

Continuous extension in topological digital spaces

ERIK MELIN

ABSTRACT. We give necessary and sufficient conditions for the existence of a continuous extension from a smallest-neighborhood space (Alexandrov space) X to the Khalimsky line. Using this result, we classify the subsets $A \subset X$ such that every continuous function $A \rightarrow \mathbb{Z}$ can be extended to all of X . We also consider the more general case of mappings $X \rightarrow Y$ between smallest-neighborhood spaces, and prove a digital no-retraction theorem for the Khalimsky plane.

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1. INTRODUCTION

The classical Tietze extension theorem states that if X is a normal topological space and A is a closed subset of X , then any continuous map from A into the closed interval $[a, b]$ can be extended to a continuous function on all of X into $[a, b]$. In digital geometry it is more natural to study functions from a digital space (defined below) to the integers. Having equipped the spaces with suitable topologies, we may consider continuous functions and the corresponding extension problem.

We solve this problem for a fairly general class of topological digital spaces and for continuous functions with the Khalimsky line as codomain. The Khalimsky topology is a topology that appears naturally in the study of digital geometry, cf. Kong [12]. The notion of straight lines in Khalimsky spaces has been studied in [18] and more general curves and surfaces in [17]. The results in this paper are generalizations of a previous work [19], where the digital spaces considered were restricted to \mathbb{Z}^n .

Digital topology can be considered from a graph-theoretical point of view; one has a set of points and an adjacency relation, i.e., a symmetric binary

relation defining which points are adjacent. For example, in \mathbb{Z}^n one possibility is to say that points p and q are adjacent if $\|p - q\|_\infty = 1$.

In such, graph based, spaces no topology, in the usual meaning, is directly involved. Nevertheless, a theory for continuous functions has been developed in this setting, for example by Boxer [2].

Herman and Webster [5] and Kovalevsky [16] have independently argued that cellular complexes are appropriate topological spaces for digital geometry. The results of this paper applies to such spaces, since they are topologically equivalent to spaces of the type introduced in this article. See, for example, Klette [10].

We shall only consider digital spaces that are also topological spaces and where the topological notion of connectedness agrees with the notion of connectedness in digital spaces. More details on the theory of such spaces and applications to image analysis can be found in for example the survey by Kong and Rosenfeld [14], in [13] by Kong et al. and in Koppermans's [15] article.

2. BACKGROUND

This section is devoted to some basic definitions and results that we will need. Those unfamiliar with the subject might find Kiselman's [9] Lecture Notes helpful.

2.1. Smallest-neighborhood spaces. In any topological space, a finite intersection of open sets is open, whereas the stronger requirement that an arbitrary intersection of open sets is open, is not satisfied in general. In a classical article, Alexandrov [1] considers topological spaces that fulfill the stronger requirement, where arbitrary intersections of open sets are open. In his paper, Alexandrov called such spaces *Diskrete Räume* (discrete spaces). Unfortunately, this terminology is not possible today since the term discrete topology is occupied by the topology where every set is open. Instead, following Kiselman [9], we will call such spaces *smallest-neighborhood spaces*. Another name that has been used is *Alexandrov spaces*.

Let $\mathcal{N}_X(x)$ denote the intersection of all neighborhoods of a point x in a topological space X . If there is no danger of ambiguity, we will just write $\mathcal{N}(x)$. In a smallest-neighborhood space, $\mathcal{N}(x)$ is always open and thus a neighborhood of x ; clearly $\mathcal{N}(x)$ is the smallest neighborhood of x . We may note that $x \in \mathcal{N}(y)$ if and only if $y \in \mathcal{C}(x)$, where $\mathcal{C}(x) = \overline{\{x\}}$ is the topological closure of the singleton set containing x . Note that the familiar construction of the closure, as an intersection of closed sets, is dual to the construction of the smallest neighborhood.

Conversely, the existence of a smallest neighborhood around every point implies that an arbitrary intersection of open sets is open; hence this existence could have been used as an alternative definition of a smallest-neighborhood space. A topological space X is called connected if the only sets which are both closed and open are the empty set and X itself. Note that $\mathcal{N}(x)$ considered as a subspace of X is always connected and that a two-point set $\{x, y\}$ is connected

if and only if $x \in \mathcal{N}(y)$ or $y \in \mathcal{N}(x)$. A point x is called *open* if the set $\{x\}$ is open, and is called *closed* if $\{x\}$ is closed. If a point x is either open or closed it is called *pure*, otherwise it is called *mixed*.

Kolmogorov's separation axiom, also called the T_0 axiom, states that given two distinct points x and y , there is an open set containing one of them but not the other. An equivalent formulation is that $\mathcal{N}(x) = \mathcal{N}(y)$ implies $x = y$ for every x and y . This axiom is quite natural to impose; if x and y have the same neighborhood, then they are indistinguishable from a topological point of view and should perhaps be identified. The separation axiom T_1 , on the other hand, is too strong. It states that points are closed and in a smallest-neighborhood space this means that every set is closed. Hence only spaces with the discrete topology are smallest-neighborhood spaces satisfying the T_1 axiom.

Remark 2.1. There is a correspondence between smallest-neighborhood spaces and partially ordered sets; namely if we define $x \preceq y$ if and only if $y \in \mathcal{N}(x)$. This relation is always reflexive and transitive and is anti-symmetric if and only the space is T_0 . Then the order is a partial order and is called the *specialization order*. It was introduced by Alexandrov [1]. It is not hard to see that a function is continuous if and only if it is increasing for the specialization order. A consequence is that our results can be formulated in the language of partially ordered sets instead of topologies, if one prefers.

2.2. Topological digital spaces. In the graph-theoretical setting, a digital space is called *connected* if it is path connected under the adjacency relation—see [14] for details.

