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New results regarding the lattice of uniform topologies on $C(X)$

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Abstract

For a Tychonoff space X, the lattice $\mathcal{U}_X$ was introduced in L.A. Pérez-Morales, G. Delgadillo-Piñón, and R. Pichardo-Mendoza, The lattice of uniform topologies on $C(X)$, Questions and Answers in General Topology **39** *(2021), 65–71.*
In the present paper we continue the study of $\mathcal{U}_X$. To be specific, the present paper deals, in its first half, with structural and categorical properties of $\mathcal{U}_X$, while in its second part focuses on cardinal characteristics of the lattice and how these relate to some cardinal functions of the space X.

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1. Introduction

In [9] the authors define, given a completely regular Hausdorff space X, a partially ordered set $(\mathcal{U}_X, \subseteq)$ (see Section 2 for details and the corresponding definitions) which turns out to be a bounded lattice (the *lattice of uniform topologies on* $C(X)$). Here we expand some of the results obtained in that paper and explore new directions. For example, Section 3 is mainly about finding connections between order-isomorphisms and homeomorphisms, while

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the last two sections deal heavily on finding relations between some cardinal characteristics of $\mathcal{U}_X$ and highly common cardinal functions of X.

2. Preliminaries

All topological notions and all set-theoretic notions whose definition is not included here should be understood as in [1] and [7], respectively. With respect to lattices, we will follow [8] for notation and results. The same goes for Boolean algebras and [6].

The symbol ω denotes both, the set of all non-negative integers and the first infinite cardinal. Also, $\mathbb{R}$ is the real line endowed with the Euclidean topology.

Given a set S, $[S]^{<\omega}$ denotes the collection of all finite subsets of S. For a set A, the symbol ${}^A S$ is used to represent the collection of all functions from A to S. In particular, for $f \in {}^A S$, $E \subseteq A$, and $H \subseteq S$ we define $f[E] := \{f(x) : x \in E\}$ and $f^{-1}[H] := \{x \in A : f(x) \in H\}$. Moreover, if $y \in S$, $f^{-1}\{y\} := f^{-1}[\{y\}]$.

A nonempty family of sets, α, is called *directed* if for any $A, B \in \alpha$ there is $E \in \alpha$ with $A \cup B \subseteq E$. For example, $[S]^{<\omega}$ is directed, for any set S.

Assume X is a set. Hence, $\mathcal{P}(X)$ and $\mathcal{D}_X$ represent its power set and the collection of all directed subsets of $\mathcal{P}(X)$, respectively. In [10] the term *base for an ideal on* X was used to refer to members of $\mathcal{D}_X$.

Unless otherwise stated, the word *space* means *Hausdorff completely regular space* (i.e., *Tychonoff space*).

Assume X is a space. Then, τ_X and τ_X^* stand, respectively, for the families of all open and closed subsets of X. Moreover, whenever $x \in X$, $\tau_X(x)$ will be the set $\{U \in \tau_X : x \in U\}$. Now, given $A \subseteq X$, the symbol $\operatorname{cl}_X A$ (or $\overline{A}$ when the space X is clear from the context) represents the closure of A in X; similarly, $\operatorname{int}_X A$ and $\operatorname{int} A$ will be used to denote the interior of A in X.

$C(X)$ is, as usual, the subset of ${}^X\mathbb{R}$ consisting of all continuous functions. Now, given $\alpha \in \mathcal{D}_X$ we generate a topology on $C(X)$ as follows: a set $U \subseteq C(X)$ is open if and only if for each $f \in U$ there are $A \in \alpha$ and a real number $\varepsilon > 0$ with

$$V(f, A, \varepsilon) := \{g \in C(X) : \forall\, x \in A\ (|f(x) - g(x)| < \varepsilon)\} \subseteq U.$$

The resulting topological space is denoted by $C_\alpha(X)$. As it is explained in [10], $C_\alpha(X)$ is a uniformizable topological space which may not be Hausdorff. In fact, one has the following result (whose proof can be found in [10, Proposition 3.1, p. 559]).

Lemma 2.1. *For any space X and $\alpha \in \mathcal{D}_X$, $C_\alpha(X)$ is Hausdorff if and only if α has dense union, i.e., $\overline{\bigcup \alpha} = X$.*

Given a space X, set $\mathcal{U}_X := \{\tau_{C_\gamma(X)} : \gamma \in \mathcal{D}_X\}$. In order to simplify our writing, for each $\alpha \in \mathcal{D}_X$ we identify the space $C_\alpha(X)$ with its topology. Thus, expressions of the form $C_\alpha(X) \in \mathcal{U}_X$ will be common in this paper. Also, in those occasions where the ground space is clear from the context, we will suppress it from our notation, i.e., we will use C_α instead of $C_\alpha(X)$. Finally, for any $\alpha, \beta \in \mathcal{D}_X$, both, $C_\alpha(X) \leq C_\beta(X)$ and $C_\alpha \leq C_\beta$, are abbreviations of the relation $\tau_{C_\alpha(X)} \subseteq \tau_{C_\beta(X)}$.

It is shown in [9, Proposition 3.2, p. 67] that the poset $(\mathcal{U}_X, \subseteq)$ is a bounded distributive lattice; to be precise, given $\alpha, \beta \in \mathcal{D}_X$, the collections

$$\alpha \vee \beta := \{A \cup B : A \in \alpha,\ B \in \beta\} \qquad \text{and} \qquad \alpha \wedge \beta := \{\overline{A} \cap \overline{B} : A \in \alpha,\ B \in \beta\}$$

are directed and, moreover, $C_{\alpha\vee\beta}$ and $C_{\alpha\wedge\beta}$ are, respectively, the supremum and infimum of $\{C_\alpha, C_\beta\}$ in $\mathcal{U}_X$.

The topologies generated on $C(X)$ by the directed sets $\{\varnothing\}$, $[X]^{<\omega}$, and $\{X\}$ are denoted by $C_\varnothing(X)$, $C_p(X)$, and $C_u(X)$, respectively. Let us note that $C_\varnothing$ is the indiscrete topology on $C(X)$, while C_p and C_u are the topologies of pointwise and uniform convergence on $C(X)$, respectively.

The result below (see [10, Theorem 3.4, p. 560] for a proof) will be used several times in what follows.

Proposition 2.2. *If X is a space and $\alpha, \beta \in \mathcal{D}_X$, then $C_\alpha \le C_\beta$ if and only if for each $A \in \alpha$ there is $B \in \beta$ with $A \subseteq \overline{B}$.*

We finish this section by mentioning that our notation for topological cardinal functions follows [3]; in particular, all of them are, by definition, infinite.

3. Some structural and categorical results

We begin by improving the result presented in [9, Proposition 3.2, p. 67].

Proposition 3.1. *For any space X, $\mathcal{U}_X$ is a complete lattice.*

Proof. Given an arbitrary set $\mathcal{S} \subseteq \mathcal{D}_X$, define $\mathcal{A} := \{C_\delta : \delta \in \mathcal{S}\}$.

By letting α be the family of all sets of the form $\bigcup \mathcal{E}$, where $\mathcal{E} \subseteq \bigcup \mathcal{S}$ is finite, we obtain $\alpha \in \mathcal{D}_X$. Also, the fact that $\delta \subseteq \alpha$, whenever $\delta \in \mathcal{S}$, implies (see Proposition 2.2) that C_α is an upper bound for $\mathcal{A}$.

Now, assume that $\gamma \in \mathcal{D}_X$ is such that C_γ is an upper bound for $\mathcal{A}$. In order to show that $C_\alpha \le C_\gamma$, fix $A \in \alpha$. There is a finite set $\mathcal{E} \subseteq \bigcup \mathcal{S}$ satisfying $A = \bigcup \mathcal{E}$. According to Proposition 2.2, for each $E \in \mathcal{E}$ there exists $E^* \in \gamma$ with $E \subseteq \overline{E^*}$. Since γ is directed, $\bigcup\{E^* : E \in \mathcal{E}\} \subseteq G$ for some $G \in \gamma$ and, consequently, $A \subseteq \overline{G}$. In other words, $C_\alpha \le C_\gamma$.

