(x)\) for all \( j \in J\). As \((\varphi_j(F), \varphi_j(G)), (\varphi_j(F'), \varphi_j(G')) \in CP_j\) we conclude \((\varphi_j(F \land F'), \varphi_j(G \land G')) \in CP\) for all \( j \in J \) and therefore \((F \land F', G \land G') \in CP\).

It is furthermore clear that all \( \varphi_j : (X, CP) \rightarrow (X_j, CP_j)\) are Cauchy continuous. Consider finally a mapping \( \psi : (X'', CP'') \rightarrow (X, CP)\) such that \(\varphi_j \circ \psi\) are Cauchy continuous for all \( j \in J\). For \((F'', G'') \in CP''\) then \((\varphi_j(\psi(F'')), \varphi_j(\psi(G''))) = (\varphi_j(\psi(F'')), \varphi_j(\psi(G''))) \in CP_j\) for all \( j \in J\). This implies \(\psi(F''), \psi(G'')\) \in CP and \(\psi\) is Cauchy continuous. \( \square \)

**Example 4.2** (Subspace). Let \((X, CP)\) be a \( \tau \)-quasi-Cauchy space and let \( A \subseteq X \). The initial \( \tau \)-quasi-Cauchy structure on \( A \) for the embedding \( i_A : A \rightarrow X, a \rightarrow a, CP_A, \) is defined, for \((F, G) \in PF^+_1(A)\), by
\[(F, G) \in CP_A \iff ([F], [G]) = (i_A(F), i_A(G)) \in CP.\]
The following result shows that a subspace of a \( \top \)-quasi uniform space induces the subspace of the \( \top \)-quasi-Cauchy space.

**Proposition 4.3.** Let \((X, \mathcal{U})\) be a \( \top \)-quasi-uniform space and let \( A \subseteq X \). Then \((F, G) \in (\mathcal{CP}^\Lambda)_A\) if and only if \( G \otimes F \geq \mathcal{U}_{A \times A} \).

**Proof.** This follows from \([G \otimes F] = (i_A \times i_A)(G \otimes F) = i_A(G) \otimes i_A(F) = [G] \otimes [F]\) and \([G \otimes F]_{A \times A} = G \otimes F\) and \([\mathcal{U}_{A \times A}] \geq \mathcal{U}\), see e.g. [15]. \(\square\)

In a similar way, we can show the next result.

**Proposition 4.4.** Let \((X, \Lambda)\) be a \( \top \)-quasi-uniform limit space and let \( A \subseteq X \). Then \((F, G) \in (\mathcal{CP}^\Lambda)_A\) if and only if there is \( \Phi \in \Lambda \) such that \( \Phi_{A \times A} \) exists and \( \Phi \otimes \Phi \geq \Phi_{A \times A} \).

**Example 4.5** (Product space). Let \((X_j, \mathcal{CP}_j)\) be \( \top \)-quasi-Cauchy spaces for all \( j \in J \). The initial \( \top \)-quasi-Cauchy structure on the Cartesian product \( \prod_{j \in J} X_j \) with respect to the projects \( pr_i : \prod_{j \in J} X_j \to X_i, \pi - \mathcal{CP} \), is defined by

\[(F, G) \in \pi - \mathcal{CP} \iff (pr_j(F), pr_j(G)) \in \mathcal{CP}_j \forall j \in J.\]

For our next result we will assume that \( \bigvee A = \top \) for \( A \subseteq L \) implies \( \bigvee_{\alpha \in A} \alpha * \alpha = \top \). It was shown in [13] that this can e.g. be ensured if the quantale \( \Lambda \) is divisible [10], i.e. if for all \( \alpha, \beta \in L \) with \( \alpha \leq \beta \) there is \( \gamma \in L \) such that \( \alpha = \beta * \gamma \). Another sufficient condition for this is the existence of a \( \top \)-approximating sequence \( (\alpha_1, \alpha_2, \ldots) \) in \( L \) with the properties \( \bot \neq \alpha_1 \leq \alpha_2 \leq \ldots < \top \) and \( \bigvee_{k \in \mathbb{N}} \alpha_k = \top \), [15]. Here, the well-below relation (sometimes also called the totally-below relation) is defined by \( \alpha \ll \beta \) if for all subsets \( D \subseteq L \) such that \( \beta \leq \bigvee D \) there is \( \delta \in D \) such that \( \alpha \leq \delta \).

We note that \((\Delta^+, \leq, *)\) satisfies this property [15], however \((\Delta^+, \leq, *)\) is in general not divisible, see [8]. Also, \( \Lambda \) being a value quantale [5] ensures the property, see [13].

**Theorem 4.6.** Let the quantale \( \Lambda \) satisfy that \( \bigvee_{\alpha \in A} \alpha * \alpha = \top \) whenever \( \bigvee A = \top \) for \( A \subseteq L \). Then the category \( \top - \mathcal{QChy} \) is Cartesian closed.

**Proof.** We show that \( \top - \mathcal{QChy} \) has function spaces in the sense of [1]. As a well-fibred topological category, it is then Cartesian closed.

For \((X, \mathcal{CP}), (X', \mathcal{CP}') \in \top - \mathcal{QChy}\) we denote

\[H = H((X, \mathcal{CP}), (X', \mathcal{CP}')) = \{ \varphi : (X, \mathcal{CP}) \to (X', \mathcal{CP}') \text{ Cauchy continuous} \}\]

We define \( \mathcal{CP}_c \subseteq \mathcal{PF}_H^\top (H) \) by

\[(H, K) \in \mathcal{CP}_c \iff \text{ for all } (F, G) \in \mathcal{CP} \text{ we have } \{ \text{ev}(H \otimes F), \text{ev}(K \otimes G) \} \in \mathcal{CP}',\]

with the evaluation mapping \( \text{ev} : H \times X \to X', (\varphi, x) \mapsto \varphi(x) \).

We first show that \((H, \mathcal{CP}_c)\) is a \( \top \)-quasi-Cauchy space.

\((\mathcal{TQC}1)\) We note that for \( \varphi \in H \) and \( a \in L^X \) we have \( \text{ev}(\top \{ \varphi \} \otimes a)(y) = \text{ev}(\varphi, x) = \top \{ \varphi \} a(x) = \varphi(a)(y) \). Hence, for \( F \in \mathcal{PF}_H^\top (X) \) we have \( \varphi(F) = \text{ev}([\varphi] \otimes F) \) and we obtain for \((F, G) \in \mathcal{CP}\) that \( \text{ev}([\varphi] \otimes \)}
F), ev([ϕ] ⊙ G)) = (ϕ(F), ϕ(G)) ∈ CP′ by the Cauchy continuity of ϕ. This shows ([ϕ], [ϕ]) ∈ CPc.

(TQC2) follows from (ev(H′ ⊙ F), ev(K′ ⊙ G)) ≥ (ev(H ⊙ F), ev(K ⊙ G)) for a pair T-filter (H′, K′) ≥ (H, K) ∈ CPc.

(TQC3) Let (H, K, (H′, K′) ∈ CPc such that \( \bigvee_{ϕ \in H} h(ϕ) * k′(ϕ) = \top = \bigvee_{ϕ \in H} h′(ϕ) * k(ϕ) \) for all \( h \in H, h′ \in H′, k \in K, k′ \in K′ \). For \( (F, G) ∈ CP \) we have (ev(H ⊙ F), ev(K ⊙ G)), (ev(H′ ⊙ F), ev(K′ ⊙ G)) ∈ CP′ and

\[
(\text{ev}((H ∧ H′) ⊙ F), \text{ev}((K ∧ K′) ⊙ G)) = (\text{ev}(H ⊙ F) ∧ \text{ev}(H′ ⊙ F), \text{ev}(K ⊙ G) ∧ \text{ev}(K′ ⊙ G)).
\]

The assumption on the quantale yields

\[
\bigvee_{y \in X'} \text{ev}(h ⊙ f)(y) * \text{ev}(k' ⊙ f)(y)
= \bigvee_{y \in X'} \bigvee_{ϕ(x) = y} h(ϕ) * f(x) * \bigvee_{ψ(z) = y} k'(ψ) * f(z)
\geq \bigvee_{y \in X'} \bigvee_{ϕ(x) = y} h(ϕ) * f(x) * k'(ϕ) * f(x)
= \bigvee_{ϕ \in H} h(ϕ) * k'(ϕ) * \bigvee_{x \in X} f(x) * f(x)
= \top * \top = \top.
\]

Similarly, \( \bigvee_{y \in X} \text{ev}(h' ⊙ f)(y) * \text{ev}(k ⊙ f)(y) = \top \). This is true for all \( h \in H, h' \in H', k \in K, k' \in K' \) and \( f \in F \) and we conclude that \( (\text{ev}((H ∧ H') ⊙ F), \text{ev}((K ∧ K') ⊙ G)) ∈ CP' \). Hence \( (H ∧ H', K ∧ K') ∈ CPc \).

