Asymptotic proximities

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Abstract. A ballean is a set endowed with some family of subsets which are called the balls. The properties of the family of balls are postulated in such a way that the balleans can be considered as a natural asymptotic counterparts of the uniform topological spaces. We introduce and study an asymptotic proximity as a counterpart of proximity relation for uniform topological space.

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1. Introduction and preliminaries

A ball structure is a triple $B = (X, P, B)$ where $X$, $P$ are non-empty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of $X$ which is called a ball of radius $\alpha$ around $x$. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set $X$ is called the support of $B$, $P$ is called the set of radii. Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure is called

- lower symmetric if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,
  $$B^*(x, \alpha') \subseteq B(x, \alpha), B(x, \beta') \subseteq B^*(x, \beta);$$

- upper symmetric if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,
  $$B(x, \alpha) \subseteq B^*(x, \alpha'), B^*(x, \beta) \subseteq B(x, \beta');$$
• **lower multiplicative** if, for any \( \alpha, \beta \in P \), there exists \( \gamma \in P \) such that, for every \( x \in X \),
  \[
  B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);
  \]
• **upper multiplicative** if, for any \( \alpha, \beta \in P \), there exists \( \gamma \in P \) such that, for every \( x \in X \),
  \[
  B(B(x, \alpha), \beta) \subseteq B(x, \gamma).
  \]

Let \( B = (X, P, B) \) be a lower symmetric and lower multiplicative ball structure. Then the family
  \[
  \left\{ \bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \right\}
  \]
is a base of entourages for some (uniquely determined) uniformity on \( X \). On the other hand, if \( \mathcal{U} \subseteq X \times X \) is a uniformity on \( X \), then the ball structure \( (X, \mathcal{U}, B) \) is lower symmetric and lower multiplicative, where \( B(x, U) = \{ y \in X : (x, y) \in U \} \). Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

We say that a ball structure is a **ballean** if \( B \) is upper symmetric and upper multiplicative. A structure on \( X \), equivalent to a ballean, can also be defined in terminology of entourages. In this case it is called a coarse structure [5]. For motivations to study ball structures see [1],[4],[5].

Let \( B_1 = (X_1, P_1, B_1) \) and \( B_2 = (X_2, P_2, B_2) \) be ball structures. A mapping \( f : X_1 \to X_2 \) is called a \( \prec \)-**mapping** if, for every \( \alpha \in P_1 \), there exists \( \beta \in P_2 \) such that, for every \( x \in X_1 \),
  \[
  f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).
  \]

A bijection \( f : X_1 \to X_2 \) is called an **asymorphism** between \( B_1 \) and \( B_2 \) if \( f \) and \( f^{-1} \) are \( \prec \)-mappings.

Let \( B_1, B_2 \) be ball structures with common support \( X \). We say that \( B_1 \prec B_2 \) if the identity mapping \( id : X \to X \) is a \( \prec \)-mapping of \( B_1 \) to \( B_2 \). If \( B_1 \prec B_2 \) and \( B_2 \prec B_1 \), we say that \( B_1 \) and \( B_2 \) coincide and write \( B_1 = B_2 \).

Let \( B = (X, P, B) \) be a ballean. A subset \( Y \subseteq X \) is called **bounded** if there exist \( \alpha \in P \) such that \( Y \subseteq B(x, \alpha) \) for some \( x \in Y \). A family \( \mathcal{F} \) of subsets of \( X \) is called **uniformly bounded** if there exists \( \alpha \in P \) such that \( F \subseteq B(x, \alpha) \) for all \( F \in \mathcal{F}, x \in F \). We use the following observation: the ballean \( B_1 \) and \( B_2 \) with common support coincide if and only if every family of subsets of \( X \) uniformly bounded in \( B_1 \) is uniformly bounded in \( B_2 \) and vise versa.

For an arbitrary ballean \( B = (X, P, B) \) we define preorder by \( \preceq \) on the set \( P \) by the rule: \( \alpha \preceq \beta \) if and only if \( B(x, \alpha) \subseteq B(x, \beta) \) for every \( x \in X \). A subset \( P' \subseteq P \) is called **cofinal** if, for every \( \alpha \in P \), there exists \( \alpha' \in P' \) such that \( \alpha \preceq \alpha' \).