From the previous paragraphs we conclude that any subset of $\mathcal{U}_X$ has a supremum in $\mathcal{U}_X$. Now, regarding infima, let us observe that the infimum of $\varnothing$ in $\mathcal{U}_X$ is C_u. Thus, we will suppose that $\mathcal{S}$ is non-empty.

Denote by $\mathcal{E}$ the set of all choice functions of $\mathcal{S}$, i.e., $e \in \mathcal{E}$ if and only if $e : \mathcal{S} \to \bigcup \mathcal{S}$ and $e(\delta) \in \delta$, for all $\delta \in \mathcal{S}$. Now, for each $e \in \mathcal{E}$, set

$$\widetilde{e} := \bigcap \{\overline{e(\delta)} : \delta \in \mathcal{S}\}.$$

We claim that if $\beta := \{\widetilde{e} : e \in \mathcal{E}\}$, then C_β is the infimum of $\mathcal{A}$.

To show that β is directed, consider $d, e \in \mathcal{E}$. Since, for any $\delta \in \mathcal{S}$, δ is directed, we deduce that there is a set $f(\delta) \in \delta$ with $\overline{d(\delta) \cup e(\delta)} \subseteq f(\delta)$. This produces f, a choice function of $\mathcal{S}$, in such a way that $\widetilde{d} \cup \widetilde{e} \subseteq \widetilde{f}$.

The fact that C_β is a lower bound for $\mathcal{A}$ follows from the observation that for each $e \in \mathcal{E}$ and $\delta \in \mathcal{S}$, $\widetilde{e} \subseteq \overline{e(\delta)}$.

Finally, let $\gamma \in \mathcal{D}_X$ be such that C_γ is a lower bound for $\mathcal{A}$. Fix $G \in \gamma$. Then, for any $\delta \in \mathcal{S}$ there is $e(\delta) \in \delta$ with $G \subseteq \overline{e(\delta)}$. As a consequence, we obtain e, a choice function of $\mathcal{S}$, with $G \subseteq \widetilde{e}$. □

As in [8], we will use the symbol $\Sigma(E)$ to the represent the collection of all topologies on a fixed set E. It is well-known that when we order $\Sigma(E)$ by direct inclusion, the resulting structure is a complete lattice. In particular, the supremum of $\mathcal{A} \subseteq \Sigma(E)$ is the topology on E *generated by* $\bigcup \mathcal{A}$ (i.e., it has the collection $\bigcup \mathcal{A}$ as a subbase).

Clearly, $\mathcal{U}_X$ is a suborder of $\Sigma(C(X))$. Thus, a natural question is, given a family $\mathcal{A} \subseteq \mathcal{U}_X$, is the supremum (respectively, infimum) of $\mathcal{A}$ as calculated in $\mathcal{U}_X$ the same as the supremum (respectively, infimum) of $\mathcal{A}$ as obtained in $\Sigma(C(X))$? We have a positive answer for suprema.

Corollary 3.2. *If X is a space and $\mathcal{A} \subseteq \mathcal{U}_X$, then $\bigvee \mathcal{A}$, the supremum of $\mathcal{A}$ in $\mathcal{U}_X$, is the topology on $C(X)$ which has $\bigcup \mathcal{A}$ as a subbase.*

Proof. Fix $\mathcal{S} \subseteq \mathcal{D}_X$ in such a way that $\mathcal{A} = \{C_\beta : \beta \in \mathcal{S}\}$ and denote by σ the topology on $C(X)$ generated by $\bigcup \mathcal{A}$. Since $\bigvee \mathcal{A}$ is an upper bound of $\mathcal{A}$ in $\Sigma(C(X))$, we obtain $\sigma \subseteq \bigvee \mathcal{A}$.

Now, let $f \in U \in \bigvee \mathcal{A}$ be arbitrary. According to the proof of Proposition 3.1, there are $\varepsilon > 0$ and $\mathcal{E}$, a finite subset of $\bigcup \mathcal{S}$, with $V(f, A, \varepsilon) \subseteq U$, where $A := \bigcup \mathcal{E}$. When $\mathcal{E} = \varnothing$, we deduce that $U = C(X) \in \sigma$. Hence, let us assume that $\mathcal{E} \neq \varnothing$.

For each $E \in \mathcal{E}$ let $\beta(E) \in \mathcal{S}$ be such that $E \in \beta(E)$. By setting $\mathcal{W} := \{\operatorname{int}_{C_{\beta(E)}} V(f, E, \varepsilon) : E \in \mathcal{E}\}$ we produce a finite subset of $\bigcup \mathcal{A}$ which satisfies $f \in \bigcap \mathcal{W} \subseteq V(f, A, \varepsilon) \subseteq U$. In conclusion, $\bigvee \mathcal{A} \subseteq \sigma$. □

Recall that if E is a set and $\sigma, \tau \in \Sigma(E)$, the infimum of $\{\sigma, \tau\}$ in $\Sigma(E)$ is $\sigma \cap \tau$; consequently, for any space X and $\alpha, \beta \in \mathcal{D}_X$, $C_\alpha \wedge C_\beta \subseteq C_\alpha \cap C_\beta$. Now, assume that X is a non-empty space which is *resolvable* (i.e., it can be written as the union of two disjoint dense subsets of it). In [9, Proposition 4.5, p. 69], it is shown that there are two Hausdorff topologies $\sigma, \tau \in \mathcal{U}_X$ with $\sigma \wedge \tau = C_\varnothing$. Consequently, $\sigma \cap \tau$ is a T_1 topology, but $\sigma \wedge \tau$ fails to be T_0. Hence, the question posed in the paragraph preceding Corollary 3.2 has a negative answer for infima.

Problem 3.3. *Given a space X, find conditions on $\alpha, \beta \in \mathcal{D}_X$ in order to obtain $C_\alpha \wedge C_\beta = C_\alpha \cap C_\beta$.*

As in [9], the symbol $\mathcal{C}_X$ represents the collection of all members of $\mathcal{U}_X$ which have a complement in $\mathcal{U}_X$. Thus, from the fact that $\mathcal{U}_X$ is a bounded distributive lattice, we deduce that $\mathcal{U}_X$ is a Boolean algebra if and only if $\mathcal{U}_X = \mathcal{C}_X$. Our next result shows that this condition is attained only in trivial cases.

Proposition 3.4. *For any space X, $\mathcal{U}_X$ is a Boolean algebra if and only if X is finite.*

Proof. Firstly observe that, in virtue of [9, Proposition 3.3, p. 68], we only need to show that X is a finite space if and only if for each $\alpha \in \mathcal{D}_X$ there is $E \in \alpha$ with $\overline{E} \in \tau_X$ and $\bigcup \alpha \subseteq \overline{E}$. Now, evidently any finite X satisfies the latter condition. For the converse let us assume that X is infinite. Since X is Hausdorff, there is $\{U_n : n < \omega\}$, a family of non-empty open subsets of X, with $U_m \cap U_n = \varnothing$, whenever $m < n < \omega$. By setting $\alpha := \{\bigcup_{k=0}^{n} U_k : n < \omega\}$ we obtain a member of $\mathcal{D}_X$ in such a way that, for each $E \in \alpha$, there is $m < \omega$ with $U_m \cap E = \varnothing$ and thus, $\bigcup \alpha \nsubseteq \overline{E}$. □

For our next results we will need some auxiliary concepts. First of all, assume that f is function from the space X into a space Y. One easily verifies that for any $\alpha \in \mathcal{D}_X$ the family

$$f^*\alpha := \{f[A] : A \in \alpha\}$$

belongs to $\mathcal{D}_Y$ and so, we have the following notion (recall that for any space Z and $\gamma \in \mathcal{D}_Z$ we are identifying the space $C_\gamma(Z)$ with its topology).