Next we show that ev : \((H, CPc) × (X, CP) → (X', CP') \) is Cauchy continuous. To this end, let \((H, K) ∈ CPc × CP \). Then \((pr_H(H), pr_H(K)) ∈ CPc \) and \((pr_X(H), pr_X(K)) ∈ CP \) with the corresponding projection mappings. By the definition of CPc we obtain \((ev(pr_H(H) ⊙ pr_H(K)), ev(pr_H(K) ⊙ pr_X(K))) ∈ CP' \) and as \( ev(H), ev(K) \) \( ≥ \) \( ev(pr_H(H) ⊙ pr_X(H)), ev(pr_H(K) ⊙ pr_X(K)) \) we deduce \((ev(H), ev(K)) ∈ CP' \).

Finally, let \( ϕ : (X, CP) × (X', CP') → (X'', CP'') \) be a Cauchy continuous mapping. For \( x \in X \), we define \( ϕ_x : X' → X'' \) by \( ϕ_x(x') = ϕ(x, x') \). Let \( (F', G') ∈ CP' \). As \( ([x], [x]) ∈ CP \) we see that \( ([x] ⊙ F', [x] ⊙ G') ∈ CP × CP' \) and hence, by continuity of \( ϕ \), \( ϕ(x ⊙ F', x ⊙ G') \) \( ∈ CP'' \). It is not difficult to show that \( (ϕ_x(F'), ϕ_x(G')) ≥ (ϕ([x] ⊙ F'), ϕ([x] ⊙ G')) \) and therefore \( ϕ_x \) is Cauchy continuous. We can hence define \( ϕ^* : X → H((X', CP'), (X'', CP'')) \) by \( ϕ^*(x) = ϕ_x \) and we need to show that \( ϕ^* \) is Cauchy continuous. Let \( (F, G) ∈ CP \). Then, for all \( (H, K) ∈ CP' \) we have \( (F ⊙ H, G ⊙ K) ∈ CP × CP' \). From \( ev(ϕ^* × id_X) = ϕ \) we obtain with the Cauchy continuity of \( ϕ \),

\[
(\text{ev}(ϕ^*(F) ⊙ H), \text{ev}(ϕ^*(G) ⊙ K)) = (ϕ(F ⊙ H), ϕ(G ⊙ K)) ∈ CP''.
\]

This shows that \( (ϕ^*(F), ϕ^*(G)) ∈ CPc \) and the proof is complete. □
5. Convergence

For a $\top$-quasi-Cauchy space $(X, CP)$ we say that a pair $\top$-filter $(F, G) \in PF_1^{\top}(X)$ converges to $x \in X$, and we write $x \in q^{CP}(F, G)$, if $(F \land [x], G \land [x]) \in CP$. We note that convergent pair $\top$-filters are Cauchy pair $\top$-filters by (TQCS2).

Example 5.1. Let $(X, U)$ be a $\top$-quasi-uniform space. In [25], see also [15], it is defined that a pair $\top$-filter $(F, G)$ converges to $x$ if and only if $[x] \otimes F \geq U$ and $G \otimes [x] \geq U$. In [15] it was shown that this requirement is equivalent to $(F \land [x], G \land [x])$ being a Cauchy pair $\top$-filter.

Example 5.2. Let $(X, \Lambda)$ be a $\top$-quasi-uniform limit space. We say that a pair $\top$-filter $(F, G)$ converges to $x$ if $[x] \otimes F \in \Lambda$ and $G \otimes [x] \in \Lambda$. It is not difficult to show that this is equivalent to $(F \land [x], G \land [x])$ being a Cauchy pair $\top$-filter.

Proposition 5.3. Let $(X, CP)$ be a $\top$-quasi-Cauchy space. Convergence of pair $\top$-filters has the following properties. For all $(F, G), (F', G') \in PF_1^{\top}(X)$ and all $x \in X$ we have:

(TQL1) $x \in q^{CP}([x], [x])$;
(TQL2) $(F, G) \leq (F', G')$ implies $q^{CP}(F, G) \subseteq q^{CP}(F', G')$;
(TQL3) $q^{CP}(F, G) \cap q^{CP}(F', G') \subseteq q^{CP}(F \land F', G \land G')$.

Proof. We only show (TQL3). We have $(F \land [x], G \land [x]), (F' \land [x], G' \land [x]) \in CP$. The axiom (TQC3) then yields $(F \land F' \land [x], G \land G' \land [x]) \in CP$.

Proposition 5.4. Let $(X, CP)$ and $(X', CP')$ be $\top$-quasi-Cauchy spaces and let $\varphi : (X, CP) \rightarrow (X', CP')$ be Cauchy continuous. Then $x \in q^{CP}(F, G)$ implies $\varphi(x) \in q^{CP'}(\varphi(F), \varphi(G))$.

Proof. If $(F \land [x], G \land [x]) \in CP$, then by Cauchy continuity, $(\varphi(F) \land [\varphi(x)], \varphi(G) \land [\varphi(x)]) = (\varphi(F \land [x]), \varphi(G \land [x])) \in CP'$.

Definition 5.5. Let $(X, P)$ be a $\top$-quasi-Cauchy space and let $A \subseteq X$. We define the closure of $A$, $\overline{A} = \overline{A}^{CP}$, by $x \in \overline{A}$ if there is a pair $\top$-filter $(F, G) \in PF_1^{\top}(A)$ such that $x \in q^{CP}([F], [G])$.

The following result shows that this notion of closure coincides with the definition of closure in $\top$-quasi-uniform spaces given by Wang and Yue [23].

Proposition 5.6. Let $(X, U)$ be a $\top$-quasi-uniform space and let $A \subseteq X$. Then $x \in \overline{A}^{CP}$ if, and only if, for all $u \in U$ we have $\forall_{a \in A} u(x, a) \ast u(a, x) = \top$.

Proof. Let $x \in \overline{A}^{CP}$. By the definition of $CP$ there is a pair $\top$-filter $(F, G)$ on $A$ such that $[x] \otimes [F] \geq U$ and $[G] \otimes [x] \geq U$. Hence, for $u \in U$, we have
\( T = \bigvee_{f \in F} [T_x \otimes f, u] = \bigvee_{g \in G} [g \otimes T_x, u]. \) We conclude
\[
T = \bigvee_{f \in F, g \in G, a \in A} \bigcap (f(a) \to u(x, a)) \ast (g(a) \to u(a, x)) \\
\leq \bigvee_{f \in F, g \in G, a \in A} \bigcap ((f(a) \ast g(a)) \to (u(x, a) \ast u(a, x))) \\
\leq \bigvee_{f \in F, g \in G, a \in A} \bigcap ((\bigvee_{a \in A} f(a) \ast g(a)) \to (\bigvee_{a \in A} u(x, a) \ast u(a, x))) \\
= \bigvee_{a \in A} (u(x, a) \ast u(a, x))
\]

because \((F, G)\) is a pair \(T\)-filter on \(A\).

Conversely, let \(\bigvee_{a \in A} u(x, a) \ast u(a, x) = T\) for all \(u \in U\). We define \(F\) as the \(T\)-filter on \(A\) with \(T\)-filter basis on \(A\), \(\{u(x, \cdot) : u \in U_{A \times A}\}\). Here, \(u(x, \cdot)(a) = u(x, a)\) for \(a \in A\), that is, \(u(x, \cdot) \in LA\). Likewise, \(G\) is the \(T\)-filter on \(A\) with \(T\)-filter basis \(\{u(\cdot, x) : u \in U_{A \times A}\}\). The given condition guarantees that \((F, G)\) is a pair \(T\)-filter on \(A\). Moreover, we have \([x] \otimes [F] \geq U\), as for \(u \in U\) we have \(\bigvee_{x \in A} u(x, \cdot)(s, t) \leq u(s, t)\) for \(s, t \in X\). Similarly, \([G] \otimes [x] \geq U\).

