Applied General Topology
© Universidad Politécnica de Valencia
Volume 9, No. 2, 2008
pp. 189-195

Asymptotic proximities

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Abstract. A ballean is a set endowed with some family of subsets which are called the balls. The properties of the family of balls are postulated in such a way that the balleans can be considered as a natural asymptotic counterparts of the uniform topological spaces. We introduce and study an asymptotic proximity as a counterpart of proximity relation for uniform topological space.

2000 AMS Classification: 54E05, 54E15.

Keywords: ballean, determining covering, proximity.

1. Introduction and preliminaries

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$ where X, P are non-empty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii*. Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B^*(x, \alpha') \subseteq B(x, \alpha), B(x, \beta') \subseteq B^*(x, \beta);$$

- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), B^*(x, \beta) \subseteq B(x, \beta');$$

- *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\gamma),\gamma) \subseteq B(x,\alpha) \cap B(x,\beta);$$

- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family

$$\Big\{ \bigcup_{x \in X} B(x,\alpha) \times B(x,\alpha) : \alpha \in P \Big\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x,y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

We say that a ball structure is a *ballean* if $\mathcal{B}$ is upper symmetric and upper multiplicative. A structure on X, equivalent to a ballean, can also be defined in terminology of entourages. In this case it is called a coarse structure [5]. For motivations to study balleans see [1],[4],[5].

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta).$$

A bijection $f : X_1 \longrightarrow X_2$ is called an *asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ if f and f^{-1} are $\prec$-mappings.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be balleans with common support X. We say that $\mathcal{B}_1 \prec \mathcal{B}_2$ if the identity mapping id:$X \to X$ is a $\prec$-mapping of $\mathcal{B}_1$ to $\mathcal{B}_2$. If $\mathcal{B}_1 \prec \mathcal{B}_2$ and $\mathcal{B}_2 \prec \mathcal{B}_1$, we say that $\mathcal{B}_1$ and $\mathcal{B}_2$ coincide and write $\mathcal{B}_1 = \mathcal{B}_2$.

Let $\mathcal{B} = (X, P, B)$ be a ballean. A subset $Y \subseteq X$ is called *bounded* if there exist $\alpha \in P$ such that $Y \subseteq B(x,\alpha)$ for some $x \in Y$. A family $\mathcal{F}$ of subsets of X is called *uniformly bounded* if there exists $\alpha \in P$ such that $F \subseteq B(x,\alpha)$ for all $F \in \mathcal{F}, x \in F$. We use the following observation: the ballean $\mathcal{B}_1$ and $\mathcal{B}_2$ with common support coincide if and only if every family of subsets of X uniformly bounded in $\mathcal{B}_1$ is uniformly bounded in $\mathcal{B}_2$ and vise versa.

For an arbitrary ballean $\mathcal{B} = (X, P, B)$ we define preordering $\leqslant$ on the set P by the rule: $\alpha \leqslant \beta$ if and only if $B(x,\alpha) \subseteq B(x,\beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that $\alpha \leqslant \alpha'$.

A ballean $\mathcal{B}$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x,\alpha)$. A connected ballean $\mathcal{B}$ is called *ordinal* if there exists a well-ordered by $\leqslant$ subset P' of P.

Every metric space (X, d) determines the *metric ballean* $(X, \mathbb{R}^+, B_d)$ where $B_d(x, r) = \{y \in X : d(x, y) \leq r\}$. A ballean is called *metrizable* if it is asymorphic to some metric ballean. By [4, Theorem 9.1], a ballean $\mathcal{B} = (X, P, B)$ is metrizable if and only if $\mathcal{B}$ is connected and P has a countable cofinal subset. Clearly, every metrizable ballean is ordinal.

