Cancellation of 3-Point Topological Spaces

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ABSTRACT. The cancellation problem, which goes back to S. Ulam [2], is formulated as follows:
Given topological spaces $X, Y, Z$, under what circumstances does $X \times Z \approx Y \times Z$ (≈ meaning homeomorphic to) imply $X \approx Y$?
In [1] it is proved that, for $T_0$ topological spaces and denoting by $S$ the Sierpinski space, if $X \times S \approx Y \times S$ then $X \approx Y$.
This note concerns all nine (up to homeomorphism) 3-point spaces, which are given in [4].

2000 AMS Classification: 54B10
Keywords: Homeomorphism, Cancellation problem, 3-point spaces.

1. TWO CANCELLATION RESULTS

Below $X$ and $Y$ denote $T_1$ topological spaces.

Proposition 1.1. Let $S$ be a topological space with a unique closed singleton $\{p\}$. If there is a homeomorphism $\phi : X \times S \to Y \times S$ then $\phi(X \times \{p\}) = Y \times \{p\}$.

Proof. We shall show that $\phi(X \times \{p\}) \subset Y \times \{p\}$ which, using similar arguments, will be enough to prove that $\phi(X \times \{p\}) = Y \times \{p\}$ and, consequently, that $X \approx Y$.
Let us suppose that for some $x \in X, y \in Y$ and $q \in S \setminus \{p\}$ we have $\phi(x, p) = (y, q)$. Then $\{(y, q)\}$ is closed and, therefore, $(Y \times S) \setminus \{(y, q)\}$ is open.

Let $r$ belong to the topological closure of $\{q\}, r \neq q$. Then $(y, r) \in (Y \times S) \setminus \{(y, q)\}$ and we must have open sets $U_y, U_r$, containing $y$ and $r$, respectively, such that $U_y \times U_r \subset (Y \times S) \setminus \{(y, q)\}$. We reach a contradiction since $(y, q)$ belongs to $U_y \times U_r$. \[\square\]

*The second named author gratefully acknowledges financial support from Fundação para a Ciência e Tecnologia, Lisboa, Portugal.
An example of such an $S$ is obtained as follows. Let $S$ be a set with 4 elements at least. Let $a, b \in S$ and denote by $S_1$ the complement of the subset they form. Take then as basis for a topology on $S$ the set $\{\{a\}, \{a, b\}, S_1\}$. If $S$ happens to have just 4 points then it is the only minimal, universal space with such a number of elements [3].

**Proposition 1.2.** Let $S$ be a topological space with a dense, open singleton $\{p\}$ and such that, for every $q \in S \setminus \{p\}$, the topological closure of $\{q\}$ is finite. If there is a homeomorphism $\phi : X \times S \to Y \times S$ then $\phi(X \times \{p\}) = Y \times \{p\}$.

**Proof.** Let $\{p\}$ be an open, dense singleton in $S$. We will show that $\phi(X \times \{p\}) = Y \times \{p\}$ which, as observed before, is enough to conclude that $X \approx Y$.

Assume that for some $x \in X, y \in Y$ and $q \neq p$ we have $\phi(x, p) = (y, q)$. Consider the closed set $\overline{\{y\}} \times \overline{\{q\}}$, the bar denoting closure, its image $\phi^{-1}(\overline{\{y\}} \times \overline{\{q\}})$, which is also closed, and suppose that $\overline{\{q\}}$ has $s$ elements. Also, observe that $p \notin \overline{\{q\}}$.

Since $(x, p)$ belongs to $\phi^{-1}(\overline{\{y\}} \times \overline{\{q\}})$ and this set has $s$ elements, there is an $r$ in $\overline{\{q\}}$ such that $(x, r)$ does not belong to this set. There are then open sets $U_x, U_r$, containing $x$ and $r$, respectively, with $U_x \times U_r \subset (X \times S) \setminus \phi^{-1}(\overline{\{y\}} \times \overline{\{q\}})$. We have a contradiction since $(x, p) \in U_x \times U_r$. □

An example for $S$ can be the following *Door* space. Let $S$ be a set and fix $p \in S$. Define $U \subset S$ to be open if it is empty or contains $p$.

2. 3-POINT SPACES

We go on assuming that $X, Y$ are $T_1$ topological spaces though such assumption is not used in Propositions 2.1 and 2.2 below.

If we now consider $S = \{a, b, c\}$ to be one of the 3-point spaces [4], we see that Propositions 1.1 and 1.2 of §1 allow us to deduce immediately that $S$ can be cancelled except in the following cases

- $S$ is discrete,
- $S$ has $\{\{a\}, \{b\}, \{a, c\}\}$ as a topological basis,
- $S$ is trivial.