This implies \(x \in q^{CP_T}([F], [G])\), that is, \(x \in \overline{A}^{CP_T}\).

We note that for an \(L\)-metric space \((X, d)\), the closure of \(A\) in \((X, [d])\) is characterized by \(x \in \overline{A}^{CP_T}\) if and only if \(\bigvee_{a \in A} d(x, a) \ast d(a, x) = T\). This is a characterization of closure in \((X, d)\) used in [11].

Using the concept of convergence, we can introduce the following separation axiom. We call a \(T\)-quasi-Cauchy space \((X, CP)\) separated if \(x, y \in q^{CP}(F, G)\) implies \(x = y\).

Separation for \(T\)-quasi-uniform spaces was defined in [23]. The following result shows that our definition applies also there.

**Proposition 5.7.** Let \((X, U)\) be a \(T\)-quasi-uniform space. Then \((X, CP)\) is separated if and only if \(x = y\) whenever \(u(x, y) = T = u(y, x)\) for all \(u \in U\).

**Proof.** Let \((X, CP)\) be separated and let \(u(x, y) = T = u(y, x)\) for all \(u \in U\). Then \([x] \otimes [y] \geq U\) and \([y] \otimes [x] \geq U\). This implies \(x \in q^{CP}([y], [y])\) and because \(y \in q^{CP}([y], [y])\) we obtain \(x = y\).

For the converse, let \(x, y \in q^{CP}(F, G)\). Then \([x] \otimes F \geq U\) and \(G \otimes [y] \geq U\) and with (TU2), we conclude \(U \leq U \circ U \leq (G \otimes [y]) \circ ([x] \otimes F) = [x] \otimes [y]\), as \((F, G)\) is a pair \(T\)-filter. Similarly, we see \([y] \otimes [x] \geq U\). Hence, for all \(u \in U\) we have \(u(x, y) = T\) and \(u(y, x) = T\) and therefore \(x = y\).

For the special case of an \(L\)-metric space \((X, d)\), we have that \((X, CP^{[d]}(F, G))\) is separated if and only if \(x = y\) whenever \(d(x, y) = T = d(y, x)\). Again, this is a characterization of separation of an \(L\)-metric space in [11].
6. Completeness and completion

We say that a pair \( \top \)-filter \((F, G)\) is convergent in \((X, CP)\) if there is \( x \in X \) such that \( x \in q^{CP}(F, G) \), that is, if \((F \land \{x\}, G \land \{x\}) \in CP\). Otherwise, we call \((F, G)\) non-convergent. A \( \top \)-quasi-Cauchy space \((X, CP)\) is called complete if every \((F, G) \in CP\) is convergent.

**Proposition 6.1.** Let \((X_j, CP_j)\) be complete \( \top \)-quasi-Cauchy spaces for all \( j \in J \). Then also the product space \((\prod_{j \in J} X_j, \pi - CP)\) is complete.

**Proof.** Let \((F, G) \in \pi - CP\). Then for all \( j \in J \), \((pr_j(F), pr_j(G)) \in CP_j\) and hence there is \( x_j \in X_j \) such that \((pr_j(F) \land \{x_j\}, pr_j(G) \land \{x_j\}) \in CP_j\). We define \( x = (x_j)_{j \in J} \). Then \((pr_j(F \land \{x\}), pr_j(G \land \{x\})) = (pr_j(F) \land \{x_j\}, pr_j(G) \land \{x_j\}) \in CP_j\) for all \( j \in J \) and hence \((F \land \{x\}, G \land \{x\}) \in \pi - CP\). \( \square \)

A completion \(((X^+, CP^+), \phi)\) of a \( \top \)-quasi-Cauchy space \((X, CP)\) is a complete \( \top \)-quasi-Cauchy space \((X^+, CP^+)\) and a dense Cauchy embedding \( \phi : (X, CP) \rightarrow (X^+, CP^+)\). This means that \( \phi \) is injective and that we have \((F, G) \in CP\) if, and only if, \((\phi(F), \phi(G)) \in CP^+\) and that \( \phi(X)^{CP^+} = X^+\).

For two completions \(((X^+, CP^+), \phi), ((X^-, CP^-), \psi)\) of \((X, CP)\) we call \(((X^+, CP^+), \phi)\) finer than \(((X^-, CP^-), \psi)\), and we write \(((X^+, CP^+), \phi) \geq ((X^-, CP^-), \psi)\), if there is a Cauchy continuous mapping \( h : (X^+, CP^+) \rightarrow (X^-, CP^-)\) such that \( h \circ \phi = \psi\).

We are now going to construct a completion of a non-complete \( \top \)-quasi-Cauchy space \((X, CP)\). To this end, the following relation on \( CP\) is useful. Let \((F, G), (F', G') \in CP\). We define
\[
(F, G) \sim (F', G') \iff (F \land F', G \land G') \in CP.
\]

It is clear that for \((F, G) \in CP\) we have \((F, G) \sim ([x], [x])\) if and only if \( x \in q^{CP}(F, G)\).

**Proposition 6.2.** Let \((X, CP)\) be a \( \top \)-quasi-Cauchy space. The relation \( \sim \) is an equivalence relation.

**Proof.** Reflexivity and symmetry of the relation are clear. We check the transitivity. Let \((F, G) \sim (F', G')\) and \((F', G') \sim (F'', G'')\). As \( F \land F' \leq F'\) and \( G' \land G'' \leq G'\) we see that for \( f \in F \land F'\) and \( g \in G' \land G''\) we have \( \bigvee_{x \in X} f(x) * g(x) = \top\), because \((F', G')\) is a pair \( \top \)-filter. Similarly, we see that for \( g \in G \land G'\) and \( f \in F' \land F''\) we have \( \bigvee_{x \in X} f(x) * g(x) = \top\). Hence, from (TQC3) we obtain \((F \land F', G \land G' \land G'') \in CP\) and, using (TQC2), we conclude \((F \land F', G \land G' \land G'') \in CP\), that is, \((F, G) \sim (F'', G'')\). \( \square \)

We denote the equivalence class of \((F, G) \in CP\) by \((\langle F, G \rangle)\).

The equivalence relation allows simple proofs of the following results.

**Lemma 6.3.** Let \((X, CP)\) be a \( \top \)-quasi-Cauchy space and let \((F, G), (F', G') \in CP\). If \((F', G') \leq (F, G)\) and if \( x \in q^{CP}(F, G)\), then also \( x \in q^{CP}(F', G')\).

**Proof.** If \((F', G') \leq (F, G)\), then \((F', G') \sim (F, G) \sim ([x], [x])\) and, by transitivity, \((F', G') \sim ([x], [x])\). \( \square \)
Lemma 6.4. Let \((X, \mathcal{C}P)\) be a \(T\)-quasi-Cauchy space. If there is a pair \(T\)-filter \((F, G)\) converging to both \(x\) and \(y\), then \(\{ (H, K) \in PF^T_L (X) : x \in q_{CP} (H, K) \} = \{ (H, K) \in PF^T_L (X) : y \in q_{CP} (H, K) \}\).

Proof. Clearly, \((F, G) \in \mathcal{C}P\) and we have \(\langle [y], [y] \rangle \sim \langle (F, G) \sim \langle [x], [x] \rangle \). So if \((H, K) \sim \langle [x], [x] \rangle\), then by transitivity \((H, K) \sim \langle [y], [y] \rangle\) and vice versa. □

Lemma 6.5. Let \((X, \mathcal{C}P)\) be a \(T\)-quasi-Cauchy space. If \(x \in q_{CP}(F, G)\) and \((F', G') \in \mathcal{C}P\) satisfies that for all \(f \in F, g \in G, f' \in F', g' \in G'\) we have \(\forall x \in X f'(x) * g(x) = \top = \forall x \in X f(x) * g'(x)\), then \(x \in q_{CP}(F', G')\).