We begin the proper exposition with characterization (section 2) of families of coverings of a set X which determine a ballean on X. Then we introduce and study (section 3) an asymptotic proximity as an equivalence relation σ on the family $\mathcal{P}(X)$ of all subsets of a set X such that $Y \subseteq Z \subseteq Y'$ and $Y \sigma Y'$ imply $Y \sigma Z$. Every proximity σ determines some ballean $\mathcal{B}(\sigma)$ on X. Given a ballean $\mathcal{B} = (X, P, B)$, we say that the subsets Y, Z of X are close if there exists $\alpha \in P$ such that $Y \subseteq B(Z, \alpha)$, $Z \subseteq B(Y, \alpha)$. The closeness relation is a prototype for the asymptotic proximity. We show (Theorem 3.1) that, given an asymptotic proximity σ on $\mathcal{P}(X)$, the closeness σ' defined by $\mathcal{B}(\sigma)$ is finner then σ. On the other hand (Theorem 3.4), if $\mathcal{B} = (X, P, B)$ is a ballean and σ is a closeness on $\mathcal{P}(X)$ determined by $\mathcal{B}$, then $\sigma = \sigma'$ where σ' is closeness determined by $\mathcal{B}(\sigma)$. In Section 4 we examine the question whether the closeness on $\mathcal{P}(X)$ arising from a ballean $\mathcal{B} = (X, P, B)$ determines $\mathcal{B}$. In general case this is not so, but our main result (Theorem 4.2) gives a positive answer in the case of ordinal (in particular, metrizable) balleans.

2. Determining coverings

Let X be a set, $\mathcal{F}$ be a family of subsets of X, $Y \subseteq X$. We put

$$st(Y, \mathcal{F}) = \bigcup \{F \in \mathcal{F} : Y \bigcap F \neq \varnothing\}.$$

Given any $x \in X$, we write $st(x, \mathcal{F})$ instead of $st(\{x\}, \mathcal{F})$.

For two families $\mathcal{F}, \mathcal{F}'$ of subsets of X, we put

$$st(\mathcal{F}, \mathcal{F}') = \{st(F, \mathcal{F}') : F \in \mathcal{F}\}.$$

A family $\mathcal{F}$ of subsets of $X$is called *hereditary* if, for any subsets F, F' of X such that $F \in \mathcal{F}$ and $F' \subseteq F$, we have $F' \subseteq \mathcal{F}$.

A family $\mathcal{F}$ of subsets of X is called a *covering* if $\bigcup \mathcal{F} = X$.

We say that a family $\{\mathcal{F}_\alpha, \alpha \in P\}$ of hereditary coverings of X is *star stable* if, for any $\alpha, \beta \in P$, there exist $\gamma \in P$ such that

$$st(\mathcal{F}_\alpha, \mathcal{F}_\beta) \subseteq \mathcal{F}_\gamma.$$

Let $\{\mathcal{F}_\alpha : \alpha \in P\}$ be a family of star stable coverings of X. We consider a ball structure $\mathcal{B} = (X, P, B)$, where

$$B(x, \alpha) = st(x, \mathcal{F}_\alpha),$$

and show that $\mathcal{B}$ is a ballean.

Given any $x \in X$ and $\alpha \in P$, we have

$$B(x, \alpha) = \{y \in X : y \in st(x, \mathcal{F}_\alpha)\}, \quad B^*(x, \alpha) = \{y \in X : x \in st(y, \mathcal{F}_\alpha)\}.$$

Since $y \in st(x, \mathcal{F}_\alpha)$ if and only if $x \in st(y, \mathcal{F}_\alpha)$, then $B^*(x, \alpha) = B(x, \alpha)$, so $\mathcal{B}$ is upper symmetric.

Given any $x \in X$ and $\alpha, \beta \in P$, we choose $\alpha' \in P$ and $\gamma \in P$ such that

$$st(\mathcal{F}_\alpha, \mathcal{F}_\alpha) \subseteq \mathcal{F}_{\alpha'} \ \ and \ \ st(\mathcal{F}_{\alpha'}, \mathcal{F}_\beta) \subseteq \mathcal{F}_\gamma.$$

Then we have

$$B(B(x,\alpha),\beta) = st(st(x,\mathcal{F}_\alpha),\mathcal{F}_\beta) \subseteq st(x,\mathcal{F}_\gamma) = B(x,\gamma),$$

so $\mathcal{B}$ is upper multiplicative.