A ballean \( B \) is called **connected** if, for any \( x, y \in X \), there exists \( \alpha \in P \) such that \( y \in B(x, \alpha) \). A connected ballean \( B \) is called **ordinal** if there exists a well-ordered by \( \preceq \) subset \( P' \) of \( P \).
Every metric space \((X, d)\) determines the \textit{metric ballean} \((X, \mathbb{R}^+, B_d)\) where \(B_d(x, r) = \{y \in X : d(x, y) \leq r\}\). A ballean is called \textit{metrizable} if it is isomorphic to some metric ballean. By [4, Theorem 9.1], a ballean \(B = (X, P, B)\) is metrizable if and only if \(B\) is connected and \(P\) has a countable cofinal subset. Clearly, every metrizable ballean is ordinal.

We begin the proper exposition with characterization (section 2) of families of coverings of a set \(X\) which determine a ballean on \(X\). Then we introduce and study (section 3) an asymptotic proximity as an equivalence relation \(\sigma\) on the family \(\mathcal{P}(X)\) of all subsets of a set \(X\) such that \(Y \subseteq Z \subseteq Y'\) and \(Y \sigma Y'\) imply \(Y \sigma Z\). Every proximity \(\sigma\) determines some ballean \(\mathcal{B}(\sigma)\) on \(X\). Given a ballean \(B = (X, P, B)\), we say that the subsets \(Y, Z\) of \(X\) are close if there exists \(\alpha \in P\) such that \(Y \subseteq B(Z, \alpha)\), \(Z \subseteq B(Y, \alpha)\). The closeness relation is a prototype for the asymptotic proximity. We show (Theorem 3.1) that, given an asymptotic proximity \(\sigma\) on \(\mathcal{P}(X)\), the closeness \(\sigma'\) defined by \(B(\sigma)\) is finer than \(\sigma\). On the other hand (Theorem 3.4), if \(B = (X, P, B)\) is a ballean and \(\sigma\) is a closeness on \(\mathcal{P}(X)\) determined by \(B\), then \(\sigma = \sigma'\) where \(\sigma'\) is closeness determined by \(B(\sigma)\). In Section 4 we examine the question whether the closeness on \(\mathcal{P}(X)\) arising from a ballean \(B = (X, P, B)\) determines \(B\). In general case this is not so, but our main result (Theorem 4.2) gives a positive answer in the case of ordinal (in particular, metrizable) balleans.

2. Determining coverings

Let \(X\) be a set, \(\mathcal{F}\) be a family of subsets of \(X, Y \subseteq X\). We put
\[
\text{st}(Y, \mathcal{F}) = \bigcup \{F \in \mathcal{F} : Y \cap F \neq \emptyset\}.
\]

Given any \(x \in X\), we write \(\text{st}(x, \mathcal{F})\) instead of \(\text{st}\{\{x\}, \mathcal{F}\}\).

For two families \(\mathcal{F}, \mathcal{F}'\) of subsets of \(X\), we put
\[
\text{st}(\mathcal{F}, \mathcal{F}') = \{\text{st}(F, \mathcal{F}') : F \in \mathcal{F}\}.
\]

A family \(\mathcal{F}\) of subsets of \(X\) is called \textit{hereditary} if, for any subsets \(F, F'\) of \(X\) such that \(F \in \mathcal{F}\) and \(F' \subseteq F\), we have \(F' \subseteq \mathcal{F}\).

A family \(\mathcal{F}\) of subsets of \(X\) is called a \textit{covering} if \(\bigcup \mathcal{F} = X\).

We say that a family \(\{\mathcal{F}_\alpha : \alpha \in P\}\) of hereditary coverings of \(X\) is \textit{star stable} if, for any \(\alpha, \beta \in P\), there exist \(\gamma \in P\) such that
\[
\text{st}(\mathcal{F}_\alpha, \mathcal{F}_\beta) \subseteq \mathcal{F}_\gamma.
\]

Let \(\{\mathcal{F}_\alpha : \alpha \in P\}\) be a family of star stable coverings of \(X\). We consider a ball structure \(B = (X, P, B)\), where
\[
B(x, \alpha) = \text{st}(x, \mathcal{F}_\alpha),
\]
and show that \(B\) is a ballean.