Proof. We have \((F \wedge F', G \wedge G') \in \mathcal{C}P\) and \((F, G) \leq (F', G')\). According to Lemma 6.3, \(x \in q_{CP}(F \wedge F', G \wedge G') \subseteq q_{CP}(F', G')\). □

We define now \(X^* = X \cup \{ \langle (F, G) : (F, G) \in \mathcal{C}P\}\) non-convergent\} and we denote \(j : X \rightarrow X^*, x \mapsto x\) the embedding injection of \(X\) into \(X^*\). We define \(\mathcal{C}P^* \subseteq PF^T_L (X^*)\) as follows. \((H, K) \in \mathcal{C}P^*\) if there is \((F, G) \in \mathcal{C}P\) convergent such that \(H \geq j(F)\) and \(K \geq j(G)\) or if there is \((F, G) \in \mathcal{C}P\) non-convergent such that \(H \geq j(F) \wedge \langle [(F, G)] \rangle\) and \(K \geq j(G) \wedge \langle [(F, G)] \rangle\).

Theorem 6.6. Let \((X, \mathcal{C}P)\) be a \(T\)-quasi-Cauchy space. Then \((X^*, \mathcal{C}P^*)\) \(j\) is a completion of \((X, \mathcal{C}P)\).

Proof. We first show that \((X^*, \mathcal{C}P^*)\) is a \(T\)-quasi-Cauchy space. The axiom (TQC1) follows, as for \(x \in X\) we have \(j([x]) = [j(x)]\). For \((F, G) \in \mathcal{C}P\) non-convergent, we have \(\langle [(F, G)] \rangle \geq j(F) \wedge \langle [(F, G)] \rangle\) and \(\langle [(F, G)] \rangle \geq j(G) \wedge \langle [(F, G)] \rangle\) and hence also \(\langle [(F, G)] \rangle, \langle [(F, G)] \rangle \in \mathcal{C}P^*\).

The axiom (TQC2) is obvious.

For the axiom (TQC3), let \((H, K), (H', K') \in \mathcal{C}P^*\) such that \(\forall z \in X^*, h(z)^* * k'(z)^* = \top = \forall z \in X^*, h'(z)^* * k(z)^* \) for all \(h \in H, h' \in H', k \in K, k' \in K'\). We distinguish four cases.

Case 1: There are \((F, G), (F', G') \in \mathcal{C}P\) convergent such that \(H \geq j(F), K \geq j(G)\) and \(H' \geq j(F'), K' \geq j(G')\). We then have
\[
\top = \bigvee_{z \in X^*} j(f)(z)^* * j(g')(z)^*
= \bigvee_{z \in X} j(f)(z) * j(g')(z)
\vee \bigvee_{(F, G) \in \mathcal{C}P \text{ non-conv.}} j(f) \langle [(F, G)] \rangle * j(g') \langle [(F, G)] \rangle
= \bigvee_{z \in X} f(z)^* * g'(z)
\]
Similarly, \(\forall z \in X f'(z)^* * g(z) = \top\) and hence \((F \wedge F', G \wedge G') \in \mathcal{C}P\) convergent, from Lemma 6.3. Therefore \(j(F \wedge F'), j(G \wedge G') \leq (H \wedge H', K \wedge K')\) and we obtain \((H \wedge H', K \wedge K') \in \mathcal{C}P^*\).

Case 2: There is \((F, G) \in \mathcal{C}P\) convergent such that \(H \geq j(F)\) and \(K \geq j(G)\) and there is \((F', G') \in \mathcal{C}P\) non-convergent such that \(H' \geq j(F') \wedge \langle [(F', G')] \rangle\) and
\[ K' \geq j(G') \land [((F', G')] \]  
If \( \forall x \in X f'(x) \ast g(x) = T = \bigvee_{x \in X} f(x) \ast g'(x) \) for all \( f \in F, f' \in F', g \in G, g' \in G' \), then \( (F', G') \) would be convergent by Lemma 6.5. Hence we may assume without loss of generality that there are \( f' \in F', g \in G \) such that \( \bigvee_{x \in X} f'(x) \ast g(x) \neq T \). As \( j(f') \lor T((F', G')) \in j(F') \land [((F', G'))] \leq H_0 \) we conclude, with \( j(g) \in j(G) \leq K \) that \( \bigvee_{x \in X} (j(f') \lor T((F', G'))(x')) \ast j(g)(x') = \bigvee_{x \in X} f'(x) \ast g(x) \neq T, \) a contradiction. Hence this case does not occur.

**Case 3:** There is \( (F', G') \in CP \) convergent such that \( H_0 \geq j(F') \) and \( K' \geq j(G') \) and there is \( (F, G) \in CP \) non-convergent such that \( H \geq j(F) \land [(F, G)] \) and \( K \geq j(G) \land [(F, G)] \). The arguments of case 2 can be used to show that also this case does not occur.

**Case 4:** There are \( (F, G), (F', G') \in CP \) non-convergent such that \( H \geq j(F) \land [((F, G))] \), \( K \geq j(G) \land [((F, G))] \) and \( H' \geq j(F') \land [((F', G'))], K' \geq j(G') \land [((F', G'))] \). Then we have, for \( f \in F, f' \in F', g \in G, g' \in G' \),

\[
T = \bigvee_{x^* \in X^*} (j(f) \lor T((F,G)))(x^*) \ast (j(g') \lor T((F',G')))(x^*) \\
= \bigvee_{x \in X} f(x) \ast g'(x) \lor \bigvee_{x^* \in X^* \setminus X} T((F,G))(x^*) \ast T((F',G'))(x^*),
\]

and similarly,

\[
T = \bigvee_{x \in X} f'(x) \ast g(x) \lor \bigvee_{x^* \in X^* \setminus X} T((F',G'))(x^*) \ast T((F,G))(x^*).
\]

If \( (F, G) \neq (F', G') \), then \( \bigvee_{x \in X} f(x) \ast g'(x) = T = \bigvee_{x \in X} f'(x) \ast g(x) \) and \( (F \land F', G \land G') \in CP \). As this pair \( T \text{-filter} \) is \( (F, G), (F', G') \) we conclude \( (F, G) = (F \land F', G \land G') = (F', G') \), a contradiction. Hence \( (F, G) = (F', G') \). Then \( (F, G) \sim (F', G') \) and hence \( (F \land F', G \land G') \in CP \) is non-convergent, too. Therefore \( H \land H' \geq j(F \land F') \land [((F, G))] \) and \( K \land K' \geq j(G \land G') \land [((F', G'))] \) and we have \( (H \land H', K \land K') \in CP \).

The mapping \( j : (X, CP) \rightarrow (X^*, CP^*) \) is Cauchy continuous, as for \( (F, G) \in CP \) either \( (F, G) \) is convergent and then \( j(F), j(G) \in CP^* \). Or \( (F, G) \) is non-convergent and then, because \( j(F) \geq j(F) \land [((F, G))] \) and \( j(G) \geq j(F) \land [((F, G))] \), again \( j(F), j(G) \in CP^* \).

Conversely, if \( j(F), j(G) \in CP^* \), then either there is \( (F', G') \in CP \) convergent such that \( F \geq F' \) and \( G \geq G' \) and hence \( (F, G) \in CP \) by \( TQC2 \). Or there is \( (F', G') \in CP \) non-convergent such that \( j(F) \geq j(F') \land [((F', G'))] \) and \( j(G) \geq j(G') \land [((F', G'))] \). This implies \( F' \leq F \) and \( G' \leq G \) and with \( TQC2 \), \( (F, G) \in CP \).

Furthermore, \( (X^*, CP^*) \) is complete. If \( (G, K) \in CP^* \) such that \( H \geq j(F) \) and \( K \geq j(G) \) with \( (F, G) \in CP \) convergent to \( x \in X \), then \( (H, K) \) is convergent to \( x \) in \( (X^*, CP^*) \). If \( (G, K) \in CP^* \) such that \( H \geq j(F) \land [((F, G))] \) and \( K \geq j(G) \land [((F, G))] \) with \( (F, G) \in CP \) non-convergent, then \( (H \land [((F, G))], K \land [((F, G))] \in CP^* \), that is \( (H, K) \) converges to \( (F, G) \) in \( (X^*, CP^*) \).
Finally, we show that $j(X)^{CP^*} = X^*$. Let $x^* \in X^*$. If $x \in X$, then $(j([x]), j([x]))$ converges to $x$ in $(X^*, CP^*)$. If $x^* = (F, G) \in CP$ non-convergent, then $(j(F) \wedge [(F, G)]), j(G) \wedge [(F, G)]) \in CP^*$ and hence $(j(F), j(G))$ converges to $x^*$ in $(X^*, CP^*)$. Hence $x^* \in j(X)^{CP^*}$ and the proof is complete.