We note that a subset Y of X is bounded in $\mathcal{B}$ if and only if $Y \in \mathcal{F}_\alpha$ for some $\alpha \in P$. A family $\mathcal{F}$ of subsets of X is bounded in $\mathcal{B}$ if and only if there exists $\alpha \in P$ such that $\mathcal{F} \subseteq \mathcal{F}_\alpha$.

Thus we have shown that every star stable family of coverings of X determines some ballean on X. On the other hand, let $\mathcal{B} = (X, P, B)$ be an arbitrary ballean on X. For every $\alpha \in P$, we put

$$\mathcal{F}_\alpha = \{F \subseteq X : F \subseteq B(x,\alpha) \ for \ some \ x \in X\}.$$

Then the ballean on X determined by the star stable family $\{\mathcal{F}_\alpha : \alpha \in P\}$ of coverings of X coincides with $\mathcal{B}$.

3. Proximities and closeness

Let X be a set, $\mathcal{P}(X)$ be a family of all subsets of X. Let σ be an equivalence on $\mathcal{P}(X)$ such that, for all $Y, Y', Z \in \mathcal{P}(X)$,

$$Y \subseteq Z \subseteq Y', \ \ Y\sigma Y' \Longrightarrow Y\sigma Z.$$

We say that σ is (*an asymptotic*) *proximity* and describe a way in which σ defines some ballean $\mathcal{B}(\sigma)$ on X.

We call a family $\mathcal{F}$ of subsets of X to be *non-expanding* with respect to σ if, for every subset Y of X, we have

$$Y\sigma(Y \bigcup st(Y,\mathcal{F})).$$

We note that every subfamily of non-expanding family is non-expanding.

Let $\mathcal{F}_1, \mathcal{F}_2$ be non-expanding with respect to σ families of subsets of X. We show that the family $st(\mathcal{F}_1, \mathcal{F}_2)$ is also non-expanding with respect to σ.

We fix an arbitrary subset Y of X and put

$$\mathcal{F}_2' = \{F' \in \mathcal{F}_2 : Y \bigcap F' \neq \varnothing\}.$$

Since $\mathcal{F}_2'$ is non-expanding, we have

$$Y\sigma(Y \bigcup \bigcup \mathcal{F}_2').$$

We put $Z = Y \bigcup \bigcup \mathcal{F}_2'$ and

$$\mathcal{F}_1' = \{F \in \mathcal{F}_1 : F \bigcap F' \neq \varnothing \ for \ some \ F' \in \mathcal{F}_2'\}.$$

Since $\mathcal{F}_1'$ is non-expanding, we have

$$Z\sigma(Z \bigcup \bigcup \mathcal{F}_1').$$

We put $T = Z \bigcup \bigcup \mathcal{F}'_1$. Since $\mathcal{F}_2$ is non-expanding, we have

$$T\sigma(T\bigcup(\bigcup\{F \in \mathcal{F}_2 : F\bigcap T \neq \varnothing\})).$$

We put $H = T\bigcup(\bigcup\{F \in \mathcal{F}_2 : F\bigcap T \neq \varnothing\})$. Then $Y\sigma H$ and $Y \subseteq H$. By the construction of H, we have

$$Y \subseteq Y\bigcup(\bigcup\{S \in st(\mathcal{F}_1, \mathcal{F}_2) : S\bigcap Y \neq \varnothing\}) \subseteq H.$$

Since σ is a proximity, we conclude

$$Y\sigma(Y\bigcup(\bigcup\{S \in st(\mathcal{F}_1, \mathcal{F}_2) : S\bigcap Y \neq \varnothing\})).$$

In particular, we proved that the family of all non-expanding (with respect to σ) hereditary covering of X is star stable. Following Section 2, we define $\mathcal{B}(\sigma)$ by means this family of coverings.