Given any \(x \in X\) and \(\alpha \in P\), we have
\[
B(x, \alpha) = \{y \in X : y \in \text{st}(x, \mathcal{F}_\alpha)\}, \quad B^*(x, \alpha) = \{y \in X : x \in \text{st}(y, \mathcal{F}_\alpha)\}.
\]

Since \(y \in \text{st}(x, \mathcal{F}_\alpha)\) if and only if \(x \in \text{st}(y, \mathcal{F}_\alpha)\), then \(B^*(x, \alpha) = B(x, \alpha)\), so \(B\) is upper symmetric.
Given any $x \in X$ and $\alpha, \beta \in P$, we choose $\alpha' \in P$ and $\gamma \in P$ such that
\[ st(F_\alpha, F_\alpha) \subseteq F_{\alpha'} \text{ and } st(F_{\alpha'}, F_\beta) \subseteq F_\gamma. \]

Then we have
\[ B(B(x, \alpha), \beta) = st(st(x, F_\alpha), F_\beta) \subseteq st(x, F_\gamma) = B(x, \gamma), \]
so $B$ is upper multiplicative.

We note that a subset $Y$ of $X$ is bounded in $B$ if and only if $Y \in F_\alpha$ for some $\alpha \in P$. A family $F$ of subsets of $X$ is bounded in $B$ if and only if there exists $\alpha \in P$ such that $F \subseteq F_\alpha$.

Thus we have shown that every star stable family of coverings of $X$ determines some ballean on $X$. On the other hand, let $B = (X, P, B)$ be an arbitrary ballean on $X$. For every $\alpha \in P$, we put
\[ F_\alpha = \{ F \subseteq X : F \subseteq B(x, \alpha) \text{ for some } x \in X \}. \]

Then the ballean on $X$ determined by the star stable family $\{ F_\alpha : \alpha \in P \}$ of coverings of $X$ coincides with $B$.

3. Proximities and closeness

Let $X$ be a set, $P(X)$ be a family of all subsets of $X$. Let $\sigma$ be an equivalence on $P(X)$ such that, for all $Y, Y', Z \in P(X)$,
\[ Y \subseteq Z \subseteq Y', \ Y \sigma Y' \Longrightarrow Y \sigma Z. \]

We say that $\sigma$ is (an asymptotic) proximity and describe a way in which $\sigma$ defines some ballean $B(\sigma)$ on $X$.

We call a family $F$ of subsets of $X$ to be non-expanding with respect to $\sigma$ if, for every subset $Y$ of $X$, we have
\[ Y \sigma (Y \bigcup st(Y, F)). \]

We note that every subfamily of non-expanding family is non-expanding.

Let $F_1, F_2$ be non-expanding with respect to $\sigma$ families of subsets of $X$. We show that the family $st(F_1, F_2)$ is also non-expanding with respect to $\sigma$.

We fix an arbitrary subset $Y$ of $X$ and put
\[ F'_2 = \{ F' \in F_2 : Y \bigcap F' \neq \emptyset \}. \]

Since $F'_2$ is non-expanding, we have
\[ Y \sigma (Y \bigcup F'_2). \]

We put $Z = Y \bigcup \bigcup F'_2$ and
\[ F'_1 = \{ F \in F_1 : F \bigcap F' \neq \emptyset \text{ for some } F' \in F'_2 \}. \]

Since $F'_1$ is non-expanding, we have
\[ Z \sigma (Z \bigcup F'_1). \]
We put $T = Z \bigcup \mathcal{F}_1$. Since $\mathcal{F}_2$ is non-expanding, we have
$$T \sigma (T \bigcup \{ F \in \mathcal{F}_2 : F \cap T \neq \emptyset \}).$$

We put $H = T \bigcup \{ F \in \mathcal{F}_2 : F \cap T \neq \emptyset \})$. Then $Y \sigma H$ and $Y \subseteq H$. By the construction of $H$, we have
$$Y \subseteq Y \bigcup \{ S \in \text{st}(\mathcal{F}_1, \mathcal{F}_2) : S \cap Y \neq \emptyset \} \subseteq H.$$

Since $\sigma$ is a proximity, we conclude
$$Y \sigma (Y \bigcup \{ S \in \text{st}(\mathcal{F}_1, \mathcal{F}_2) : S \cap Y \neq \emptyset \})).$$

In particular, we proved that the family of all non-expanding (with respect to $\sigma$) hereditary covering of $X$ is star stable. Following Section 2, we define $B(\sigma)$ by means this family of coverings.