**Theorem 6.7.** Let $(X, CP), (X', CP')$ be $\top$-quasi-Cauchy spaces, let $(X', CP')$ be complete and let $\varphi : (X, CP) \rightarrow (X', CP')$ be Cauchy continuous. Then there exists a Cauchy continuous mapping $\varphi^* : (X^*, CP^*) \rightarrow (X', CP')$ such that $\varphi = \varphi^* \circ j$.

**Proof.** We define $\varphi^*$ as follows. For $x^* = j(x)$ with $x \in X$, we define $\varphi^*(x^*) = \varphi(x)$ and for $x^* = (F, G) \in CP$ non-convergent, we define $\varphi^*(x^*) = y$, where $y$ is one of the limits of $(\varphi(F), \varphi(G))$ in $(X', CP')$. We note that we consider $y$ as a fixed choice and that it does not depend on the representative of $(\varphi(F), \varphi(G))$. For if $(F, G) \sim (F', G')$ then $(F \wedge F', G \wedge G') \in CP$ and is $\leq (F, G)$. Hence, if $y \in q^{CP} (\varphi(F), \varphi(G))$, with Lemma 6.3 also $y \in q^{CP} (\varphi(F) \wedge \varphi(G), \varphi(G \wedge G'))$ and using (TQC2) also $y \in q^{CP} (\varphi(F'), \varphi(G'))$.

Clearly, with this definition, we have $\varphi^* \circ j = \varphi$ and we have to show that $\varphi^*$ is Cauchy continuous. To this end, let $(H, K) \in CP^*$. We distinguish two cases.

1. **Case 1:** There is $(F, G) \in CP$ convergent such that $H \geq j(F)$ and $K \geq j(G)$. Let $x \in q^{CP} (F, G)$. Then $\varphi^*(H) \geq \varphi(F), \varphi^*(K) \geq \varphi(G)$ and $x(\varphi^*) \in q^{CP} (\varphi(F), \varphi(G))$. This means $(\varphi(F) \wedge [\varphi(x)], \varphi(G) \wedge [\varphi(x)]) \in CP'$ and (TQC2) yields $(\varphi^*(H), \varphi^*(K)) \in CP'$.

2. **Case 2:** There is $(F, G) \in CP$ non-convergent such that $\mathbb{H} \geq j(F) \wedge [(F, G)]$ and $\mathbb{K} \geq j(G) \wedge [(F, G)]$. We conclude $\varphi^*(\mathbb{H}) \geq \varphi^*(j(F)) \wedge [\varphi^*(j(F))] = \varphi(F) \wedge [y]$ and, similarly, $\varphi^*(\mathbb{K}) \geq \varphi(G) \wedge [y]$, where $y \in q^{CP} (\varphi(F), \varphi(G))$. Hence $(\varphi(F) \wedge [y], \varphi(G) \wedge [y]) \in CP'$ and, again with (TQC2), $(\varphi^*(\mathbb{H}), \varphi^*(\mathbb{K})) \in CP'$.

**Corollary 6.8.** Let $(X, CP)$ be a $\top$-quasi-Cauchy space. Then $((X^*, CP^*), j)$ is the finest completion of $(X, CP)$.

**Proof.** If $((\mathbb{X}^\sim, CP^\sim), \psi)$ is a further completion, then $\psi : (X, C) \rightarrow (\mathbb{X}^\sim, CP^\sim)$ is Cauchy continuous and $(\mathbb{X}^\sim, CP^\sim)$ is complete. Theorem 6.7 ensures that there exists a Cauchy continuous mapping $\psi^* : (X^*, CP^*) \rightarrow (\mathbb{X}^\sim, CP^\sim)$ such that $\psi^* \circ j = \psi$. Hence, $((X^*, CP^*), j) \succeq ((\mathbb{X}^\sim, CP^\sim), \psi)$.

**Corollary 6.9.** Let $(X, CP_X), (Y, CP_Y)$ be $\top$-quasi-Cauchy spaces and let $\varphi : (X, CP_X) \rightarrow (Y, CP_Y)$ be Cauchy continuous. We denote the finest completions of $(X, CP_X)$ and $(Y, CP_Y)$ by $(X^*, CP^*_X)$ and $(Y^*, CP^*_Y)$, respectively. Then there exists a Cauchy continuous mapping $\varphi^* : (X^*, CP^*_X) \rightarrow (Y^*, CP^*_Y)$ such that $\varphi^* \circ j_X = j_Y \circ \varphi$.

**Proof.** We consider the Cauchy continuous mapping $g = j_Y \circ \varphi : (X, CP) \rightarrow (Y^*, CP^*_Y)$. Then there exists a Cauchy continuous mapping $\varphi^* : (X^*, CP^*) \rightarrow (Y^*, CP^*_Y)$ such that $\varphi^* \circ j_X = g$. □
Finally, we are showing that separation carries over from \((X, CP)\) to \((X^*, CP^*)\).

**Proposition 6.10.** Let \((X, CP)\) be a separated \(\top\)-quasi-Cauchy space. Then also \((X^*, CP^*)\) is separated.

**Proof.** Let \(x^*, y^* \in q^{CP^*}(\mathbb{H}, \mathbb{K})\). We distinguish three cases.

**Case 1:** \(x^* = j(x), y^* = j(y)\) with \(x, y \in X\). Then, by definition of \(q^{CP^*}\), we have \((\mathbb{H} \land [j(x)], \mathbb{K} \land [j(x)]) \in CP^*\) and \((\mathbb{H} \land [j(y)], \mathbb{K} \land [j(y)]) \in CP^*\).

If \(\mathbb{H} \land [j(x)] \geq j(\mathbb{F}) \land [((\mathbb{F}, \mathbb{G}))]\) and \(\mathbb{K} \land [j(x)] \geq j(\mathbb{G}) \land [((\mathbb{F}, \mathbb{G}))]\) with \((\mathbb{F}, \mathbb{G}) \in CP\) non-convergent, then for \(f \in \mathbb{F}\) we have that \(j(f) \lor \top \land [((\mathbb{F}, \mathbb{G}))] \in \mathbb{H} \land [j(x)]\).

Hence
\[ T = \bigvee_{h \in \mathbb{H}} [h \lor \top \land j(x), j(f) \lor \top \land [((\mathbb{F}, \mathbb{G}))]] \leq j(f)(j(x)) = f(x), \]

and we have \(\mathbb{F} \leq [x]\). Similarly, we obtain \(\mathbb{G} \leq [x]\) and hence \((\mathbb{F}, \mathbb{G})\) converges to \(x\) in \((X, CP)\), a contradiction. Hence we must have \(\mathbb{H} \land [j(x)] \geq j(\mathbb{F})\) and \(\mathbb{K} \land [j(x)] \geq j(\mathbb{G})\) with \((\mathbb{F}, \mathbb{G}) \in CP\) convergent. If we assume that \((\mathbb{F}, \mathbb{G})\) converges to \(\mathfrak{x}\), then \(\mathbb{H} \land [j(x)] \geq j(\mathbb{F}) \land [j(\mathfrak{x})]\) and for \(f \in \mathbb{F}\) we have \(j(f) \lor \top \land j(\mathfrak{x}) \in \mathbb{H} \land [j(x)]\). This implies
\[ T = \bigvee_{h \in \mathbb{H}} [h \lor \top \land j(x), j(f) \lor \top \land j(\mathfrak{x})] \leq j(f)(j(x)) \lor \top j(\mathfrak{x})(j(x)). \]

If \(j(\mathfrak{x}) \neq j(x)\), then \(f(x) = \top\) and we see that \(\mathbb{F} \leq [x]\). Similarly, \(\mathbb{G} \leq [x]\) and hence \((\mathbb{F}, \mathbb{G})\) converges also to \(x\). As \((X, CP)\) is separated, we obtain \(x = \mathfrak{x}\).