Definition 3.5. If X, Y, and f are as in the previous paragraph, the phrase *φ is the f-induced relation* means that

$$\varphi = \{(C_\alpha(X), C_{f^*\alpha}(Y)) : \alpha \in \mathcal{D}_X\} \subseteq \mathcal{U}_X \times \mathcal{U}_Y.$$

With the notation used above, the domain of φ, $\mathrm{dom}(\varphi)$, is equal to $\mathcal{U}_X$ and its range, $\mathrm{ran}(\varphi)$, is a subset of $\mathcal{U}_Y$.

Proposition 3.6. *If X and Y are spaces and $f : X \to Y$, then f is continuous if and only if φ, the f-induced relation, is an order-preserving function.*

Proof. Let us begin by assuming that f is continuous and prove the statement below.

$$\forall\, \alpha, \beta \in \mathcal{D}_X\ (C_\alpha \le C_\beta\ \to\ C_{f^*\alpha} \le C_{f^*\beta}). \tag{3.1}$$

Given $\alpha, \beta \in \mathcal{D}_X$ with $C_\alpha \le C_\beta$, fix $A \in f^*\alpha$. There is $B \in \alpha$ with $A = f[B]$ and so (see Proposition 2.2), for some $E \in \beta$, $B \subseteq \mathrm{cl}_X E$. Finally, f's continuity produces $A = f[B] \subseteq f[\mathrm{cl}_X E] \subseteq \mathrm{cl}_Y f[E]$ and, clearly, $f[E] \in f^*\beta$.

The final step for this implication is to note that the properties required for φ are consequences of (3.1).

Suppose that φ is an order-preserving function and fix $A \subseteq X$. According to Proposition 2.2, $C_{\mathrm{cl}_X A} \le C_A$ and so,

$$C_{f[\mathrm{cl}_X A]} = \varphi(C_{\mathrm{cl}_X A}) \le \varphi(C_A) = C_{f[A]},$$

i.e., $f[\mathrm{cl}_X A] \subseteq \mathrm{cl}_Y f[A]$. □

For the rest of the paper, given a space X, a point $x \in X$, and a set $A \subseteq X$, we use the symbols $C_x(X)$ and $C_A(X)$ to represent the topological spaces $C_{\{\{x\}\}}(X)$ and $C_{\{A\}}(X)$, respectively. As expected, if the space X is clear from the context, we only write C_x and C_A; also, as we have done before, C_x and C_A are, as well, the topologies of the corresponding spaces.

A function f from the space X into the space Y is called *open onto its range* if, for any $U \in \tau_X$, $f[U] \in \tau_{f[X]}$. Note that if f is one-to-one, then f is open onto its range if and only if f is *closed onto its range* (i.e., whenever G is a closed subset of X, $f[G]$ is a closed subset of the subspace $f[X]$).

Proposition 3.7. *Assume X and Y are spaces. For any $f : X \to Y$, the following are equivalent.*

(1) *f is one-to-one and open onto its range.*
(2) *φ^{-1}, the inverse relation of the f-induced relation, is an order-preserving function.*

Proof. Observe that for the implication (1) $\to$ (2), it suffices to prove that the statement

$$\forall\, \alpha, \beta \in \mathcal{D}_X \; (C_{f^*\alpha} \leq C_{f^*\beta} \; \to \; C_\alpha \leq C_\beta) \tag{3.2}$$

follows from (1). Thus, suppose (1) and fix $\alpha, \beta \in \mathcal{D}_X$ with $C_{f^*\alpha} \leq C_{f^*\beta}$. Given $A \in \alpha$, Proposition 2.2 guarantees the existence of $B \in \beta$ with $f[A] \subseteq \mathrm{cl}_Y f[B]$, i.e., $A \subseteq f^{-1}[\mathrm{cl}_Y f[B]]$. Thus, we only need to show that $f^{-1}[\mathrm{cl}_Y f[B]] \subseteq \mathrm{cl}_X B$. If $x \in f^{-1}[\mathrm{cl}_Y f[B]]$ and $U \in \tau_X(x)$ are arbitrary, then $f(x) \in f[X] \cap \mathrm{cl}_Y f[B] = \mathrm{cl}_{f[X]} f[B]$ and $f[U] \in \tau_{f[X]}(f(x))$; consequently, $f[U] \cap f[B] \neq \varnothing$. Since f is one-to-one, $f[U \cap B] \neq \varnothing$ and so, $U \cap B \neq \varnothing$, as required.

For the rest of the argument, assume (2). In order to verify that f is one-to-one, let $x, y \in X$ be such that $f(x) = f(y)$. Hence, $C_{f(x)} = C_{f(y)}$ and, as a consequence, $C_x = \varphi^{-1}(C_{f(x)}) = \varphi^{-1}(C_{f(y)}) = C_y$. The use of Proposition 2.2 produces $x = y$.

Given that f is one-to-one, we only need to argue that f is closed onto its range. Suppose G is a closed subset of X. By letting $E := \mathrm{cl}_Y f[G]$ and $A := f^{-1}[E]$, we deduce that $f[A] = E \cap f[X] = \mathrm{cl}_{f[X]} f[G]$. Therefore, $C_{f[A]} \leq C_E \leq C_{f[G]}$ and so, $C_A = \varphi^{-1}(C_{f[A]}) \leq \varphi^{-1}(C_{f[G]}) = C_G$. Hence, $A \subseteq \mathrm{cl}_X G = G$ and, consequently, $\mathrm{cl}_{f[X]} f[G] = f[A] \subseteq f[G]$, i.e., $f[G]$ is a closed subset of $f[X]$. $\square$

Proposition 3.8. *If X and Y are spaces and $f : X \to Y$, then f is onto if and only if $\mathrm{ran}(\varphi) = \mathcal{U}_Y$, where φ is the f-induced relation.*

Proof. When f is onto and $\alpha \in \mathcal{D}_Y$, the collection $\beta := \{f^{-1}[A] : A \in \alpha\}$ belongs to $\mathcal{D}_X$ and $f^*\beta = \alpha$. Thus, $(C_\beta, C_\alpha) \in \varphi$ and so, $C_\alpha \in \mathrm{ran}(\varphi)$.

For the remaining implication, fix $y \in Y$ and note that $C_y \in \mathcal{U}_Y = \mathrm{ran}(\varphi)$, i.e., for some $\alpha \in \mathcal{D}_X$, $(C_\alpha, C_y) \in \varphi$. Now, our definition of φ produces $\beta \in \mathcal{D}_X$ with $C_\alpha = C_\beta$ and $C_y = C_{f^*\beta}$. Since $C_y \leq C_{f^*\beta}$, there is $B \in \beta$ in such a way that $y \in \mathrm{cl}_X f[B]$ and so, $B \neq \varnothing$. From the relation $C_{f^*\beta} \leq C_y$ we obtain $f[B] \subseteq \mathrm{cl}_Y \{y\} = \{y\}$ and therefore, $\varnothing \neq B \subseteq f^{-1}\{y\}$. $\square$

Since any topological embedding is a continuous one-to-one function that is open onto its range, we obtain the following result.

Corollary 3.9. *If Y is a space which can be embedded into a space X, then there is an order-embedding from $\mathcal{U}_Y$ into $\mathcal{U}_X$. In particular, $|\mathcal{U}_Y| \leq |\mathcal{U}_X|$.*

Assume X and Y are spaces for which there is $\varphi : \mathcal{U}_X \to \mathcal{U}_Y$, an (order) isomorphism. According to [9, Proposition 5.1, p. 70], for each $x \in X$, $C_x(X)$ is an atom of $\mathcal{U}_X$ (i.e., a minimal element of $\mathcal{U}_X \setminus \{C_\varnothing\}$) and so, $\varphi(C_x(X))$ happens to be an atom of $\mathcal{U}_Y$; consequently (see [9, Proposition 5.1, p. 70]), there exists a point $y \in Y$ with $\varphi(C_x(X)) = C_y(Y)$. Moreover, as one easily deduces from Proposition 2.2, y is the only member of Y with this property.