Given a topological space X we may try to identify it with a digital space in this sense. It is natural to define the binary adjacency relation such that distinct points x and y are adjacent if and only if $\{x, y\}$ is connected (as a subspace of X).

If X is not connected in the topological sense, it is obvious that it cannot be path connected under the adjacency relation above. On the other hand, the following lemma shows that a connected smallest-neighborhood space is indeed also a connected digital space in the graph-theoretical sense. The result is also stated in for example [4, Lemma 4.2.1], however, the present proof is shorter.

Lemma 2.2. *Let X be a connected smallest-neighborhood space. Then for any pair of points x and y of X there is a finite sequence (x_0, \dots, x_n) such that $x = x_0$ and $y = x_n$ and $\{x_j, x_{j+1}\}$ is connected for $i = 0, 1, \dots, n - 1$.*

Proof. Let x be a point in X , and denote by Y the set of points which can be connected to x by such a finite sequence. Obviously $x \in Y$. Suppose that $y \in Y$. It follows that $\mathcal{N}(y) \subset Y$ and $\mathcal{C}(y) \subset Y$. Thus Y is open, closed and nonempty. Since X is connected this reads $Y = X$. \square

With this lemma as a motivation, we give the following definition: a *topological digital space* is a connected smallest-neighborhood space.

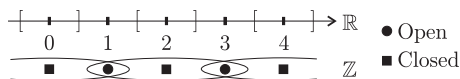


FIGURE 1. Construction of the Khalimsky line.

2.3. Examples of topological digital spaces. We shall present some examples of digital spaces and examine some properties of these examples. Eckhardt and Latecki [3] and Kong [11] give more examples of topologies on \mathbb{Z}^2 and \mathbb{Z}^3 .

Example 2.3. Every finite topological space is obviously a smallest-neighborhood space; therefore every connected finite space is a digital space. Properties of finite spaces are studied by Stong [20]. Many results there are also true for infinite smallest-neighborhood spaces.

Example 2.4 (The Khalimsky line [6, 7]). We shall construct a topology on the digital line, \mathbb{Z} , originally introduced by Efim Khalimsky. Let us identify with each even integer m the closed, real, interval $[m - 1/2, m + 1/2]$ and with each odd integer the open interval $]m - 1/2, m + 1/2[$. These intervals form a partition of the Euclidean line \mathbb{R} and we may therefore consider the quotient space. Identifying each interval with the corresponding integer gives us the *Khalimsky topology* on \mathbb{Z} (see Figure 1). Since \mathbb{R} is connected, the Khalimsky line is connected. It follows easily that an even point is closed and that an odd point is open. In terms of smallest neighborhoods, we have $\mathcal{N}(m) = \{m\}$ if m is odd and $\mathcal{N}(n) = \{n \pm 1, n\}$ if n is even.

As will be seen in the next section, the Khalimsky topology is of particular importance in the study of topological digital spaces. Therefore we will here make some further definitions. Let a and b be integers. A *Khalimsky interval* is an interval $[a, b] \cap \mathbb{Z}$ of integers with the topology induced from the Khalimsky line. We will denote such an interval by $[a, b]_{\mathbb{Z}}$. A *Khalimsky arc* is a homeomorphic image of a Khalimsky interval into any topological space.

Unless otherwise stated, we will assume that \mathbb{Z} is equipped with the Khalimsky topology from now on. This makes it meaningful to consider continuous functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$. We will discuss some properties of such functions; more details can be found in [9]. Suppose that f is continuous. Since $M = \{m, m + 1\}$ is connected it follows that $f(M)$ is connected, but this is the case only if $|f(m) - f(m + 1)| \leq 1$. Hence f is Lipschitz with Lipschitz constant 1; we say f is Lip-1. Lip-1, however, is not sufficient for continuity. If $y = f(x)$ is odd, then $U = f^{-1}(\{y\})$ must be open, so if x is even then $x \pm 1 \in U$. This means that $f(x \pm 1) = f(x)$. It is straightforward to prove the following, see [19, Lemma 2].

Proposition 2.5. $f: \mathbb{Z} \rightarrow \mathbb{Z}$ is continuous if and only if for every $p, q \in \mathbb{Z}$ where $p \neq q$, one of the following conditions holds

- (1) $|f(q) - f(p)| < |q - p|$
- (2) $|f(q) - f(p)| = |q - p|$ and $p \equiv f(p) \pmod{2}$.

Corollary 2.6. *A function $f: A \rightarrow \mathbb{Z}$, $A \subset \mathbb{Z}$ can be extended to a continuous function $F: \mathbb{Z} \rightarrow \mathbb{Z}$ if and only if the condition in the above proposition holds for every $p, q \in A$ where $p \neq q$.*

The following two examples give spaces with topologies that are derived from the Khalimsky topology.

Example 2.7 (Khalimsky n -space). The *Khalimsky plane* is the Cartesian product of two Khalimsky lines and in general, *Khalimsky n -space* is \mathbb{Z}^n with the product topology. Points with all coordinates even are closed and points with all coordinates odd are open. Points with both even and odd coordinates are mixed.

Example 2.8. We may consider a quotient space $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ for some even integer $m \geq 2$ (if m is odd, we will end up identifying open and closed points, which results in a space with the indiscrete topology, sometimes called the chaotic topology). Such a space is called a *Khalimsky circle*. If $m \geq 4$, \mathbb{Z}_m is a compact space which is locally homeomorphic to the Khalimsky line. In a similar manner, we may construct a *Khalimsky torus*, by identifying the edges of a Khalimsky rectangle, i.e., a product of two Khalimsky intervals (of even length).

Our final example shows that there are topological digital spaces of arbitrarily high cardinality.

Example 2.9. Let J be any index set and let K be the disjoint union of J copies of the Khalimsky line. This space is of course not connected, but we may identify the point 0 in every copy and then consider the quotient space K_0 . It is easy to see that K_0 is connected. Note also that $\mathcal{N}(0)$ may be a very large set in this space.