As a result, we have that the convergence of \((\mathbb{H}, \mathbb{K})\) to \(j(x)\) in \((X^*, CP^*)\) implies the existence of \((\mathbb{F}, \mathbb{G}) \in CP\) converging to \(x\), such that \(\mathbb{H} \geq j(\mathbb{F})\) and \(\mathbb{K} \geq j(\mathbb{G})\). In the same way, \((\mathbb{F}', \mathbb{G}') \in CP\) exists, converging to \(y\) and \(\mathbb{H} \geq j(\mathbb{F}')\) and \(\mathbb{K} \geq j(\mathbb{G}')\). We conclude that \(j^+(\mathbb{H})\) and \(j^+(\mathbb{K})\) exist. We show this for \(j^+(\mathbb{H})\). For \(f \in \mathbb{F}\) and \(h \in \mathbb{H}\) we have \(h \land j(f) \in \mathbb{H}\) and as \(\mathbb{H}\) is a \(\top\)-filter, we conclude
\[ T = \bigvee_{x^* \in X^*} (h \land j(f))(x^*) \leq \bigvee_{x \in X} h(j(x)) = \bigvee_{x \in X} j^+(h)(x). \]

We conclude \(j^+(\mathbb{H}) \geq j^+(\mathbb{K}) \geq \mathbb{G}\) and also \(j^+(\mathbb{H}) \geq \mathbb{F}'\) and \(j^+(\mathbb{K}) \geq \mathbb{G}'\).

We note that \((\mathbb{H}, \mathbb{K})\) is a pair \(\top\)-filter and \(\mathbb{H} \geq j(\mathbb{F}), \mathbb{K} \geq j(\mathbb{G})\) and we fix \(f \in \mathbb{F}, g \in \mathbb{G}\). Then for \(h \in \mathbb{H}, k \in \mathbb{K}\) we have \(h \land j(f) \in \mathbb{H}, k \land j(g) \in \mathbb{K}\) and hence,
\[ T = \bigvee_{x^* \in X^*} (h \land j(f))(x^*) \lor (k \land j(g))(x^*) \leq \bigvee_{x \in X} h(j(x)) \lor k(j(x)) = \bigvee_{x \in X} j^+(h)(x) \lor j^+(k)(x) \]

and hence \((j^+(\mathbb{H}), j^+(\mathbb{K}))\) is a pair \(\top\)-filter converging to both \(x\) and \(y\) in \((X, CP)\). Separation yields \(x = y\), i.e. \(x^* = y^*\).

**Case 2:** \(x^* = ((\mathbb{F}, \mathbb{G})), y^* = ((\mathbb{F}', \mathbb{G}'))\) with \((\mathbb{F}, \mathbb{G}),(\mathbb{F}', \mathbb{G}') \in CP\) non-convergent. Then \((\mathbb{H} \land [((\mathbb{F}, \mathbb{G}))], \mathbb{K} \land [((\mathbb{F}, \mathbb{G}))]) \in CP^*\). If we assume \(\mathbb{H} \land [((\mathbb{F}, \mathbb{G}))], \mathbb{K} \land [((\mathbb{F}, \mathbb{G}))]) \in CP^*\).
\[ \langle ([F, G]) \rangle \geq j(F) \text{ and } \langle ([F, G]) \rangle \geq j(G) \text{ with } (F, G) \in CP \text{ convergent, then for } \overline{F} \in F \text{ we had } T = \bigvee_{h \in H} [h \vee T_{\langle ([F, G]) \rangle} \langle \overline{F} \rangle \rangle \leq j(\overline{F}) \langle ([F, G]) \rangle = 1, \text{ a contradiction. Hence we must have } H \wedge \langle ([F, G]) \rangle \geq j(F) \wedge \langle ([F, G]) \rangle \text{ with } (F, G) \in CP \text{ non-convergent. For } \overline{F} \in F \text{ then} \]
\[ T = \bigvee_{h \in H} [h \vee T_{\langle ([F, G]) \rangle} \langle \overline{F} \rangle \rangle \leq T_{\langle ([F, G]) \rangle} \langle ([F, G]) \rangle. \]

Hence \( ([F, G]) = \langle ([F, G]) \rangle \) and we conclude \( H \geq j(F) \wedge \langle ([F, G]) \rangle, K \geq j(G) \wedge \langle ([F, G]) \rangle. \) In the same way we see that also \( H \geq j(F') \wedge \langle ([F', G']) \rangle, K \geq j(G') \wedge \langle ([F', G']) \rangle. \) As \((H, K)\) is a pair \(T\)-filter, we conclude, for \(f \in F\) and \(g' \in G'\) that
\[ T = \bigvee_{x^* \in X^*} \langle f(x) \rangle \wedge T_{\langle ([F, G]) \rangle} \langle x^* \rangle = \bigvee_{x \in X} f(x) \wedge g'(x) \wedge \bigvee_{(F, G) \in CP \text{ non-conv.}} T_{\langle ([F, G]) \rangle} \langle ([F, G]) \rangle \wedge T_{\langle ([F', G']) \rangle} \langle ([F, G]) \rangle. \]

If \( ([F, G]) \neq \langle ([F', G']) \rangle \), then \( \bigvee_{x \in X} f(x) \wedge g'(x) = T \) and, with analogous arguments, \( \bigvee_{x \in X} f'(x) \wedge g(x) = T \) for \( f' \in F', g \in G. \) Therefore, \((F \wedge F', G \wedge G') \in CP, \leq ([F, G], [F', G'])\) and we conclude \((F, G) \sim (F \wedge F', G \wedge G') \sim (F', G')\) and we have also in this case \( x^* = (F, G) = (F', G') = y^*. \)

**Case 3:** \( x^* = j(x) \) with \( x \in X \) and \( y^* = ([F, G]) \) with \((F, G) \in CP\) non-convergent. As we have seen before, then \( H \geq j(F') \) and \( K \geq j(G') \) with \((F', G') \in CP\) convergent and \( H \geq j(F) \wedge \langle ([F, G]) \rangle \) and \( H \geq j(G) \wedge \langle ([F, G]) \rangle. \) For \( f' \in F' \) and \( g \in G \) we have \( j(f') \in H \) and \( j(g) \wedge T_{\langle ([F, G]) \rangle} \in K \) and because \((H, H)\) is a \(T\)-pair filter we obtain
\[ T = \bigvee_{x^* \in X^*} j(f')(x^*) \wedge j(g) \wedge T_{\langle ([F, G]) \rangle} \langle x^* \rangle = \bigvee_{x \in X} f'(x) \wedge g(x). \]

Hence, \( F', G \) are linked. Similarly, we can show that also \( F, G \) are linked and hence \((F \wedge F', G \wedge G') \in CP. \) As this pair \(T\)-filter is \( \leq (F', G')\) it is convergent and therefore also \((F, G)\) is convergent, a contradiction. Hence this case cannot occur. \( \square \)

### 7. Symmetry — \(T\)-Cauchy spaces

Let \((X, CP)\) be a \(T\)-quasi-Cauchy space. We call \((X, CP)\) symmetric if the axiom
(TQCS) \((F, G) \in CP \) implies \((G, F) \in CP \)

is satisfied.

For \(T\)-quasi-uniform spaces and \(L\)-metric spaces, we obtain the usual concepts.

**Proposition 7.1.** Let \((X, U)\) be a \(T\)-quasi-uniform space. Then \((X, CP^U)\) is symmetric if, and only if, \(U \leq U^{-1}. \)
Proof. Let first \((X, \mathcal{CP}^d)\) be symmetric. It was shown in [25] that, for each \(x \in X\), \((\mathcal{U}(x, \cdot), \mathcal{U}(\cdot, x)) \in \mathcal{CP}^d\) where \(\mathcal{U}(x, \cdot)\) is generated by the \(\top\)-filter basis \(\{u(x, \cdot) : u \in \mathcal{U}\}\) and \(\mathcal{U}(\cdot, x)\) is generated by the \(\top\)-filter basis \(\{u(\cdot, x) : u \in \mathcal{U}\}\). A \(\top\)-filter basis for \(\bigwedge_{x \in X} \mathcal{U}(\cdot, x) \otimes \mathcal{U}(x, \cdot)\) is given by the \(L\)-sets \(b = \bigvee_{x \in X} u(x, \cdot) \otimes u(\cdot, x)\) with \(u \in \mathcal{U}\) and we have \(b(s, t) \geq u(s, s) * u(t, t) = u(s, t)\). Therefore, we have \(\mathcal{A} \in \mathcal{U} \leq \bigwedge_{x \in X} \mathcal{U}(\cdot, x) \otimes \mathcal{U}(x, \cdot) \leq \mathcal{U}\).

Hence we have \(\mathcal{U} \leq \bigwedge_{(F, G) \in \mathcal{CP}^d} (G \otimes F) \leq \bigwedge_{(G, F) \in \mathcal{CP}^d} F \otimes G = \mathcal{U}\).