Definition 3.10. Let X and Y be a pair of spaces. If $\varphi : \mathcal{U}_X \to \mathcal{U}_Y$ is an isomorphism, we will say that $f : X \to Y$ is the *φ-induced function* if

$$\text{for each } x \in X,\ \varphi(C_x(X)) = C_{f(x)}(Y). \tag{3.3}$$

Observe that if f is a homeomorphism from a space X onto a space Y and φ is the f-induced relation, the previous results imply that φ is an isomorphism. Now, when g is the φ-induced function, we obtain that, for each $x \in X$,

$$\varphi(C_x) = C_{f^*\{\{x\}\}} = C_{f(x)} \qquad \text{and} \qquad \varphi(C_x) = C_{g(x)},$$

i.e., $f(x) = g(x)$. In conclusion, $f = g$. Hence, the following is a natural question.

Problem 3.11. *Assume X and Y are spaces for which there is an isomorphism $\varphi : \mathcal{U}_X \to \mathcal{U}_Y$. If f is the φ-induced function and ψ is the f-induced relation, do we get $\varphi = \psi$?*

With the idea in mind of giving a positive answer to this question for a class of spaces (zero-dimensional spaces), we will present some auxiliary results.

Lemma 3.12. *Assume $\varphi : \mathcal{U}_X \to \mathcal{U}_Y$ is an isomorphism, where X and Y are spaces. If f is the φ-induced function, then the following statements hold.*

1. *f is a bijection and f^{-1} is the φ^{-1}-induced function.*
2. *If $A \subseteq X$ and $\beta \in \mathcal{D}_Y$ satisfy $\varphi(C_A(X)) = C_\beta(Y)$, then $f[\mathrm{cl}_X A] \subseteq \bigcup \overline{\beta}$.*

Proof. For (1), let g be the φ^{-1}-induced function. Given $x \in X$, the relation $\varphi(C_x) = C_{f(x)}$ implies that $C_x = \varphi^{-1}(C_{f(x)}) = C_{g(f(x))}$ and so, $g \circ f$ is the identity function on X. Similarly, $f \circ g$ is the identity function on Y.

Given $x \in \overline{A}$, Proposition 2.2 produces $C_x \leq C_A$ and so, $C_{f(x)} = \varphi(C_x) \leq \varphi(C_A) = C_\beta$; hence, $f(x) \in \bigcup \overline{\beta}$. ☐

Proposition 3.13. *Let X and Y be spaces in such a way that there is an isomorphism $\varphi : \mathcal{U}_X \to \mathcal{U}_Y$. Denote by f the φ-induced function and consider the following statements.*

1. *φ is the f-induced relation.*
2. *For any $A \subseteq X$, $\varphi(C_A(X)) = C_{f[A]}(Y)$.*
3. *Whenever G is a closed subset of X, $\varphi(C_G(X)) = C_{f[G]}(Y)$.*

Then, (1) is equivalent to (2) and if f is continuous, (2) and (3) are equivalent.

Proof. The implications (1)→(2) and (2)→(3) are immediate. On the other hand, it follows from the work done in the first paragraphs of the proof of Proposition 3.1 that, for any $\alpha \in \mathcal{D}_X$,

$$C_\alpha = \bigvee\{C_A : A \in \alpha\} \qquad \text{and} \qquad C_{f^*\alpha} = \bigvee\{C_{f[A]} : A \in \alpha\};$$

therefore, by assuming (2) we obtain

$$\varphi(C_\alpha) = \bigvee\{\varphi(C_A) : A \in \alpha\} = \bigvee\{C_{f[A]} : A \in \alpha\} = C_{f^*\alpha},$$

i.e., (1) holds.

Now suppose f is continuous and (3) is true. In order to prove (2), fix $A \subseteq X$ and set $G := \overline{A}$. According to Proposition 2.2, $C_A = C_G$ and, consequently, $\varphi(C_A) = \varphi(C_G) = C_{f[G]}$. From the relation $f[A] \subseteq f[G]$ we deduce that $C_{f[A]} \le C_{f[G]}$. The continuity of f produces $f[G] \subseteq \overline{f[A]}$ and so, $C_{f[G]} \le C_{f[A]}$. In conclusion, $\varphi(C_A) = C_{f[G]} = C_{f[A]}$, as needed. □

Recall that for any space Z, $\mathrm{CO}(Z)$ is the collection of all subsets of Z which are closed and open in Z. Consequently, Z is *zero-dimensional* when $\mathrm{CO}(Z)$ is a base for Z.

Lemma 3.14. *Assume X and Y are spaces for which there is $\varphi : \mathcal{U}_X \to \mathcal{U}_Y$, an isomorphism. If f is the φ-induced function, the following statements hold.*

1. *For each $A \in \mathrm{CO}(X)$, $f[A] \in \mathrm{CO}(Y)$ and $\varphi(C_A(X)) = C_{f[A]}(Y)$.*
2. *If Y is zero-dimensional, f is continuous.*

Proof. Given $A \in \mathrm{CO}(X)$, the proof of [9, Proposition 3.3, p. 68] shows that C_A and $C_{X\setminus A}$ are complements of each other in $\mathcal{U}_X$ and so, $\varphi(C_A)$ and $\varphi(C_{X\setminus A})$ have the same relation in $\mathcal{U}_Y$. Then, according to [9, Proposition 5.3, p. 70], there exists $B \in \mathrm{CO}(Y)$ with $\varphi(C_A) = C_B$ and $\varphi(C_{X\setminus A}) = C_{Y\setminus B}$. From Lemma 3.12(2), $f[\overline{A}] \subseteq \overline{B}$ and $f[\overline{X \setminus A}] \subseteq \overline{Y \setminus B}$, i.e., $f[A] \subseteq B$ and $Y \setminus B \supseteq f[X \setminus A] = Y \setminus f[A]$. Thus, $f[A] = B$.

For the second part, fix $B \in \mathrm{CO}(Y)$. According to Lemma 3.12(1), f^{-1} is the φ^{-1}-induced function and so, we can apply part (1) of this lemma to f^{-1} in order to get $f^{-1}[B] \in \tau_X$. Thus, the assumption that $\mathrm{CO}(Y)$ is a base for Y gives f's continuity. □

Lemma 3.15. *Let X and Y be spaces, with X zero-dimensional. If φ is an isomorphism from $\mathcal{U}_X$ onto $\mathcal{U}_Y$ and f is the φ-induced function, then $\varphi(C_G) \le C_{f[G]}$, whenever G is a closed subset of X.*

Proof. Given G, a closed subset of X, there are $\mathcal{A} \subseteq \mathrm{CO}(X)$ and $\beta \in \mathcal{D}_X$ in such a way that $G = \bigcap \mathcal{A}$ and $\varphi(C_G) = C_\beta$. Let us argue that

$$\text{for all } A \in \mathcal{A} \text{ and } B \in \beta,\ B \subseteq f[A]. \tag{3.4}$$

Suppose $A \in \mathcal{A}$ and $B \in \beta$ are arbitrary. Since $G \subseteq A$, we deduce that $C_G \le C_A$ and, consequently, the use of Lemma 3.14(1) gives

$$C_\beta = \varphi(C_G) \le \varphi(C_A) = C_{f[A]};$$

in particular, $B \subseteq \overline{f[A]}$. To complete this part, invoke lemmas 3.12(1) and 3.14(2) in order to get the continuity of f^{-1}, i.e., the closedness of f.