2.4. Arc-connected spaces. Let X be a topological digital space satisfying the T_0 axiom and let $x, y \in X$. The following theorem shows that we can actually find a Khalimsky arc in X with endpoints x and y , i.e., there is an interval $I = [a, b]_{\mathbb{Z}}$ and a function $\psi: I \rightarrow X$ such that $\psi(a) = x$, $\psi(b) = y$ and ψ is a homeomorphism of I and $\psi(I)$. We call a topological space having this property *Khalimsky arc-connected* or just *arc-connected* if it is clear from the context what is intended. Note that an arc-connected space is always connected; this can be proved with the same argument as is used to prove that a path-connected space (in the usual topological sense) is connected.

Remark 2.10. In [7] the result below was proved for finite spaces only. Another difference in their version is that it was not required there that the space in question satisfied the T_0 -axiom. This is because they allowed an arc with two points $\{x, y\}$ to have the indiscrete topology $\{\emptyset, \{x, y\}\}$, which is clearly necessary if $\mathcal{N}(x) = \mathcal{N}(y)$ in X . With our definition, this type of two-point arc is not allowed; hence we must require the space to be T_0 .

Theorem 2.11. *A T_0 topological digital space is Khalimsky arc-connected.*

Proof. Let X be a topological digital space and let $a, b \in X$. First, use Lemma 2.2 to get a finite, connected sequence of points (x_0, \dots, x_n) such that $x_0 = a$ and $x_n = b$. Then use the finiteness of the sequence to choose a subsequence $Y = (y_0, \dots, y_m)$ that is minimal with respect to connectedness, and such that $y_0 = a$ and $y_m = b$. With this, we mean that Y (considered as a subset of X) has no connected proper subset containing a and b .

Suppose that $i > j$ and that $\{y_i, y_j\}$ is connected. Then $i = j + 1$; for if not, the sequence $(y_0, \dots, y_j, y_i, \dots, y_m)$ would be a connected proper subset of the minimal sequence, which is a contradiction.

By the result of the above paragraph, there are two possibilities for $\mathcal{N}_Y(y_0)$: either $\mathcal{N}_Y(y_0) = \{y_0\}$ or $\mathcal{N}_Y(y_0) = \{y_0, y_1\}$. (If $x \in \mathcal{N}_Y(y)$, then x and y are connected.) Let $s = 1$ in the first case and $s = 0$ in the second case. For a clearer notation, let us agree to re-index the sequence $\{y_i\}$, so that y_s is its first element and y_{m+s} its last. We will show that the function

$$\psi: I = [s, s + m]_{\mathbb{Z}} \rightarrow Y, i \mapsto y_i$$

is a homeomorphism.

Note that Y is T_0 since X is T_0 . Clearly ψ is surjective, and by minimality also injective. To show that ψ is a homeomorphism, it suffices to show that

$$(2.1) \quad \mathcal{N}_Y(y_i) = \psi(\mathcal{N}_I(i)) \text{ for every } i \in I.$$

For $i = s$ this holds by the choice of s . We use finite induction. Suppose that $s < k \leq s + m$, and that (2.1) holds for $i = k - 1$. We consider two cases:

Case 1: k is odd. Then $\mathcal{N}_I(k) = \{k\}$. We must show that y_{k-1} and y_{k+1} are not in $\mathcal{N}_Y(y_k)$. But since Eq. (2.1) holds for $i = k - 1$, and $k \in \mathcal{N}_I(k - 1)$, we know that $y_k \in \mathcal{N}_Y(y_{k-1})$. It follows that $y_{k-1} \notin \mathcal{N}_Y(y_k)$ (Y is T_0) and also that $y_{k+1} \notin \mathcal{N}_Y(y_k)$, since $\mathcal{N}_Y(y_{k-1}) \cap \mathcal{N}_Y(y_k)$ is a neighborhood of y_k which does not include y_{k+1} .

Case 2: k is even. We must show that $y_{k-1} \in \mathcal{N}_Y(y_k)$ and $y_{k+1} \in \mathcal{N}_Y(y_k)$ provided $k < s + m$. Now, $\mathcal{N}_Y(y_{k-1}) = \{y_{k-1}\}$ by assumption, so clearly y_{k-1} belongs to $\mathcal{N}_Y(y_k)$. If $k < s + m$ and $y_k \in \mathcal{N}_Y(y_{k+1})$, then $\mathcal{N}_Y(y_k) \cap \mathcal{N}_Y(y_{k+1})$ would be a neighborhood of y_k not containing y_{k-1} . This is a contradiction, and hence $y_{k+1} \in \mathcal{N}_Y(y_k)$. \square

This result shows the fundamental importance of Khalimsky's topology in the theory of T_0 digital spaces; in any such space, a minimal connected subset containing two points x and y has the topological structure of a finite Khalimsky interval. We state this important result formally:

Corollary 2.12. *A subspace of a T_0 digital space is a minimal connected subspace containing points x and y if and only if it is a Khalimsky arc with endpoints x and y .*

Note that this result implies that the Khalimsky topology is the only topology on \mathbb{Z} (up to translation) such that \mathbb{Z} is connected in the intuitive sense (i.e., $\{m, m + 1\}$ is connected for every m) and such that removing one point separates the digital line into two components.

3. THE ARC METRIC

Let X be a T_0 topological digital space and let A be a Khalimsky arc in X . We define the *length of A* , denoted $L(A)$, to be the number of points in A minus one, $L(A) = |A| - 1$. This means that an arc consisting of just one element has length zero. The *arc metric* is defined to be the minimal length of an arc between two points x and y .

Definition 3.1. *Suppose that X is a T_0 digital space. Then we define the **arc metric** ρ on X to be*

$$\rho_X(x, y) = \min\{L(A); A \subset X \text{ is a Khalimsky arc between } x \text{ and } y\}.$$

If there is no danger of ambiguity, we may write just $\rho(x, y)$.