For the converse, let \((F, G) \in \mathcal{CP}^d\). Then \(G \otimes F \geq \mathcal{U}\) and hence \(F \otimes G = (G \otimes F)^{-1} \geq U^{-1} \geq U\), that is, \((G, F) \in \mathcal{CP}^d\). \(\square\)

Corollary 7.2. Let \((X, d)\) be an \(L\)-metric space. Then \((X, \mathcal{CP}^d)\) is symmetric if, and only if, \(d(x, y) = d(y, x)\) for all \(x, y \in X\).

In the sequel we are going to show that symmetric \(\top\)-Cauchy spaces can be identified with \(\top\)-Cauchy spaces. Reid and Richardson [21] gave the following definition. A \(\top\)-Cauchy space \((X, \mathcal{C})\) is a set \(X\) with a set of \(\top\)-filters \(\mathcal{C} \subseteq F^1(X)\) such that

- \((\text{TC1}) \ [x] \in \mathcal{C}\) for all \(x \in X\);
- \((\text{TC2}) \ \emptyset \subseteq F \subseteq C, \text{ then } G \subseteq C;\)
- \((\text{TC3}) \ F \land G \subseteq C\) whenever \(F, G \in \mathcal{C}\) and \(\bigvee_{x \in X} f(x) * g(x) = \top\) for all \(f \in F, g \in G\).

A mapping \(\varphi : (X, \mathcal{C}) \to (X', \mathcal{C}')\) is called Cauchy continuous if \(\varphi(F) \in \mathcal{C}'\) whenever \(F \in \mathcal{C}\). We denote the category of \(\top\)-Cauchy spaces with Cauchy continuous mappings as morphisms by \(\top\text{-Chy}\).

Let now \((X, \mathcal{CP})\) be a \(\top\)-quasi-Cauchy space. We define \(\mathcal{C}_\mathcal{CP} = \{[\mathcal{H}] \in F^1(X) : ([\mathcal{H}], [\mathcal{H}]) \in \mathcal{CP}\}\).

The proofs of the following propositions are straightforward and not shown.

Proposition 7.3. Let \((X, \mathcal{CP})\) be a \(\top\)-quasi-Cauchy space. Then \((X, \mathcal{C}_\mathcal{CP})\) is a \(\top\)-Cauchy space.

Proposition 7.4. If \(\varphi : (X, \mathcal{CP}) \to (X', \mathcal{CP}')\) is Cauchy continuous, then \(\varphi : (X, \mathcal{C}_\mathcal{CP}) \to (X', \mathcal{C}_\mathcal{CP}')\) is Cauchy continuous.

Hence we have a functor \(F : \top\text{-QChy} \to \top\text{-Chy}\) which maps a \(\top\)-quasi-Cauchy space \((X, \mathcal{CP})\) to the \(\top\)-Cauchy space \((X, \mathcal{C}_\mathcal{CP})\) and leaves morphisms unchanged.

Let now \((X, \mathcal{C})\) be a \(\top\)-Cauchy space. We define \(\mathcal{CP}_C = \{(F, G) \in PF^1(X) : \exists [\mathcal{H}] \in \mathcal{C} \text{ s.t. } ([\mathcal{H}], [\mathcal{H}]) \leq (F, G)\}\).

Again, we omit the straightforward proofs of the following results.
Proposition 7.5. Let \((X, C)\) be a \(\top\)-Cauchy space. Then \((X, CP_C)\) is a symmetric \(\top\)-quasi-Cauchy space.

Proposition 7.6. If \(\varphi : (X, C) \to (X', C')\) is Cauchy continuous, then \(\varphi : (X, CP_C) \to (X', CP_{C'})\) is Cauchy continuous.

Hence we have another functor \(G : \top\text{-}Chy \to \top\text{-}QChy\) which maps a \(\top\)-Cauchy space \((X, C)\) to the \(\top\)-quasi-Cauchy space \((X, CP_C)\) and leaves morphisms unchanged.

Proposition 7.7. We have \(F \circ G = id_{\top\text{-}Chy}\) and \(G \circ F \geq id_{\top\text{-}QChy}\).

Proof. For a \(\top\)-Cauchy space \((X, C)\), we have \(F \in C(\top\text{-}CP_C)\) if, and only if, \((F, F) \in CP_C\), if, and only if, there is \(H \in C\) such that \((H, H) \leq (F, F)\) if, and only if, there is \(H \in C\) such that \(H \leq F\). This is equivalent to \(F \in C\).

For a \(\top\)-quasi-Cauchy space \((X, CP)\), we have \((F, G) \in CP(\top\text{-}CP)\) if, and only if, there is \(H \in CP\) such that \((H, H) \leq (F, G)\) if, and only if, there is \(H \in F(X)\) such that \((H, H) \in CP\) and \((H, H) \leq (F, G)\). This implies \((F, G) \in CP\). Hence we have shown \(CP(\top\text{-}CP) \subseteq CP\). We note that if \((X, CP)\) is symmetric, we even have equality, as \((F, G) \in CP\) implies \((G, F) \in CP\) and hence \((F \land G, F \land G) \in CP\) and we can choose \(H = F \land G\).

Corollary 7.8. The category \(\top\text{-}Chy\) is isomorphic to a reflective subcategory of the category \(\top\text{-}QChy\).

We denote the subcategory of symmetric \(\top\)-quasi-Cauchy spaces by \(\top\text{-}sQChy\). We restrict the domain of the functor \(F\) to this subcategory, and note that the codomain of \(G\) is automatically in this subcategory. Denoting these resulting functors again by \(F, G\), we even have \(F \circ G = id_{\top\text{-}Chy}\) and \(G \circ F = id_{\top\text{-}QChy}\).

Corollary 7.9. The categories \(\top\text{-}Chy\) and \(\top\text{-}sQChy\) are isomorphic.

In this way we can identify symmetric \(\top\)-quasi-Cauchy spaces and \(\top\)-Cauchy spaces.

Proposition 7.10. Let \((X, CP)\) be a \(\top\)-quasi-Cauchy space and \(((X^*, CP^*), j)\) be the finest completion. If \((X, CP)\) is symmetric, then so is \((X^*, CP^*)\).

Proof. This follows directly from the definition of \(CP^*\). \(\square\)

We will finally outline, that for a symmetric \(\top\)-quasi-Cauchy space \((X, CP)\), we can construct \((X^*, CP^*)\) in a different way. We first need some definitions [21].

Let \((X, C)\) be a \(\top\)-Cauchy space. A \(\top\)-filter \(F \in F^\top_1(X)\) converges to \(x \in X\) if \(F \land [x] \in C\). The space \((X, C)\) is called complete if each \(F \in C\) converges to some \(x \in X\). For a subset \(A \subseteq X\) we define the \(C\)-closure of \(A\) by \(x \in \overline{A}^C\) if there is \(F \in F^\top_1(A)\) such that \(F \land [x] \in C\). A completion \(((X^+, C^+), \phi)\) of a \(\top\)-Cauchy space \((X, C)\) is a complete \(\top\)-Cauchy space \((X^+, C^+)\) with a dense embedding \(\phi : X \to X^+\), that is, we have \(\phi(F) \in C^+\) if, and only if, \(F \in C\) and \(\overline{\phi(X)}^{C^+} = X^+\).
Lemma 7.11. A \( \top \)-Cauchy space \((X, \mathcal{C})\) is complete if, and only if, \((X, CP_{\mathcal{C}})\) is complete.

Proof. If \((X, \mathcal{C})\) is complete and \((F, G) \in CP_{\mathcal{C}}\) then there is \(H \in \mathcal{C}\) such that \((H, H) \leq (F, G)\). Hence there is \(x \in X\) such that \(H \land [x] \in \mathcal{C}\) and \((H \land [x], H \land [x]) \leq (F \land [x], G \land [x])\). This means that \((F, G)\) converges to \(x\) in \((X, CP_{\mathcal{C}})\). Hence, \((X, CP_{\mathcal{C}})\) is complete.