From (3.4) and the fact that f is one-to-one, we obtain that, for any $B \in \beta$,

$$B \subseteq \bigcap\{f[A] : A \in \mathcal{A}\} = f\left[\bigcap \mathcal{A}\right] = f[G].$$

In other words, $C_\beta \leq C_{f[G]}$, as claimed. □

Proposition 3.16. *Let X, Y, φ, f, and ψ be as in Problem 3.11. If X and Y are zero-dimensional, then $\varphi = \psi$.*

Proof. First of all, lemmas 3.14(2) and 3.12(1) guarantee that f is a homeomorphism.

With the idea in mind of verifying condition (3) of Proposition 3.13, fix G, a closed subset of X. According to Lemma 3.15, $\varphi(C_G) \leq C_{f[G]}$. On the other hand, $f[G]$ is a closed subset of Y and so, by applying Lemma 3.15 to φ^{-1} and f^{-1}, we obtain $\varphi^{-1}(C_{f[G]}) \leq C_{f^{-1}[f[G]]} = C_G$, i.e., $C_{f[G]} \leq \varphi(C_G)$. Thus, $\varphi(C_G) = C_{f[G]}$.

We conclude that φ is the f-induced relation or, in other words, $\varphi = \psi$. □

Corollary 3.17. *Let X and Y be a pair of zero-dimensional spaces. For any function $\varphi : \mathcal{U}_X \to \mathcal{U}_Y$, the following statements are equivalent.*

(1) *φ is an isomorphism.*
(2) *For some homeomorphism $f : X \to Y$, φ is the f-induced relation.*

Problem 3.18. *Is the assumption of zero-dimensionality necessary in Corollary 3.17? To be more precise, are there non-homeomorphic spaces X and Y for which the lattices $\mathcal{U}_X$ and $\mathcal{U}_Y$ are isomorphic?*

4. Some cardinal characteristics

Definition 4.1. For a space X, set $\mathcal{U}_X^+ := \mathcal{U}_X \setminus \{C_\varnothing\}$. Also, given a family $\mathcal{S} \subseteq \mathcal{U}_X^+$, we say that

(1) $\mathcal{S}$ is *an antichain in* $\mathcal{U}_X$ if for any $\sigma, \tau \in \mathcal{S}$, the condition $\sigma \neq \tau$ implies that $\sigma \wedge \tau = C_\varnothing$;
(2) $\mathcal{S}$ is *dense in* $\mathcal{U}_X$ if for each $\sigma \in \mathcal{U}_X^+$ there is $\tau \in \mathcal{S}$ with $\tau \leq \sigma$.

For a space X, *the cellularity of* $\mathcal{U}_X$, $c(\mathcal{U}_X)$, is the supremum of all cardinals of the form $|\mathcal{W}|$, where $\mathcal{W}$ is an antichain in $\mathcal{U}_X$. The *density of* $\mathcal{U}_X$, $\pi(\mathcal{U}_X)$, is the minimum size of a dense subset of $\mathcal{U}_X$.

Proposition 4.2. *If X is a space, then $c(\mathcal{U}_X) = \pi(\mathcal{U}_X) = |X|$.*

Proof. As one easily verifies, $\mathcal{A} := \{C_x : x \in X\}$ is an antichain in $\mathcal{U}_X$. Thus, $|X| \leq c(\mathcal{U}_X)$. On the other hand, if $\alpha \in \mathcal{D}_X$ satisfies $C_\alpha \in \mathcal{U}_X^+$, then $C_\alpha \not\leq C_\varnothing$, i.e., there are $A \in \alpha$ and $z \in A$. Therefore, $C_z \leq C_\alpha$ and, consequently, $\mathcal{A}$ is a dense subset of $\mathcal{U}_X$. Hence, $\pi(\mathcal{U}_X) \leq |X|$.

In order to prove that $c(\mathcal{U}_X) \leq \pi(\mathcal{U}_X)$, let us fix $\mathcal{W}$, an antichain in $\mathcal{U}_X$, and $\mathcal{S}$, a dense subset of $\mathcal{U}_X$. Then, there is $e : \mathcal{W} \to \mathcal{S}$ such that $e(\tau) \leq \tau$, whenever $\tau \in \mathcal{W}$. Given $\sigma, \tau \in \mathcal{W}$ with $\sigma \neq \tau$, one gets $e(\sigma) \wedge e(\tau) \leq \sigma \wedge \tau = C_\varnothing$

and so, $e(\sigma) \neq e(\tau)$; in other words, e is one-to-one and, as a consequence, $|\mathcal{W}| \leq |\mathcal{S}|$. $\square$

Now we turn our attention to $|\mathcal{U}_X|$ and $|\mathcal{D}_X|$, for an arbitrary space X. With this in mind, given a cardinal κ, let us recursively define $\beth_0(\kappa) := \kappa$ and, for each integer n, $\beth_{n+1}(\kappa) := 2^{\beth_n(\kappa)}$.

Proposition 4.3. *The following statements hold for any finite space X.*

(1) *When $|X| = 1$, $|\Sigma(X)| < 2^{|X|} < |\mathcal{D}_X| = \beth_2(|X|)$.*
(2) *If X has at least two points, then $2^{|X|} \leq |\Sigma(X)| < |\mathcal{D}_X| < \beth_2(|X|)$.*
(3) *$|\mathcal{U}_X| = 2^{|X|}$.*

Proof. If X has exactly one element, then

$$\Sigma(X) = \{\{\varnothing, X\}\} \quad \text{and} \quad \mathcal{D}_X = \{\varnothing, \{\varnothing\}, \{X\}, \{\varnothing, X\}\}.$$

With respect to (2), since the function $\eta : \mathcal{P}(X) \setminus \{\varnothing\} \to \Sigma(X)$ given by $\eta(A) := \{\varnothing, A, X\}$ is one-to-one, we deduce that $2^{|X|} - 1 = |\operatorname{ran}(\eta)| \leq |\Sigma(X)|$. Let us fix $p, q \in X$ with $p \neq q$. From the fact that $\{\varnothing, \{p\}, \{q\}, \{p, q\}, X\}$ is a member of $\Sigma(X) \setminus \operatorname{ran}(\eta)$, it follows that $2^{|X|} \leq |\Sigma(X)|$.

The relations $\Sigma(X) \subseteq \mathcal{D}_X$ and $\{X\} \in \mathcal{D}_X \setminus \Sigma(X)$ clearly imply that $|\Sigma(X)| < |\mathcal{D}_X|$. Lastly, the inequality $|\mathcal{D}_X| < \beth_2(|X|)$ follows from the facts $\mathcal{D}_X \subseteq \mathcal{P}(\mathcal{P}(X))$ and $C_p \vee C_q \in \mathcal{P}(\mathcal{P}(X)) \setminus \mathcal{D}_X$.

In order to prove (3), start by noticing that from $|X| < \omega$ one gets $C_p = C_u$. Thus, [9, Proposition 5.2, p. 70] implies that $\mathcal{P}(X)$, ordered by direct inclusion, and the closed interval $[C_\varnothing, C_u]$, equipped with the order it inherits from $\mathcal{U}_X$, are order-isomorphic. Finally, (1) in [9, Proposition 3.2, p. 67] guarantees that $\mathcal{U}_X = [C_\varnothing, C_u]$. $\square$

Given a space X, let us denote by $\operatorname{RO}(X)$ the collection of all regular open subsets of X. According to [6, Theorem 1.37, p. 26], when we order $\operatorname{RO}(X)$ by direct inclusion, the resulting structure is a complete Boolean algebra.

Proposition 4.4. *The following relations hold for any infinite topological space X.*

(1) *$|\mathcal{D}_X| = \beth_2(|X|)$.*
(2) *$\max\{2^{|X|}, 2^{|\operatorname{RO}(X)|}\} \leq |\mathcal{U}_X| \leq 2^{o(X)}$, where $o(X) := |\tau_X|$.*

Proof. The inequality $|\mathcal{D}_X| \leq \beth_2(|X|)$ follows from the relation $\mathcal{D}_X \subseteq \mathcal{P}(\mathcal{P}(X))$. On the other hand, according to [5, Theorem 7.6, p. 75], there are $\beth_2(|X|)$ filters on the set X and, naturally, each one of them is a member of $\mathcal{D}_X$. This proves (1).