We note that a subset Y of X is bounded in $\mathcal{B}(\sigma)$ if and only if the family $\{Y\}$ is non-expanding, equivalently, $\{y\}\sigma Y$ for every $y \in Y$. A family $\mathcal{F}$ of subsets of X is uniformly bounded in $\mathcal{B}(\sigma)$ if and only if $\mathcal{F}$ is non-expanding.

Let $\mathcal{B} = (X, P, B)$ be a ballean. We consider a relation σ on $\mathcal{P}(X)$ defined by the rule: $Y\sigma Z$ if and only if there exists $\alpha \in P$ such that $Y \subseteq B(Z, \alpha)$, $Z \subseteq B(Y, \alpha)$. It is easy to see that σ is a proximity; we call it a *closeness* defined by $\mathcal{B}$. We note that Y, Z are close if and only if there exists a uniformly bounded covering $\mathcal{F}$ of X such that

$$\bigcup\{F \in \mathcal{F} : F\bigcap Y \neq \varnothing\} = \bigcup\{F \in \mathcal{F} : F\bigcap Z \neq \varnothing\}.$$

Theorem 3.1. *Let X be a set, σ be a proximity on $\mathcal{P}(X)$, σ' be a closeness defined by $\mathcal{B}(\sigma)$. Then $\sigma' \subseteq \sigma$.*

Proof. We remind that a family $\mathcal{F}$ of subsets of X is uniformly bounded in $\mathcal{B}(\sigma)$ if and only if $\mathcal{F}$ is non-expanding with respect to σ. Let $Y, Z \in \mathcal{P}(X)$ and $Y\sigma' Z$. Then there exists a non-expanding (with respect to σ) family $\mathcal{F}$ of subsets of X such that $Y \subseteq \bigcup\mathcal{F}$, $Z \subseteq \bigcup\mathcal{F}$ and $Y\bigcap F \neq \varnothing$, $Z\bigcap F \neq \varnothing$ for every $F \in \mathcal{F}$. It follows that $Y\sigma(\bigcup\mathcal{F})$ and $Z\sigma(\bigcup\mathcal{F})$, so $Y\sigma Z$. □ □

The following two examples show that the proximity σ from Theorem 3.1 could be much more coarse than σ'.

Example 3.2. Let X be an infinite set. We define an equivalence σ on $\mathcal{P}(X)$ by the rule: $Y\sigma Z$ if and only if either Y, Z are finite, or Y, Z are infinite. Then a subset Y of X is bounded in $\mathcal{B}(\sigma)$ if and only if Y is finite; a family $\mathcal{F}$ of subsets of X is uniformly bounded in $\mathcal{B}(\sigma)$ if and only if each subset $F \in \mathcal{F}$ is finite and, for every $x \in X$, the set $\{F \in \mathcal{F} : x \in F\}$ is finite. We show that $Y\sigma' Z$ if and only if either Y, Z are finite, or Y, Z are infinite and $|Y| = |Z|$. We should only check that if Y, Z are infinite and $|Y| = |Z|$ then $Y\sigma' Z$. To this end we fix some bijection $f : Y \longrightarrow Z$, and put $\mathcal{F} = \{\{y, f(y)\} : y \in Y\}$. Then $\mathcal{F}$ is uniformly bounded in $\mathcal{B}(\sigma)$, $Y\sigma'(\bigcup\mathcal{F})$ and $Z\sigma'(\bigcup\mathcal{F})$, so $Y\sigma' Z$. Now if X is uncountable than σ is coarser than σ'. □

Example 3.3. Let X be a well-ordered set. We define an equivalence σ on $\mathcal{P}(X)$ by the rule: $Y\sigma Z$ if and only if $\min Y = \min Z$. Then a subset Y is bounded in $\mathcal{B}(\sigma)$ if and only if Y is a singleton. It follows that $Y\sigma' Z$ if and only if $Y = Z$. □