The set of Khalimsky arcs between two points is not empty by Theorem 2.11 and their lengths form a discrete set, so the minimum does exist. Therefore ρ is well defined and it is very easy to check that it satisfies the axioms for a metric.

Remark 3.2. If Y is a subspace of X , the arc metric on Y need not equal the restriction of the arc metric on X . We only have the inequality: $\rho_X(x, y) \leq \rho_Y(x, y)$ for every $x, y \in Y$. For example, consider a Khalimsky circle K with at least 6 points. Take any $b \in K$ and consider its two neighbors x and y . Then $\rho_K(x, y) = 2$ but $\rho_{K \setminus \{b\}}(x, y) = \text{card}(K) - 2 > 2$.

The following proposition will give some idea of the relation between this metric and continuous functions on X .

Proposition 3.3. *Suppose that X is a smallest-neighborhood space and that $f: X \rightarrow \mathbb{Z}$ is continuous. Then f is Lip-1 for the arc metric.*

Proof. Suppose that $x, y \in X$ and let A be an arc of minimal length connecting x and y . By definition, A is a homeomorphic image of a Khalimsky interval; there exist a homeomorphism $\eta: I \rightarrow A \subset X$. Clearly the composition $f \circ \eta: I \rightarrow \mathbb{Z}$ is continuous and hence Lip-1. But the length of I equals $L(A)$ and by combining these facts it follows that $|f(x) - f(y)| \leq L(A) = \rho(x, y)$, i.e., f is Lip-1 for the arc metric. \square

Remark 3.4. On the Khalimsky line we have $\rho(m, n) = |m - n|$. As we know already from the study of continuous functions $\mathbb{Z} \rightarrow \mathbb{Z}$ (Proposition 2.5) the Lip-1 condition is not sufficient for continuity.

Let A be a Khalimsky arc with $L(A) \geq 1$ in a topological space X . As a subspace, we may think of A as a Khalimsky interval, or rather it is—up to homeomorphism. Therefore it makes sense to speak of open and closed points of A and every point in A is either open or closed. Note that a point, open in A , may very well be closed in another Khalimsky arc B . If, however, the point is pure in X , then it is either always open or always closed. The following generalized concept turns out to be important in what follows.

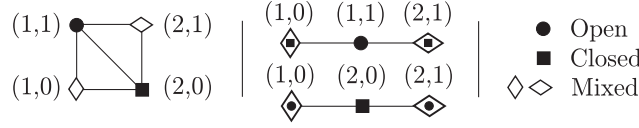


FIGURE 2. The point $(1,0)$ is closed in one minimal arc connecting it to the point $(2,1)$ and open in another.

Definition 3.5. Let X be a T_0 topological digital space and let $x, y \in X$ be distinct points in X . We say that x **is open with respect to** y if there is a Khalimsky arc A between x and y of length $\rho(x, y)$ such that x is open in A , and we say that x **is closed with respect to** y if there is a Khalimsky arc B between x and y of length $\rho(x, y)$ such that x is closed in B .

Clearly a point x is either open or closed w.r.t. y , but can it be both? The answer is in the affirmative; a simple example can be found in the Khalimsky plane \mathbb{Z}^2 . Let $x = (1, 0)$ and $y = (2, 1)$. Then $\rho(x, y) = 2$, and there are two arcs of length 2 between x and y . In the one passing $(1, 1)$, x is closed, in the one passing through $(2, 0)$, x is open. This is illustrated in Figure 2.

4. EXTENSIONS OF CONTINUOUS FUNCTIONS

The following definition gives a condition on a function which is stronger than Lip-1. It is a generalization of Definition 7 in [19].

Definition 4.1. Let X be a T_0 topological digital space, let $A \subset X$ and consider a function $f: A \rightarrow \mathbb{Z}$. Suppose x and y are two distinct points in A . If one of the the following conditions hold

- (1) $|f(x) - f(y)| < \rho(x, y)$ or
- (2) $|f(x) - f(y)| = \rho(x, y)$ and x is open (closed) w.r.t. y implies that $f(x)$ is odd (even),

then we say that the function is **strongly Lip-1 (in X) with respect to (the points) x and y** . If the function is strongly Lip-1 (in X) with respect to every pair of distinct points in A then we simply say that f is **strongly Lip-1 (w.r.t. X)**.

We emphasize that (w.r.t X) means with respect to the (global) arc metric on X and not to the arc metric of A . Confer Remark 3.2.

The relation f is strongly Lip-1 w.r.t. x and y is symmetric in x and y , and the statement can intuitively be thought of as a guarantee that there is enough distance between x in y along the shortest arcs (and thus any arc) connecting these points for the function to change continuously from $f(x)$ and $f(y)$ along these arcs, cf. Proposition 2.5.

Proposition 4.2. Let X be a T_0 topological digital space and $f: X \rightarrow \mathbb{Z}$ a function. Then f is continuous if and only if f is strongly Lip-1 w.r.t. X .

Proof. Assume first that f is continuous. The proof that f then is strongly Lip-1 is practically the same as for Proposition 3.3. Suppose that x is open (closed) w.r.t. y and let A be a minimal arc connecting these points such that x is open (closed) in A . Let η and I be as in the proof of Proposition 3.3. As before, $f \circ \eta$ is continuous. Hence either $|f(x) - f(y)| < \rho(x, y)$ or $|f(x) - f(y)| = \rho(x, y)$ and $\eta^{-1}(x)$ agrees in parity with $f(x)$, which by assumption means that $f(x)$ is odd (even). But then f is strongly Lip-1 w.r.t. x and y and therefore f is strongly Lip-1.