With respect to (2), recall that τ_X^* is the collection of all closed subsets of X. Clearly, $|\tau_X^*| = o(X)$. An immediate consequence of Proposition 2.2 is that for each $\alpha \in \mathcal{D}_X$ the family $\overline{\alpha} := \{\overline{A} : A \in \alpha\}$ is a directed set and $C_\alpha = C_{\overline{\alpha}}$. Therefore, $\mathcal{U}_X$ is equal to $\{C_\beta : \beta \in \mathcal{D}_X \ \wedge \ \beta \subseteq \tau_X^*\}$, which, in turn, implies that $|\mathcal{U}_X| \leq |\mathcal{P}(\tau_X^*)| = 2^{o(X)}$.

Now, [9, Proposition 5.2, p. 70] guarantees the existence of a one-to-one map from $\mathcal{P}(X)$ into $\mathcal{U}_X$ and so, $2^{|X|} \leq |\mathcal{U}_X|$.

For the remaining inequality we need some notation. First, given a finite function $p \subseteq \mathrm{RO}(X) \times 2$, set

$$p^{\sim} := p^{-1}\{0\} \cup \{-x : x \in p^{-1}\{1\}\},$$

where $-x$ is the Boolean complement of $x \in \mathrm{RO}(X)$. Hence, a set $\mathcal{A} \subseteq \mathrm{RO}(X)$ is called *independent* if for any finite function $p \subseteq \mathcal{A} \times 2$ one has $\bigwedge p^{\sim} \neq \varnothing$.

The fact that X is an infinite Tychonoff space implies that $\mathrm{RO}(X)$ is infinite as well and so, by Balcar-Franěk's Theorem (see [6, Theorem 13.6, p. 196]), there is an independent set $\mathcal{A} \subseteq \mathrm{RO}(X)$ with $|\mathcal{A}| = |\mathrm{RO}(X)|$.

Let us argue that, for each $d : \mathcal{A} \to 2$, the collection

$$\alpha(d) := \left\{ \bigvee p^{\sim} : p \in [d]^{<\omega} \right\}$$

is a member of $\mathcal{D}_X$. Indeed, if $p, q \in [d]^{<\omega}$, then $r := p \cup q$ is a finite subset of d with $\bigvee r^{\sim} = (\bigvee p^{\sim}) \vee \bigvee q^{\sim}$ and since $\mathrm{RO}(X)$ is ordered by direct inclusion, we conclude that $\bigvee r^{\sim}$ is an element of $\alpha(d)$ which is a superset of $\bigvee p^{\sim}$ and $\bigvee q^{\sim}$.

Claim. If $d, e \in {}^{\mathcal{A}}2$ and $U \in \mathcal{A}$ satisfy $d(U) = 0$ and $e(U) = 1$, then, for any $V \in \alpha(e)$, $U \not\subseteq V$.

Before we present the proof of our Claim, let's assume it holds and fix $d, e \in {}^{\mathcal{A}}2$ with $d \neq e$. Without loss of generality, we may assume that, for some $U \in \mathcal{A}$, $d(U) = 0$ and $e(U) = 1$. Thus, $U \in \alpha(d)$ and if V were a member of $\alpha(e)$ with $U \subseteq \overline{V}$, we would get $U = \operatorname{int} U \subseteq \operatorname{int} \overline{V} = V$, a contradiction to the Claim. As a consequence of this argument, we obtain that the function from ${}^{\mathcal{A}}2$ into $\mathcal{U}_X$ given by $d \mapsto C_{\alpha(d)}$ is one-to-one and so, $2^{|\mathrm{RO}(X)|} = 2^{|\mathcal{A}|} \leq |\mathcal{U}_X|$.

Suppose d, e, and U are as in the Claim. Seeking a contradiction, let us assume that $U \subseteq \bigvee p^{\sim}$, for some $p \in [e]^{<\omega}$. We affirm that if $q := p \restriction (\mathrm{dom}(p) \setminus \{U\})$ (the restriction of the function p to the given set), then

$$U \subseteq \bigvee q^{\sim}. \tag{4.1}$$

Indeed, when $U \notin \mathrm{dom}(p)$, $p = q$. On the other hand, if $U \in \mathrm{dom}(p)$, the relation $p \subseteq e$ gives $p(U) = 1$ and so, $\bigvee p^{\sim} = (-U) \vee \bigvee q^{\sim}$ which, clearly, implies (4.1).

Let us define $r : \mathrm{dom}(q) \cup \{U\} \to 2$ by $r(V) = 1 - q(V)$, whenever $V \in \mathrm{dom}(q)$, and $r(U) = 0$. Obviously, $r \subseteq \mathcal{A} \times 2$ is a finite function and thus, the independence of $\mathcal{A}$ and the De Morgan's laws produce

$$\varnothing \neq \bigwedge r^{\sim} = U \wedge \left(- \bigvee q^{\sim} \right),$$

a contradiction to (4.1). □

Let us recall that a T_6-*space* (equivalently, *perfectly normal space*) is a Hausdorff normal space in which all open sets are of type F_σ.

Corollary 4.5. *If X is a T_6-space, then $|\mathcal{U}_X| = 2^{o(X)}$.*

Proof. We only need to mention that, according to [3, Theorem 10.5, p. 40], $|\mathrm{RO}(X)| = o(X)$. □

Our next result is a direct consequence of corollaries 4.5 and 3.9 (recall that any infinite Tychonoff space contains a copy of the discrete space of size ω).

Corollary 4.6. *If Y is an infinite discrete subspace of a space X, $\beth_2(|Y|) \leq |\mathcal{U}_X|$. In particular, when X is infinite, $2^{\mathfrak{c}} \leq |\mathcal{U}_X|$.*

Standard arguments show that if X is an arbitrary space and D is a dense subspace of it, then the function from $\mathrm{RO}(X)$ into $\mathcal{P}(D)$ given by $U \mapsto U \cap D$ is one-to-one. Therefore (recall that $d(X)$ is the density of X),

$$\text{for any space } X,\ |\,\mathrm{RO}(X)| \leq 2^{d(X)}. \tag{4.2}$$

Regarding the accuracy of the bounds presented in Proposition 4.4(2), we have the result below.

Proposition 4.7. *The following statements are true.*

(1) *If X is the Moore-Niemytzki plane* (*see* [1, Example 1.2.4, p. 21])*, then $|X| = |\,\mathrm{RO}(X)| = \mathfrak{c}$ and $o(X) = 2^{\mathfrak{c}}$.*
(2) *When X is the Stone-Čech compactification of the integers, $|\,\mathrm{RO}(X)| = \mathfrak{c}$ and $|X| = o(X) = 2^{\mathfrak{c}}$.*
(3) *If X is the Arens-Fort space,* [1, Example 1.6.19, p. 54]*, then $|X| = \omega$ and $|\,\mathrm{RO}(X)| = o(X) = \mathfrak{c}$.*

Proof. Let us prove (1). Clearly, $|X| = \mathfrak{c}$. The equality $|\,\mathrm{RO}(X)| = \mathfrak{c}$ follows from the facts, (i) property (4.2) (recall that X is separable) and (ii) the canonical base for X consists of $\mathfrak{c}$ many regular open sets. Note that from (ii) we also deduce the relation $o(X) \leq 2^{\mathfrak{c}}$. Finally, since $X \setminus (\mathbb{R} \times \{0\})$ is an open subset of X which is homeomorphic to an open subspace of the Euclidean plane, we conclude that $2^{\mathfrak{c}} \leq o(X)$.