If $S$ is discrete the situation is not as simple as one might be led to think.

Let us take the following example. Let $S = Z$, here $Z$ stands for the integers with the discrete topology, and consider the discrete spaces $X = \{0, 1, \ldots, n - 1\}, n \geq 2, Y = \{0\}$. Now define $\phi : \{0, 1, \ldots, n - 1\} \times Z \to \{0\} \times Z$ by $\phi(x, r) = (0, nr + x)$. This map is a homeomorphism and however $Z$ cannot be cancelled.

We can say something when the spaces $X, Y$ have a finite number of connected components.

**Proposition 2.1.** Let $S$ be a finite discrete space and assume that $X$ has a finite number of connected components. If $X \times S \approx Y \times S$ then $X \approx Y$. 


Proof. The connected components of $X \times S$ or $Y \times S$ are of the type $X' \times \{x\}$, $Y' \times \{y\}$, where $X', Y'$ are components of $X$ and $Y$, respectively. It follows that $Y$ has the same number of components as $X$.

Let us consider in the sets of connected components of $X$ and connected components of $Y$ the homeomorphism equivalence relation and take an equivalence class of components of $X$, say $\{X_1, \ldots, X_k\}$. The subspace $\bigcup_{i=1}^k X_i \times S$ has $kn$ components, where $n$ is the cardinal of $S$. The same happens with $\phi(\bigcup_{i=1}^k X_i \times S)$, where $\phi$ is a homeomorphism between $X \times S$ and $Y \times S$.

Let $p \in S$. For every $i = 1, \ldots, k$, $\phi(X_i \times \{p\}) = Y_i \times \{q_i\}$, where the $q_i$'s belong to $S$ and the $Y_i$'s are components of $Y$ homeomorphic to the $X_i$'s.

Assume that the equivalence class to which the $Y_i$'s belong is $\{Y_1, \ldots, Y_l\}$. Then $\phi(\bigcup_{i=1}^k X_i \times \{p\}) \subset \bigcup_{j=1}^l Y_j \times S$. Consequently, also $\phi(\bigcup_{i=1}^k X_i \times S) \subset \bigcup_{j=1}^l Y_j \times S$.

Using the inverse homeomorphism $\phi^{-1}$, we are led to conclude that the reverse inclusion holds and, therefore, $\phi(\bigcup_{i=1}^k X_i \times S) = \bigcup_{j=1}^l Y_j \times S$. So $\bigcup_{i=1}^k X_i \times S$ and $\bigcup_{j=1}^l Y_j \times S$ have the same number of components and it follows that $k = l$.

From each component class in $X$ choose a representative and use $\phi$ to establish a homeomorphism between that representative and a component in $Y$. These homeomorphisms can then be used to conclude that every component of $X$ is homeomorphic to a component of $Y$. Since components are closed and finite in number, $X$ is homeomorphic to $Y$. $\square$

Proposition 2.2. Let $X$ and $Y$ be topological spaces with the same finite number of connected components and $S$ be a discrete space. Assume, moreover, that neither space has two homeomorphic components. If $X \times S \approx Y \times S$ then $X \approx Y$.

Proof. Let $X_i, i = 1, \ldots, n$, be the components of $X$ and fix $p \in S$.

If $\phi$ is a homeomorphism between $X \times S$ and $Y \times S$ then there are $q_i \in S, i = 1, \ldots, n$, such that $\phi(X_i \times \{p\}) = Y_i \times \{q_i\}, i = 1, \ldots, n$, where, due to our assumption on the non-existence of homeomorphic components, the $Y_i$'s are the components of $Y$. Hence $\phi$ induces a homeomorphism $\phi_i : X_i \rightarrow Y_i, i = 1, \ldots, n$.

Again, since the number of components is finite and they are closed, the $\phi_i$'s can be used to obtain a homeomorphism between $X$ and $Y$. $\square$

Proposition 2.3. Let $S$ have $\{\{a\}, \{b\}, \{a, c\}\}$ as basis. If $\phi : X \times S \rightarrow Y \times S$ is a homeomorphism then $\phi(X \times \{b\}) = Y \times \{b\}$.
Proof. Let \( \pi_S : Y \times S \to S \) denote the standard projection. The image \( \pi_S(\phi(X \times \{b\})) \) is open and, therefore, it is either \( \{b\} \) or contains a.