To prove the converse, assume that f is strongly Lip-1. It is sufficient to show that $\mathcal{N}_X(x) \subset f^{-1}(\mathcal{N}_{\mathbb{Z}}(f(x)))$ for every $x \in X$. Suppose therefore that $y \in \mathcal{N}_X(x)$. Note that $\{x, y\}$ is an arc connecting x and y and that x is closed w.r.t. y . We have to consider two cases:

Case 1: $f(x)$ is odd. Since x is closed w.r.t. y , $|f(x) - f(y)| < \rho(x, y) = 1$. It follows that $f(y) = f(x) \in \mathcal{N}_{\mathbb{Z}}(f(x))$.

Case 2: $f(x)$ is even. Then $|f(x) - f(y)| \leq \rho(x, y) = 1$. But $f(x)$ even also implies $\mathcal{N}_{\mathbb{Z}}(x) = \{f(x), f(x) \pm 1\}$, so that $f(y) \in \mathcal{N}_{\mathbb{Z}}(f(x))$. \square

Now we will show that a strongly Lip-1 function defined on a subset of X can be extended to a strongly Lip-1 function defined on all of X . We start with the following lemma.

Lemma 4.3. *Suppose that x and y are two distinct points in a T_0 topological digital space X , $f: \{x, y\} \rightarrow \mathbb{Z}$ a function that is strongly Lip-1 w.r.t. X . Then it is possible, for any point $p \in X$, to extend the function to $F: \{x, y, p\} \rightarrow \mathbb{Z}$ so that F is strongly Lip-1 w.r.t. X .*

Proof. Suppose for definiteness that $f(x) \leq f(y)$. Let $A = (a_1, \dots, a_m)$ be a minimal arc connecting x and p and let $B = (b_1, \dots, b_n)$ be a minimal arc connecting y and p . If $f(x)$ is even and x is open w.r.t. p , we can take A such that x is open in A (and if $f(x)$ is odd, ensure that x is closed in A if x is closed w.r.t. p) and similarly for B and y .

Consider now the set $A \cup B$. It is connected and hence it includes a (minimal) arc C connecting x and y . If p is included in this arc we may argue as follows: By minimality, $C = (a_1, \dots, a_{m-1}, p, b_{n-1}, \dots, b_1)$ and by the strongly Lip-1 assumption, f can be continuously extended along the arc; in particular a value is assigned to p and this value will be our $F(p)$. Since we have chosen A to be the arc yielding the strongest restriction possible for F to be strongly Lip-1 w.r.t. x and p in X , and it succeeds in A by Proposition 4.2, F must be strongly Lip-1 w.r.t. x and p in X . Similarly F is strongly Lip-1 w.r.t. y and p in X , so F is indeed the required extension.

By the triangle inequality, we always have $\rho(x, y) \leq \rho(x, p) + \rho(y, p)$. Suppose that we have equality. This implies, with C defined as above that $C = A \cup B$ and we are done. Next consider the possibility $\rho(x, y) = \rho(x, p) + \rho(y, p) - 1$. Then it may happen that p is not included in C ; we may have $C = (a_1, \dots, a_{m-1}, b_{n-1}, \dots, b_1)$. We can still do a continuous extension along C , say that it takes the value α at a_{m-1} and β at b_{n-1} . If $\alpha = \beta$ it is clear that we can set $F(p) = \alpha$, but what if $\beta = \alpha + 1$? Assume for definiteness that α is

odd and β is even. If we set $F(p) = \alpha$, we will have no problem along A , but we may have along B , namely if p is closed in B . We now show that if it is so, then we can set $F(p) = \beta$, or in other words that if p is closed in B , then p is also closed in A . Assume therefore, on the contrary, that p is open in A . Our assumptions can be summarized to the following: $a_{m-1} \in \mathcal{N}_C(b_{n-1}) \subset \mathcal{N}(b_{n-1})$ (since α is odd and the extension along C is not constant at a_{m-1} it follows that a_{m-1} is open), $p \in \mathcal{N}_A(a_{m-1}) \subset \mathcal{N}(a_{m-1})$ (since p is assumed to be open in A) and $b_{n-1} \in \mathcal{N}_B(p) \subset \mathcal{N}(p)$ (since p is closed in B) and the latter implies $p \notin \mathcal{N}(b_{n-1})$ as we assume the space to be T_0 . The contradiction follows from the following equation

$$p \in \mathcal{N}(a_{m-1}) = \mathcal{N}(a_{m-1}) \cap \mathcal{N}(b_{n-1}) \subset \mathcal{N}(b_{n-1}) \not\ni p$$

Thus the bad configuration cannot occur, and it only remains to consider the case $\rho(x, y) \leq \rho(x, p) + \rho(y, p) - 2$. Now, there is a maximal value M that $F(p)$ can take, if F is to be strongly Lip-1 w.r.t. x and p , and every integer between $f(x)$ and M will also make F strongly Lip-1. The following lower estimation of m is immediate; $M \geq f(x) + \rho(x, p) - 1$. Similarly, there is a minimal value N for $F(p)$, so that F is strongly Lip-1 w.r.t. y and p ; $N \leq f(y) - \rho(y, p) + 1$. It is therefore sufficient to show that $M \geq N$. Assume on the contrary that $M < N$, in other words that $f(x) + \rho(x, p) + 1 < f(y) - \rho(y, p) + 1$. But then

$$\rho(x, y) \leq \rho(x, p) + \rho(y, p) - 2 < f(y) - f(x)$$

This contradicts the fact that f is strongly Lip-1. \square

Proposition 4.4. *Suppose that X is a T_0 topological digital space, that A is a subspace of X and that $f: A \rightarrow \mathbb{Z}$ is strongly Lip-1 w.r.t. X . Then f can be extended to all of X so that the extended function is still strongly Lip-1.*

Proof. If A is the empty set or A is all of X the lemma is trivially true, so we need not consider these cases further. First we show that for any point where f is not defined we can define it so that the new function still is strongly Lip-1.