Theorem 3.4. *Let $\mathcal{B} = (X, P, B)$ be a ballean, σ be a closeness defined by $\mathcal{B}$, σ' be a closeness defined by $\mathcal{B}(\sigma)$. Then $\sigma = \sigma'$.*

Proof. By Theorem 3.1, $\sigma \subseteq \sigma'$. To see that $\sigma \subseteq \sigma'$ it suffices to note that every uniformly bounded in $\mathcal{B}$ family of subsets of X is non-expanding with respect to σ. □

4. Does closeness determine a ballean?

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be balleans with common support X, σ_1 and σ_2 be closeness on $\mathcal{P}(X)$ defined by $\mathcal{B}_1$ and $\mathcal{B}_2$. Is $\mathcal{B}_1 = \mathcal{B}_2$ provided that $\sigma_1 = \sigma_2$?

We give a negative answer to this general question, but prove one partial statement (Theorem 4.2) in positive direction.

Example 4.1. Let X be a countable set. We consider two families φ_1, φ_2 of coverings of X.

A family φ_1 is defined by the rule: $\mathcal{F} \in \varphi_1$ if and only if every subset $F \in \mathcal{F}$ is finite, and the set $\{F \in \mathcal{F} : x \in F\}$ is finite for every $x \in X$.

A family φ_2 is defined by the rule: $\mathcal{F} \in \varphi_2$ if and only if there exists a natural number n such that $|F| \le n$ for every $F \in \mathcal{F}$, and there exists a natural number m such that $|\{F \in \mathcal{F} : x \in F\}| \le m$ for every $x \in X$.

Clearly, the families φ_1 and φ_2 are star-stable. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be balleans on X determined by φ_1 and φ_2. Using arguments from Example 3.2, it is easy to see that $\mathcal{B}_1$ and $\mathcal{B}_2$ define the same closeness σ: $Y\sigma Z$ if and only if either Y, Z are finite, or Y, Z are infinite. Then we take a partition $\{F_n : n \in \omega\}$ of X such that $|F_n| = n$ for every $n \in \omega$. Clearly, $\mathcal{F}$ is uniformly bounded in $\mathcal{B}_1$, but $\mathcal{F}$ is not uniformly bounded in $\mathcal{B}_2$. It follows that $\mathcal{B}_1$ is stronger than $\mathcal{B}_2$.

It is worth to mark that Example 4.1 gives a ballean $\mathcal{B}$ with the closeness σ such that $\mathcal{B} \ne \mathcal{B}(\sigma)$. To see this, we put $\mathcal{B} = \mathcal{B}_2$ and note that $\mathcal{B}(\sigma) = \mathcal{B}_1$.

Theorem 4.2. *Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be ordinal balleans with common support and the same closeness. Then $\mathcal{B}_1 = \mathcal{B}_2$.*

Proof. We assume on the contrary that, say, $\mathcal{B}_2 \prec \mathcal{B}_1$ does not hold, and choose $\beta \in P_2$ such that, for every $\alpha \in P_1$, there exists $x(\alpha) \in X$ such that $B_2(x(\alpha), \beta) \nsubseteq B_1(x(\alpha), \alpha)$. We may suppose that P_1 is well-ordered. In the proof of Theorem 2.1 from [3] we constructed inductively a subset

$$Y = \{y(\alpha) : \alpha \in P_1\}$$

of X such that the family $\{B_1(y(\alpha), \alpha) : \alpha \in P_1\}$ is disjoint and, for every $\alpha' \in P$,

$$B_2(y(\alpha'), \beta) \nsubseteq \bigcup \{B_1(y(\alpha), \alpha) : \alpha \in P_1\}.$$

We put $Z = B_2(Y, \beta)$. Then Y, Z are close in $\mathcal{B}_2$, but Y, Z are not close in $\mathcal{B}_1$, whence a contradiction. □

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Received March 2007

Accepted December 2007

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