Conversely, if \((X, CP_{\mathcal{C}})\) is complete and \(H \in \mathcal{C}\), then \((H, H) \in CP_{\mathcal{C}}\) and there is \(x \in X\) such that \((H \land [x], H \land [x]) \in CP_{\mathcal{C}}\). By definition of \(CP_{\mathcal{C}}\) there is \(F \in \mathcal{C}\) such that \((F, F) \leq (H \land [x], H \land [x])\) which shows \(H \land [x] \in \mathcal{C}\) and \((X, \mathcal{C})\) is complete. \(\square\)

Lemma 7.12. If the \(\top\)-quasi-Cauchy space \((X, \mathcal{C})\) is complete, then also \((X, \mathcal{C}_{CP})\) is complete. If \((X, \mathcal{C})\) is symmetric, then we have equivalence.

Proof. Let \((X, \mathcal{C})\) be complete and let \(F \in \mathcal{C}_{CP}\). Then \((F, F) \in \mathcal{C}\) and hence there is \(x \in X\) such that \((F \land [x], F \land [x]) \in \mathcal{C}\). This means that \(F \land [x] \in \mathcal{C}_{CP}\) and \((X, \mathcal{C}_{CP})\) is complete.

If \((X, \mathcal{C})\) is symmetric and \((X, \mathcal{C}_{CP})\) is complete, then for \((F, G) \in \mathcal{C}\) we have \((F \land G, F \land G) \in \mathcal{C}\) and hence, \(F \land G \in \mathcal{C}_{CP}\). Therefore, there is \(x \in X\) such that \(F \land G \land [x] \in \mathcal{C}_{CP}\) and we conclude \((F \land [x], G \land [x]) \in \mathcal{C}\). Hence, \((X, \mathcal{C})\) is complete. \(\square\)

Reid and Richardson constructed a finest completion as follows [21]. With the equivalence relation on \(\mathcal{C}\), given by \(F \sim G\) if \(F \land G \in \mathcal{C}\), we denote the equivalence class of \(F \in \mathcal{C}\) by \(\langle F \rangle\) and we define \(X^+ = X \cup \{\langle F \rangle : F \in \mathcal{C}\) non-convergent\}.

Then \((X^+, \mathcal{C}^+)\) is defined by \(j^+(x) = \langle F \rangle\) for some convergent \(F \in \mathcal{C}\) or \(H \geq j^+(F)\) with some \(F \in \mathcal{C}\) non-convergent.

Proposition 7.13. Let \((X, CP)\) be a symmetric \(\top\)-quasi-Cauchy space. For \((F, G), (F', G') \in CP\) we have \((F, G) \sim (F', G')\) in \((X, CP)\) if, and only if, \(F \sim F'\) in \((X, \mathcal{C}_{CP})\).

Proof. Let first \((F, G) \sim (F', G')\). The symmetry of \((X, CP)\) ensures \(F, F' \in \mathcal{C}_{CP}\). From \((F \land F', F \land G, G \land G') \in \mathcal{C}\) we conclude, again by symmetry, that \((F \land F', F \land G \land G') \in \mathcal{C}\) and, using (TQC2), \((F \land F', F \land F') \in \mathcal{C}\), that is, \(F \sim F'\).

If \(F \land F' \in \mathcal{C}_{CP}\), then \((F \land F', F \land F') \in \mathcal{C}\). As \((F, G), (F', G') \in \mathcal{C}\), we conclude \((F \land G, F \land G, F' \land G') \in \mathcal{C}\). Applying (TQC3) twice, we conclude \((F \land F' \land G, F \land F' \land G) \in \mathcal{C}\). (TQC2) yields \((F \land F', G \land G') \in \mathcal{C}\).

We note that by symmetry we also have \(G \sim G'\) in \((X, \mathcal{C}_{CP})\) if, and only if, \((F, G) \sim (F', G') \in CP\).

Corollary 7.14. The pair \(\top\)-filter \((F, G)\) converges to \(x\) in \((X, CP)\) if, and only if, \(F\) converges to \(x\) in \((X, \mathcal{C}_{CP})\).
As a consequence, the mapping $\eta : \{((F, G)) : (F, G) \in CP \text{ non-convergent}\} \rightarrow \{F : F \in CP \text{ non-convergent}\}$ defined by $\eta((F, G)) = F$ is a bijection and we can identify in this way $X^*$ and $j$ (from the finest completion of $(X, CP)$) and $X^*$ and $j^+$ (from the finest completion of $(X, CP)$ for a symmetric $\top$-quasi-Cauchy space.

Moreover, we can deduce the following result.

**Corollary 7.15.** Let $(X, CP)$ be a symmetric $\top$-quasi-Cauchy space and let $A \subseteq X$. Then $\overline{A^{CP}} = A^{cp}$.

If we have a symmetric $\top$-quasi-Cauchy space $(X, CP)$, then we can construct the finest completion $(X^*, CP^*)$. Alternately, we can move to the $\top$-Cauchy space $(X, C_{CP})$ and construct the finest completion $(X^+, (C_{CP})^+)$ and from there move to $(X^+, CP((C_{CP})^+))$. Identifying $X^*$ with $X^+$, we will show that $(X^+, CP((C_{CP})^+)) = (X^*, CP^*)$. First of all, we notice that for $(F, G) \in CP$ we have $F, G \in C_{CP}$ and hence $j(F), j(G) \in (C_{CP})^+$ which implies $(j(F), j(G)) \in CP((C_{CP})^+)$. Conversely, if $(j(F), j(G)) \in CP((C_{CP})^+)$, then there is $H \in (C_{CP})^+$ such that $H \subseteq j(F), j(G)$. Hence $j(F), j(G) \in (C_{CP})^+$, which implies $F, G \in C_{CP}$. This means that $(F, F), (G, G) \in CP$, and as $(F, G)$ is a pair $\top$-filter, we conclude with (TQC3) $(F \wedge G, F \wedge G) \in CP$. Hence, also $(F, G) \in CP$. Corollary 7.15 and Lemma 7.11 thus show that $((X^*, CP((C_{CP})^+)), j)$ is a completion of $(X, CP)$ and therefore, $((X^*, CP^*)$, $j)$ being the finest completion, we see that $CP^* \subseteq CP((C_{CP})^+)$. For the converse subsethood relation, we take $(H, K) \in CP((C_{CP})^+)$. Then there is $H' \in (C_{CP})^+$ such that $(H', H') \leq (H, K)$. Then either there is a convergent $F \in C_{CP}$, that is $(F, F) \in CP$ such that $j(F) \leq H' \leq H, K$, which shows $(H, K) \in CP^*$. Or there is $F \in C_{CP}$ non-convergent and, upon identification $j(F) = (F, F)$, we conclude $j(F) \subseteq (F, F) \subseteq H \subseteq H, K$ which again implies $(H, K) \in CP^*$. Hence we have proven the following result.

**Proposition 7.16.** For a symmetric $\top$-quasi-Cauchy space $(X, CP)$ we have $(X^*, CP^*) = (X^*, CP((C_{CP})^+))$.

We note that in the non-symmetric case, $(X^+, CP((C_{CP})^+))$ is in general not a completion of $(X, CP)$. To see this, we consider a complete $\top$-quasi-Cauchy space $(X, CP)$. Then $(X, C_{CP})$ is a complete $\top$-Cauchy space and we get $(X^*, CP^*) = (X, CP)$ and $(X^+, (C_{CP})^+) = (X, C_{CP})$ and from there, the complete $\top$-quasi-Cauchy space $(X, CP((C_{CP})^+))$. As $(X, CP)$ is non-symmetric, there is $(F, G) \in CP$ such that $(G, F) \notin CP$. Since $CP((C_{CP})^+) \subseteq CP$, this $(F, G)$ cannot be in $CP((C_{CP})^+)$. Hence, $j = id_X : (X, CP) \rightarrow (X, CP((C_{CP})^+))$ is not Cauchy continuous.

8. Conclusions

We defined a non-symmetric framework for studying completeness and completion, generalizing the $\top$-Cauchy pair filters in a $\top$-uniform space that were used in [25] and [15]. Our category of $\top$-quasi-Cauchy spaces has nice categorical properties and covers the important examples of $\top$-quasi-uniform spaces.
and L-metric spaces. It allows a theory of completeness and completion which is patterned after the corresponding theory in the symmetric case [21]. In the future, completions with special properties, e.g. diagonal completions or regular completions, can be studied using similar techniques as in [21] and the connection to completions of $\top$-quasi-uniform (limit) spaces are of interest. It would also be interesting to see if such a non-symmetric theory of completeness and completion finds a place in the field of monoidal topology [12].

References