To this end, let p be any point in $X \setminus A$. For every $x \in A$ it is possible to extend f to f^x defined on $A \cup \{p\}$ so that the new function is strongly Lip-1 w.r.t. x and p —for example by letting $f^x(p) = f(x)$. It is also clear that there is a minimal (say N^x) and a maximal (say M^x) value that $f^x(p)$ can attain if it still is to be strongly Lip-1 w.r.t. x , and that $f^x(p)$ may also attain every integer value in between N^x and M^x . Thus the set of possible values is in fact an interval $[N^x, M^x]_{\mathbb{Z}}$. Now define

$$R = \bigcap_{x \in A} [N^x, M^x]_{\mathbb{Z}}$$

If R is empty, then there is an x and a y such that $M^x < N^y$. This means that it is impossible to extend f at p so that it is strongly Lip-1 with respect to both x and y . But this cannot happen according to Lemma 4.3 and therefore R cannot be empty. Define $\tilde{f}(p)$ to be, say, the smallest integer in R and $\tilde{f}(x) = f(x)$ if $x \in A$. Then $\tilde{f}: A \cup \{p\} \rightarrow \mathbb{Z}$ is still strongly Lip-1.

We will now use Zorn's lemma to show existence of a strongly Lip-1 extension. Let E be the set of all (graphs of) strongly Lip-1 extensions of f . An element in E is thus a subset of $X \times \mathbb{Z}$. A partial order on E is defined by set inclusion. Let c be a chain in E . An upper bound for c is given by $\bigcup c$. Thus E is inductive. By Zorn's lemma, E has a maximal element and it corresponds to a function $F: Y \rightarrow \mathbb{Z}$. It remains to be shown that $Y = X$. Suppose that $z \in X \setminus Y$. Then, by the first part of the proof, there is a strongly Lip-1 extension of $F, Y \cup \{z\} \rightarrow \mathbb{Z}$, which contradicts the maximality of F . We conclude that F must in fact be the required extension of f . \square

We now turn to the main theorem of this section.

Theorem 4.5 (Continuous extensions). *Let A be a subspace of a T_0 topological digital space X and let $f: A \rightarrow \mathbb{Z}$ be any function. Then f can be extended to a continuous function on all of X if and only if f is strongly Lip-1 w.r.t. X .*

Proof. That the function must be strongly Lip-1 follows from Proposition 4.2. For the converse, first use Proposition 4.4 to find a strongly Lip-1 extension to all of X and then Proposition 4.2 again to conclude that this extension is in fact continuous. \square

Remark 4.6. The case with a non- T_0 digital space X can be handled. It is not hard to see that if $f: X \rightarrow \mathbb{Z}$ is continuous and $\mathcal{N}(x) = \mathcal{N}(y)$, then $f(x) = f(y)$. Suppose that $A \subset X$ and that f is a function on A . If f does not satisfy the implication: if $\mathcal{N}(x) = \mathcal{N}(y)$ then $f(x) = f(y)$, then f cannot be continuously extended. Otherwise we can identify the points with the same neighborhood and consider the quotient space (which is T_0) with the induced function \tilde{f} . It follows that f can be continuously extended if and only if \tilde{f} can be extended, i.e., if and only if \tilde{f} is strongly Lip-1.

Remark 4.7. Also the case where X is not connected can be handled; we may consider extension of the function on each connected component separately.

We may include both these remarks in a modified version of Theorem 4.5. For this we need to consider the *extended arc metric* $\tilde{\rho}_X$ defined on any smallest-neighborhood space X as follows.

$$\tilde{\rho}_X(x, y) = \begin{cases} 1/2 & \text{if } x \neq y \text{ and } \mathcal{N}(x) = \mathcal{N}(y) \\ \rho_X(x, y) & \text{otherwise,} \end{cases}$$

where ρ_X is defined by the same equation as before.

$$\rho_X(x, y) = \min\{L(A); A \subset X \text{ is a Khalimsky arc between } x \text{ and } y\}.$$

Note that if X is not connected and x and y belong to different connected components, we get $\tilde{\rho}_X(x, y) = \min \emptyset = +\infty$. Hence we allow here infinite distance between points. If we replace the arc metric ρ with the extended arc metric $\tilde{\rho}$ in definition 4.1, we may speak about strongly Lip-1 functions defined on a smallest-neighborhood space X . Using this extended version of the strongly Lip-1 condition, we have the following version of Theorem 4.5.

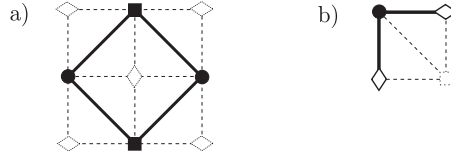


FIGURE 3. a) A completely arc-connected set. b) The set is not completely arc-connected. Note that both sets are connected, however.

Theorem 4.8. *Let A be a subspace of a smallest-neighborhood space X and let $f: A \rightarrow \mathbb{Z}$ be any function. Then f can be extended to a continuous function on all of X if and only if f is strongly Lip-1 w.r.t. X .*

5. COMPLETELY ARC-CONNECTED SETS

The goal of this section is to use Theorem 4.5 to classify the subsets of a topological digital space such that any continuous function defined there can be continuously extended to all of X . This is a digital analogue of the classical Tietze extension theorem.

Definition 5.1. *Let X be a T_0 topological digital space and A a subset of X . Then A is called **completely arc-connected in X** if A is connected, $\rho_A = \rho_X|_A$ and for every pair x, y of distinct points in A , it is true that if x is open (closed) w.r.t. y in X then x is also open (closed) w.r.t. y in A .*

It is easy to see that A is completely arc-connected in X if and only if the following conditions are satisfied for every pair of distinct points x and y : If x is open (closed) w.r.t. y in X , then A contains an arc M connecting x and y , such that x is open (closed) in M and $L(M) = \rho_X(x, y)$.