Suppose X is as in (2). From [1, Corollary 3.6.12, p. 175], $|X| = 2^{\mathfrak{c}}$. On the other hand, the relation $|\,\mathrm{RO}(X)| = \mathfrak{c}$ is a consequence of (4.2) and the fact that, according to Theorem 3.6.13 and Corollary 3.6.12 of [1, p. 175], X is a space of weight $\mathfrak{c}$ possessing a base of closed-and-open sets. This last statement also implies that $o(X) \leq 2^{\mathfrak{c}}$. Now, [1, Example 3.6.18, p. 175] guarantees that X has a pairwise disjoint family consisting of $\mathfrak{c}$ many non-empty open sets and so, $2^{\mathfrak{c}} \leq o(X)$.

Finally, when X is as in (3), one clearly gets $|X| = \omega$ and, therefore, $o(X) \leq \mathfrak{c}$. On the other hand, by definition, X has a base consisting of $\mathfrak{c}$ many closed-and-open sets; hence, $\mathfrak{c} \leq |\,\mathrm{RO}(X)| \leq o(X)$. □

In the next section we focus on the problem of calculating $|\mathcal{U}_X|$, for some spaces X.

5. The size of $\mathcal{U}_X$

Unless otherwise stated, all spaces considered from now on are infinite. Also, recall that [1] is our reference for topological cardinal functions.

In Corollary 4.5 we were able to calculate the precise value of $|\mathcal{U}_X|$ in terms of the cardinal function $o(X)$, when X belongs to the class of T_6-spaces. Here, we present some other classes of topological spaces in which the cardinality of the lattice $\mathcal{U}_X$ can be determined in a similar fashion.

Proposition 5.1. *Given a space X, if any of the following statements holds, then $|\mathcal{U}_X| = 2^{\mathfrak{c}}$.*

(1) *X is hereditarily Lindelöf and first countable.*
(2) *X admits a countable network.*
(3) *X is hereditarily separable and has countable pseudocharacter.*

Proof. From Proposition 4.4 and Figure 1 we deduce that $|\mathcal{U}_X| \leq 2^{\mathfrak{c}}$. The reverse inequality is a consequence of Corollary 4.6. □

In what follows, given a space X, we will employ the inequalities presented in Figure 1 together with Proposition 4.4(2) in order to get bounds for $|\mathcal{U}_X|$.

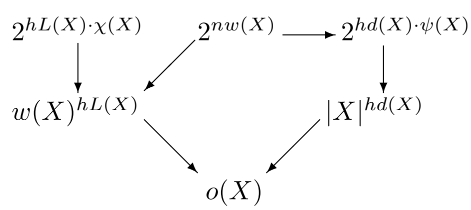

FIGURE 1. In this diagram X is an arbitrary space and the symbol $\kappa \to \lambda$ means that $\kappa \geq \lambda$. The upper right inequality can be found in [4, Theorem 7.1, p. 311] and the rest of them are basic (see [3]).

Now, regarding compact spaces we have the following results.

Lemma 5.2. *For any compact space X, $|\mathcal{U}_X| \leq \beth_2(hL(X))$.*

Proof. Given the hypotheses on X, we obtain $\chi(X) = \psi(X) \leq hL(X)$ and thus, the inequality needed follows from Figure 1 and Proposition 4.4. □

Proposition 5.3. *If X is a compact space in which every open subset of it is an F_σ-set, then $|\mathcal{U}_X| = 2^{\mathfrak{c}}$. In particular, every compact metrizable space satisfies the previous equality.*

Proof. It is sufficient to notice that our assumptions on X imply $hL(X) = \omega$. Thus, Corollary 4.6 and Lemma 5.2 give the desired result. □

Given an infinite cardinal κ, let us denote by $D(\kappa)$ and $\beta D(\kappa)$ the discrete space of size κ and its Stone-Čech compactification, respectively. The regularity of $\beta D(\kappa)$ implies that (see [3, Theorem 3.3, p. 11])

$$w(\beta D(\kappa)) \leq 2^{d(\beta D(\kappa))} = 2^{\kappa}.$$

Therefore, from Figure 1 and the compactness of $\beta D(\kappa)$ we deduce that

$$|\mathcal{U}_{\beta D(\kappa)}| \leq \beth_2\left(nw\left(\beta D(\kappa)\right)\right) = \beth_2\left(w\left(\beta D(\kappa)\right)\right) \leq \beth_3(\kappa).$$

On the other hand, since $|\beta D(\kappa)| = \beth_2(\kappa)$, Proposition 4.4(2) gives

$$\beth_3(\kappa) = 2^{|\beta D(\kappa)|} \leq |\mathcal{U}_{\beta D(\kappa)}|.$$

In conclusion, for any infinite cardinal κ, $|\mathcal{U}_{\beta D(\kappa)}| = \beth_3(\kappa)$.

Once again, let $\kappa \geq \omega$ be a cardinal. If $D(2)$ is the discrete space of size 2, then $D(2)^\kappa$ is the Cantor cube of weight κ. Clearly (see Figure 1),

$$|\mathcal{U}_{D(2)^\kappa}| \leq \beth_2\left(nw\left(D(2)^\kappa\right)\right) = \beth_2\left(w\left(D(2)^\kappa\right)\right) = \beth_2(\kappa).$$

Also, Proposition 4.4(2) produces

$$\beth_2(\kappa) = 2^{|D(2)^\kappa|} \leq |\mathcal{U}_{D(2)^\kappa}|.$$

Hence, for any infinite cardinal κ, $|\mathcal{U}_{D(2)^\kappa}| = \beth_2(\kappa)$.

Let $\mathbb{L}$ be the lexicographic square (i.e., $\mathbb{L}$ is the cartesian product $[0,1]^2$ endowed with the topology generated by the lexicographical ordering). By setting $Y := [0,1] \times \{\frac{1}{2}\}$ one gets a discrete subspace of $\mathbb{L}$ and so, according to Corollaries 4.5 and 3.9, $\beth_2(\mathfrak{c}) = |\mathcal{U}_Y| \leq |\mathcal{U}_\mathbb{L}|$. Finally, our definition of $\mathbb{L}$ gives $o(\mathbb{L}) \leq 2^\mathfrak{c}$ and, as a consequence, $|\mathcal{U}_\mathbb{L}| \leq \beth_2(\mathfrak{c})$. In other words, $|\mathcal{U}_\mathbb{L}| = \beth_2(\mathfrak{c})$.

The subspace $[0,1] \times \{0,1\}$ of $\mathbb{L}$ is called the double arrow space and we will denote it by $\mathbb{A}$. Since the subspace $(0,1) \times \{0\}$ of $\mathbb{A}$ is homeomorphic to Sorgenfrey's line, the space $\mathbb{A}^2$ contains a discrete subspace of size $\mathfrak{c}$. Therefore, as we did for $\mathbb{L}$, $|\mathcal{U}_{\mathbb{A}^2}| \geq \beth_2(\mathfrak{c})$. For the reverse inequality note that $o(\mathbb{A}^2) \leq o(\mathbb{L}^2) \leq 2^\mathfrak{c}$ and so, $|\mathcal{U}_{\mathbb{A}^2}| = \beth_2(\mathfrak{c})$.

A final note regarding $\mathbb{A}$ is pertinent. From (4.2) and the fact that $\mathbb{A}$ is separable, we deduce that $|\operatorname{RO}(\mathbb{A}^2)| \leq \mathfrak{c}$ and hence,

$$\max\{2^{|\mathbb{A}^2|}, 2^{|\operatorname{RO}(\mathbb{A}^2)|}\} = 2^\mathfrak{c} < \beth_2(\mathfrak{c}) = |\mathcal{U}_{\mathbb{A}^2}|.$$

This shows that the lower bounds for $|\mathcal{U}_X|$ presented in Proposition 4.4(2) need to be improved.