Assume that for some \( x \in X, y \in Y \) we have \( \phi(x, b) = (y, a) \). The subset \( \{(x, b)\} \) is closed and, consequently, the same happens with \( \{(y, a)\} \). Hence \( (Y \times S) \setminus \{(y, a)\} \) is open and contains \( (y, c) \). We must then have an open neighbourhood \( U_y \) of \( y \) such that \( U_y \times \{a, c\} \subset (Y \times S) \setminus \{(y, a)\} \). Again we have a contradiction and \( \phi(X \times \{b\}) = Y \times \{b\} \).

To conclude the proof that a non-discrete 3-point space can be cancelled it only remains to deal with the case where \( S \) is trivial.

Above we have an example of a homeomorphism \( \phi: X \times S \to Y \times S \) which does take a slice \( X \times \{x\} \) onto a slice \( Y \times \{y\} \). More examples can be obtained.

Take \( X = Y \), with at least 2 elements, a trivial space \( S \) with also, at least, 2 elements and let \( \psi: S \to S \) be a fixed point free bijection. Fix \( x_0 \in X \) and define \( \phi: X \times S \to X \times S \) by \( \phi(x, s) = (x, s) \), for \( x \neq x_0 \), and \( \phi(x_0, s) = (x_0, \psi(s)) \).

Then \( \phi \) is a bijection and \( \phi(\{x\} \times S) = \{x\} \times S \), for \( x \in X \). Since open sets in \( X \times S \) are of the form \( U \times S, U \) open in \( X \), and \( \phi(U \times S) = U \times S \), \( \phi \) is a homeomorphism. Obviously no slice \( X \times \{x\} \) is mapped onto a similar slice.

**Proposition 2.4.** Let \( S \) be a finite trivial space. If \( X \times S \approx Y \times S \) then \( X \approx Y \).

**Proof.** Open (closed) sets in \( X \times S \) and \( Y \times S \) are of the form \( U \times S \), where \( U \) is open (closed).

We are going to define \( f: X \to Y \) as follows. Let \( x \in X \). Then \( \{x\} \) is closed and so are \( \{x\} \times S \) and \( \phi(\{x\} \times S) \), where \( \phi: X \times S \to Y \times S \) is a homeomorphism. Hence \( \phi(\{x\} \times S) = C \times S \), for some closed set \( C \) in \( Y \). Since \( S \) is finite, \( C \) is a singleton and we make \( \{f(x)\} = C \).

This way we obtain an \( f \) which is a bijection since we began with a bijective \( \phi \).

If \( C \) is closed in \( X \), \( \phi(C \times S) = f(C) \times S \) is closed in \( Y \times S \). Consequently \( f(C) \) is closed in \( Y \). Therefore \( f \) is closed and \( f^{-1} \) is continuous.

Taking \( f^{-1} \), we would conclude that \( f \) is continuous the same way. \( \square \)

We can now state.

**Theorem 2.5.** For \( X \) and \( Y T_1 \) topological spaces and \( S \) a non-discrete 3-point topological space, if \( X \times S \approx Y \times S \) then \( X \approx Y \).

### 3. A PARTICULAR CASE

We will no longer assume \( X, Y \) to be \( T_1 \) and will suppose that \( S \) has a unique isolated point \( a \). Moreover, the singleton \( \{a\} \) will be assumed to be closed. That is, for instance, the case where \( S = \{a, b, c\} \) and \( \{(a), \{b, c\}\} \) is an open basis.
Proposition 3.1. Let $S$ have a unique isolated point $a$. Assume that $\{a\}$ is closed. For $X, Y$ connected with, at least, an isolated point each, if $\phi : X \times S \to Y \times S$ is a homeomorphism then $\phi(X \times \{a\}) = Y \times \{a\}$.

Proof. Let $\pi_S : Y \times S \to S$ denote the standard projection, as before.

The image $\pi_S(\phi(X \times \{a\}))$ is open and connected. Therefore it is either $\{a\}$ or some open, connected subset of $S$, which naturally does not contain $a$.

Let the latter be the case. If $x \in X$ is an isolated point then $\{(x, a)\}$ is open and the same happens to its image under $\pi_S \circ \phi$. This is impossible because $\{a\}$ is the unique open singleton of $S$. \qed

Examples of spaces satisfying the conditions of Proposition 3.1 are, again, some Door spaces.

Let $Z$ be a set. Fix $p \in Z$ and define $U \subset Z$ to be open if $U = Z$ or $p \notin U$.

References


Received July 2006
Accepted November 2006

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