Example 5.2. The set $\{(0, 0), (1, 1), (0, 2), (-1, 1)\}$ in the Khalimsky plane is completely arc-connected. The set $M = \{(1, 0), (1, 1), (2, 1)\}$ is not, since $(1, 0)$ is both closed and open w.r.t. $(2, 1)$ in \mathbb{Z}^2 , but only closed w.r.t. $(2, 1)$ in M . Adding the point $(2, 0)$ gives the set $M \cup \{(2, 0)\}$, which is completely arc-connected in \mathbb{Z}^2 . The sets are shown in Figure 3.

The following theorem is now very easy to prove.

Theorem 5.3. *Suppose that X is a T_0 topological digital space and that $A \subset X$ is completely arc-connected in X . If $f: A \rightarrow \mathbb{Z}$ is continuous in A , then f can be extended to a continuous function F defined on all of X .*

Proof. By Proposition 4.2, f is strongly Lip-1 w.r.t. A . We have to check that f is also strongly Lip-1 w.r.t. X . Let x and y be distinct points in A . Since $\rho_X(x, y) = \rho_A(x, y)$, the conclusion follows easily if $|f(x) - f(y)| < \rho(x, y)$. Suppose that $|f(x) - f(y)| = \rho(x, y)$. If x is open (closed) w.r.t. y in X , then x is also open (closed) w.r.t. y in A . Therefore $f(x)$ is odd (even) and thus strongly Lip-1 w.r.t. X . \square

Theorem 5.4. *Let X be a T_0 topological digital space. If $A \subset X$ is not completely arc connected then there is a continuous function $f: A \rightarrow \mathbb{Z}$ that cannot be extended to all of X .*

Proof. If A is not connected, the conclusion is immediate. Suppose therefore that A is connected but not completely arc-connected and let x and y be points in A such that some condition for arc-connectedness fails for the pair x and y . We shall define a function $g: \{x, y\} \rightarrow \mathbb{Z}$ and then we will show that g can be extended to a continuous function on A but not to a continuous function on X .

Suppose first that $\rho_A(x, y) > \rho_X(x, y)$. Let $g(x) = 0$ if x is open w.r.t. y in X , and $g(x) = 1$ otherwise. Then define $g(y) = g(x) + \rho_A(x, y) - 1$. We get

$$|g(x) - g(y)| = \rho_A(x, y) - 1 < \rho_A(x, y),$$

so g is strongly Lip-1 w.r.t. A and can therefore be continuously extended to $G: A \rightarrow \mathbb{Z}$ by Theorem 4.5. On the other hand, we have that

$$|G(x) - G(y)| = |g(x) - g(y)| = \rho_A(x, y) - 1 \geq \rho_X(x, y),$$

and by construction $G(x)$ and x do not match in parity. Therefore, again by Theorem 4.5, G cannot be extended to all of X .

The remaining case is $\rho_A(x, y) = \rho_X(x, y)$. Here x must be both open and closed w.r.t. y in X , but only open or only closed w.r.t. y in A . Define the function g as $g(x) = 0$ if x is closed w.r.t. y in A and $g(x) = 1$ otherwise. Let $g(y) = g(x) + \rho(x, y)$. Clearly $|g(x) - g(y)| = \rho_A(x, y)$ but since x and $g(x)$ match in parity, g is strongly Lip-1 w.r.t. A and can therefore be continuously extended to $G: A \rightarrow \mathbb{Z}$. We also have $|G(x) - G(y)| = \rho_X(x, y)$, however, x is both closed and open w.r.t. y in X , so G is not strongly Lip-1 in X . This completes the proof. \square

Example 5.5. The set M of Example 5.2 is not completely arc-connected. Define a function on M by $f(1, 0) = 0$, $f(1, 1) = 1$ and $f(2, 1) = 2$. Then f is continuous on M but f cannot be continuously extended to $M \cup \{(2, 0)\}$: $f(1, 0) = 0$ implies that $f(2, 0) = 0$ and $f(2, 1) = 2$ implies that $f(2, 0) = 2$, which is contradictory.

6. CONTINUOUS FUNCTIONS BETWEEN DIGITAL SPACES

Suppose that X and Y are T_0 topological digital spaces and that $A \subset X$. Let $f: A \rightarrow Y$ be a continuous function. What can we say about the possibilities to extend f to all of X ? Consider any continuous function $g: X \rightarrow Y$ and let x, y be distinct points in X . If B is a minimal arc in X connecting x and y , then $g(B)$ is a connected subset of Y —hence $g(B)$ contains an arc connecting $g(x)$ and $g(y)$. From these observations we get the following chain of inequalities

$$(6.1) \quad \rho_Y(g(x), g(y)) \leq \text{card}(g(B)) - 1 \leq \text{card}(B) - 1 = \rho_X(x, y)$$

If f is to be extended, it must therefore satisfy this inequality for every pair of points where f is defined. We state this result as a proposition:

Proposition 6.1. *Suppose that X and Y are T_0 topological digital spaces and that $g: X \rightarrow Y$ is continuous. Then g is Lip-1 for the arc metric.*

This result is a generalization of Proposition 3.3. Guided by the sharpened version of that result, Proposition 4.2, one might expect that a similar sharpening is possible here too. Indeed, suppose we have equality in (6.1). Then g is a continuous bijection between the Khalimsky arcs B and $g(B)$. But then g^{-1} is automatically continuous, as is easy to check, so B and $g(B)$ are homeomorphic. We conclude that if x is open (closed) w.r.t. y in X , then $f(x)$ is open (closed) w.r.t. $f(y)$ in Y .