Proposition 5.4. *If X is hereditarily Lindelöf, then $|\mathcal{U}_X| = 2^{o(X)}$.*

Proof. With Corollary 4.5 in mind, we only need to show that all open subsets of X are F_σ. Let $U \in \tau_X$ be arbitrary. For each $x \in U$ there is $U_x \in \tau_X$ such that $x \in U_x \subseteq \overline{U_x} \subseteq U$. Since U is Lindelöf, for some $F \in [U]^{\leq\omega}$ we obtain $U = \bigcup\left\{\overline{U_x} : x \in F\right\}$. □

We present now our findings regarding the following question.

Problem 5.5. *Given a space X, what conditions on X imply that $|\mathcal{U}_X| = 2^{o(X)}$?*

Lemma 5.6. *If X is a space with $|X|^{hd(X)} = |X|$, then $|\mathcal{U}_X| = 2^{o(X)}$.*

Proof. It follows from Figure 1 and our hypotheses that $o(X) \leq |X|$. On the other hand, the fact that X is Tychonoff clearly implies the relation $|X| \leq o(X)$. Hence, the equality we need is a consequence of Proposition 4.4(2). □

Proposition 5.7. *If X is a space for which there is a cardinal κ with $|X| = 2^\kappa$ and $\kappa \geq hd(X)$, then $|\mathcal{U}_X| = 2^{o(X)}$.*

Proof. Our choice for κ gives $|X|^{hd(X)} = |X|$ and so, the hypotheses of Lemma 5.6 are satisfied. □

As usual, the acronym GCH stands for the Generalized Continuum Hypothesis and $\text{cf}(\alpha)$ denotes the cofinality of an ordinal α.

Proposition 5.8. *Assuming* GCH, *if X is a space satisfying* $\text{cf}(|X|) > hd(X)$, *then* $|\mathcal{U}_X| = 2^{o(X)}$.

Proof. According to [7, Lemma 10.42, p. 34], $|X|^{hd(X)} = |X|$ and therefore we only need to invoke Lemma 5.6. □

Proposition 5.9. *Given a space X, if $|X|$ is a singular strong limit cardinal, then* $|\mathcal{U}_X| = 2^{o(X)}$.

Proof. The hypothesis allows us to use [2, Theorem 3, p. 22] to find a discrete set $D \subseteq X$ such that $|D| = |X|$. Hence, Proposition 4.4(2) and Corollary 4.6 imply that $|\mathcal{U}_X| = 2^{o(X)}$. □

Let us denote by A the statement "GCH holds and there are no inaccessible cardinals."

Corollary 5.10. *Assume* A *holds. Then, for any space X whose cardinality is a limit cardinal we obtain* $|\mathcal{U}_X| = 2^{o(X)}$.

With the idea in mind of finding the effect that GCH has on $|\mathcal{U}_X|$, let us recall that, for a cardinal number κ, κ^+ represents the successor cardinal of κ.

Proposition 5.11. *If* GCH *holds, then, for any space X, $|\mathcal{U}_X|$ is a regular uncountable cardinal.*

Proof. On the one hand, Corollary 4.6 implies that $|\mathcal{U}_X|$ is uncountable. On the other hand, since $2^{|X|} \leq |\mathcal{U}_X| \leq 2^{o(X)} \leq \beth_2(|X|) = (2^{|X|})^+$, we deduce that $|\mathcal{U}_X| \in \{|X|^+, (2^{|X|})^+\}$. In either case, $|\mathcal{U}_X|$ is regular. □

Proposition 5.12. *Under the assumptions $\mathfrak{c} = \omega_1$ and $2^{\mathfrak{c}} = \omega_2$, if X is a hereditarily separable space, then* $|\mathcal{U}_X| = 2^{o(X)}$.

Proof. According to [3, Theorem 4.12, p. 21], the relation $hd(X) = \omega$ guarantees that $|X| \leq 2^{\mathfrak{c}}$ and consequently, $|X| \in \{\omega, \omega_1, \omega_2\}$.

When $|X| \in \{\omega_1, \omega_2\}$, Proposition 5.7 gives us the desired equality. Finally, if $|X| = \omega$, then X admits a countable network and thus (see Proposition 5.1), $|\mathcal{U}_X| = 2^{\mathfrak{c}} = 2^{o(X)}$. □

Suppose X is a space. Since $\mathcal{U}_X$ is a subset of $\Sigma(C(X))$, we obtain $|\mathcal{U}_X| \leq |\Sigma(C(X))|$. With the idea in mind of showing two examples for which this inequality is strict, let us note first that the fact $|C(X)| \geq \omega$ implies, according to [8, Theorem 1.4, p. 179], that $|\Sigma(C(X))| = \beth_2(|C(X)|)$.

When X is an infinite discrete space, we obtain $|C(X)| = 2^{|X|}$ and so, by Proposition 4.4(2),

$$|\mathcal{U}_X| \leq \beth_2(|X|) < \beth_3(|X|) = \beth_2(|C(X)|).$$

On the other hand, if X is any infinite countable space, then it follows from Proposition 5.1(2) that

$$|\mathcal{U}_X| = 2^{\mathfrak{c}} < \beth_2(\mathfrak{c}) \leq \beth_2(|C(X)|).$$

Our final result of this section establishes some conditions for a family of topological spaces under which the corresponding Tychonoff product X satisfies the equality $|\mathcal{U}_X| = |\Sigma(C(X))|$. For this proposition we won't require for our spaces to be infinite.

Proposition 5.13. *Assume that κ is an infinite cardinal. Let X be the topological product of a family of spaces $\{X_\xi : \xi < 2^\kappa\}$. If $|X_\xi| \geq 2$ and $d(X_\xi) \leq \kappa$ for each $\xi < \kappa$, then $|\mathcal{U}_X| = |\Sigma(C(X))|$.*

Proof. Since we always have the inequality $|\mathcal{U}_X| \leq |\Sigma(C(X))|$, we only need to show that $|\mathcal{U}_X| \geq \beth_2(|C(X)|)$.

According to Proposition 4.4(2), $|\mathcal{U}_X| \geq 2^{|X|}$. Now, the fact that each X_ξ has at least two points gives $|X| \geq \beth_2(\kappa)$ and so, $2^{|X|} \geq \beth_3(\kappa)$. On the other hand, the Hewitt-Marczewski-Pondiczery Theorem (see [1, Theorem 2.3.15, p. 81]) implies that $d(X) \leq \kappa$ and therefore, from the well-known relation $2^{d(X)} \geq |C(X)|$ we deduce that $2^\kappa \geq |C(X)|$. In conclusion, $|\mathcal{U}_X| \geq \beth_3(\kappa) \geq \beth_2(|C(X)|)$, as required. □

For example, if X is a Cantor cube of the form $D(2)^{2^\kappa}$, where κ is an infinite cardinal, then $|\mathcal{U}_X| \geq \beth_2(|C(X)|)$.

We close the paper with a list of open questions.

Problem 5.14. *Does Corollary 4.5 remain true if we replace T_6 with T_5 in the hypotheses?*

Problem 5.15. *Regarding Proposition 5.4, is it true that for any compact space X, $|\mathcal{U}_X| = 2^{o(X)}$?*

Problem 5.16. *Can we drop the set-theoretic assumptions $\mathfrak{c} = \omega_1$ and $2^{\mathfrak{c}} = \omega_2$ in Proposition 5.12?*

We conjecture that, under A, the equality

$$|\mathcal{U}_X| = 2^{o(X)} \tag{5.1}$$

holds for any space X. Even though we did not prove or refute this conjecture, we were able to obtain some partial results (for example, if one assumes A, then (i) for any space X, $\beth_2(s(X)) \leq |\mathcal{U}_X|$, and (ii) we possess a short list of classes $\mathcal{S}$ in such a way that $X \in \mathcal{S}$ implies that (5.1) holds). Consequently, we pose the following problem.

Problem 5.17. *Does it follow from A that (5.1) is true for any space X?*

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