Definition 6.2. *Let us call a function $f: X \rightarrow Y$ strongly Lip-1 w.r.t. the arc metric if for every $x, y \in X$, $x \neq y$ either*

- (1) $\rho_Y(f(x), f(y)) < \rho_X(x, y)$ or
- (2) $\rho_Y(f(x), f(y)) = \rho_X(x, y)$ and x is open (closed) w.r.t. y in X implies that $f(x)$ is open (closed) w.r.t. $f(y)$ in Y .

This definition is a generalization of Definition 4.1 since $\rho_{\mathbb{Z}}(m, n) = |m - n|$ for points m and n on the Khalimsky line. We also get the following theorem, corresponding to Proposition 4.2.

Theorem 6.3. *Let X and Y be T_0 topological digital spaces. Then $f: X \rightarrow Y$ is continuous if and only if f is strongly Lip-1 w.r.t. the arc metric.*

Proof. One direction is already clear; it remains to be proved that if f is strongly Lip-1 w.r.t. the arc metric, then f is continuous. Assume that f is strongly Lip-1. It is sufficient to show that $\mathcal{N}_X(x) \subset f^{-1}(\mathcal{N}_Y(f(x)))$ for every $x \in X$. Let $y \in \mathcal{N}_X(x)$. Then $\rho_X(x, y) = 1$. If $\rho_Y(f(x), f(y)) = 0$ then $f(x) = f(y)$ and the conclusion follows. The other possibility is that $\rho_Y(f(x), f(y)) = 1$. Since $y \in \mathcal{N}_X(x)$, y is open w.r.t. x . Hence there is an arc B of length 1 in Y such that $f(y)$ is open in B . But clearly $B = \{f(x), f(y)\}$ and this means that $f(y) \in \mathcal{N}_Y(f(x))$. \square

However, strongly Lip-1 is not sufficient for an extension to exist; there is no analogue of Proposition 4.4. Proposition 6.5 below demonstrates this. We need to introduce some new notation and a result to formulate it.

Let X be any topological space. A *Khalimsky Jordan curve* in X is a homeomorphic image of a Khalimsky circle, c.f. Example 2.8. We will let X be the Khalimsky plane, \mathbb{Z}^2 . The following theorem is proved by Khalimsky et al. in [7]. Kiselman [8] has given a simpler proof.

Theorem 6.4 (Khalimsky Jordan curve theorem). *Let J be any Khalimsky Jordan curve in \mathbb{Z}^2 . Then the complement $\mathbb{Z}^2 \setminus J$ has exactly two connectivity components.*

It is easy to see that one of these components must be finite. This component together with its boundary J will be denoted D_J and called a *Khalimsky disk*. The component itself, i.e., $D_J \setminus J$, will be called the interior of D_J and is denoted by $\text{int}(D_J)$.

Proposition 6.5. *Let J be any Khalimsky Jordan curve in \mathbb{Z}^2 and suppose that $f = (f_1, f_2): D_J \rightarrow D_J$ is a continuous mapping such that $f|_J$ is the identity mapping of J . Then f is the identity mapping of D_J .*

Proof. Let $x = (x_1, x_2)$ be a point in $\text{int}(D_J)$. Let

$$a = \min\{m \in \mathbb{N}; x + (m, 0) \in J\}$$

and

$$b = \min\{m \in \mathbb{N}; x - (m, 0) \in J\}.$$

Clearly $0 < a, b < \infty$. Let $p = x + (a, 0)$ and $q = x - (b, 0)$. By assumption we have $f(p) = p$ and $f(q) = q$, so that

$$|f_1(x) - p_1| \leq \rho(f(x), p) = \rho(f(x), f(p)) \leq \rho(x, p) = |x_1 - p_1| = a$$

Here $f_1(x)$ is the first coordinate of $f(x)$, and the first inequality follows since $|x_i - y_i| \leq \rho(x, y)$ for the i :th coordinate of any two points x, y in a Khalimsky subspace. The second inequality follows from Theorem 6.3.

Similarly we have $|f_1(x) - q_1| \leq b$. But since $|p_1 - q_1| = a + b$, the only possibility is $f_1(x) = x_1$. Using the same argument in the other coordinate direction we get $f_2(x) = x_2$ and hence $f(x) = x$ as required. \square

If X is any topological space and A a subspace of X , a *retraction* of X onto A is a continuous map $\varphi: X \rightarrow A$ such that $\varphi|_A$ is the identity map of A . If such a map exist we say that A is a *retract of X* . The well-known classical no-retraction theorem states that there is no retraction of the unit disk B^2 of \mathbb{R}^2 onto the circle S^1 , or more generally of B^n onto S^{n-1} .

In view of this, Proposition 6.5 in particular states the there is no retraction from D_J onto J . Note also that $f|_J$, being the identity of J , is strongly Lip-1 w.r.t. the arc-metric on J but cannot be extended to D_J .

7. CONCLUSION

We have shown that a condition somewhat stronger than a Lipschitz condition with Lipschitz constant 1 for the arc metric is equivalent to continuity for a function $f: X \rightarrow Y$, where X and Y are T_0 digital spaces. We call such a function strongly Lip-1 w.r.t. X . Moreover, if $Y = \mathbb{Z}$ with the Khalimsky topology, then any function defined on a subset of X such that f is strongly Lip-1 w.r.t. X can be extended to a continuous defined on all of X . Using this result, we have proved a digital analogue of the classical Tietze extension theorem; the subsets of A such that any continuous function $A \rightarrow \mathbb{Z}$ can be extended to all of X are precisely the subsets A where the notion of being strongly Lip-1 w.r.t. A agrees with the notion of being strongly Lip-1 w.r.t. X . We have also proved a digital version of the no-retraction theorem for a disk in the plane. This result shows that if the codomain Y is a more general space than \mathbb{Z} , then the strongly Lip-1 condition need not be sufficient for a continuous extension to exist.

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ERIK MELIN (melin@math.uu.se)

Uppsala University, Department of Mathematics, Box 480, SE-751 06 Uppsala, Sweden