We note that a subset $Y$ of $X$ is bounded in $B(\sigma)$ if and only if the family $\{ Y \}$ is non-expanding, equivalently, $\{ y \} \sigma Y$ for every $y \in Y$. A family $\mathcal{F}$ of subsets of $X$ is uniformly bounded in $B(\sigma)$ if and only if $\mathcal{F}$ is non-expanding.

Let $B = (X, P, B)$ be a ballean. We consider a relation $\sigma$ on $\mathcal{P}(X)$ defined by the rule: $Y \sigma Z$ if and only if there exists $\alpha \in P$ such that $Y \subseteq B(Z, \alpha)$, $Z \subseteq B(Y, \alpha)$. It is easy to see that $\sigma$ is a proximity; we call it a closeness defined by $B$. We note that $Y, Z$ are close if and only if there exists a uniformly bounded covering $\mathcal{F}$ of $X$ such that
$$\bigcup \{ F \in \mathcal{F} : F \cap Y \neq \emptyset \} = \bigcup \{ F \in \mathcal{F} : F \cap Z \neq \emptyset \}.$$  

**Theorem 3.1.** Let $X$ be a set, $\sigma$ be a proximity on $\mathcal{P}(X)$, $\sigma'$ be a closeness defined by $B(\sigma)$. Then $\sigma' \subseteq \sigma$.

**Proof.** We remind that a family $\mathcal{F}$ of subsets of $X$ is uniformly bounded in $B(\sigma)$ if and only if $\mathcal{F}$ is non-expanding with respect to $\sigma$. Let $Y, Z \in \mathcal{P}(X)$ and $Y \sigma' Z$. Then there exists a non-expanding (with respect to $\sigma$) family $\mathcal{F}$ of subsets of $X$ such that $Y \subseteq \bigcup \mathcal{F}$, $Z \subseteq \bigcup \mathcal{F}$ and $Y \cap F \neq \emptyset$, $Z \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. It follows that $Y \sigma (\bigcup \mathcal{F})$ and $Z \sigma (\bigcup \mathcal{F})$, so $Y \sigma Z$. □

The following two examples show that the proximity $\sigma$ from Theorem 3.1 could be much more coarse than $\sigma'$.

**Example 3.2.** Let $X$ be an infinite set. We define an equivalence $\sigma$ on $\mathcal{P}(X)$ by the rule: $Y \sigma Z$ if and only if either $Y, Z$ are finite, or $Y, Z$ are infinite. Then a subset $Y$ of $X$ is bounded in $B(\sigma)$ if and only if $Y$ is finite; a family $\mathcal{F}$ of subsets of $X$ is uniformly bounded in $B(\sigma)$ if and only if each subset $F \in \mathcal{F}$ is finite and, for every $x \in X$, the set $\{ F \in \mathcal{F} : x \in F \}$ is finite. We show that $Y \sigma' Z$ if and only if either $Y, Z$ are finite, or $Y, Z$ are infinite and $|Y| = |Z|$. We should only check that if $Y, Z$ are infinite and $|Y| = |Z|$ then $Y \sigma' Z$. To this end we fix some bijection $f : Y \rightarrow Z$, and put $\mathcal{F} = \{ \{ y, f(y) \} : y \in Y \}$. Then $\mathcal{F}$ is uniformly bounded in $B(\sigma)$, $Y \sigma' (\bigcup \mathcal{F})$ and $Z \sigma' (\bigcup \mathcal{F})$, so $Y \sigma' Z$. Now if $X$ is uncountable than $\sigma$ is coarser than $\sigma'$. □
Example 3.3. Let \( X \) be a well-ordered set. We define an equivalence \( \sigma \) on \( \mathcal{P}(X) \) by the rule: \( Y \sigma Z \) if and only if \( \min Y = \min Z \). Then a subset \( Y \) is bounded in \( \mathcal{B}(\sigma) \) if and only if \( Y \) is a singleton. It follows that \( Y \sigma Z \) if and only if \( Y = Z \).

Theorem 3.4. Let \( \mathcal{B} = (X, P, B) \) be a ballean, \( \sigma \) be a closeness defined by \( \mathcal{B} \), \( \sigma' \) be a closeness defined by \( \mathcal{B}(\sigma) \). Then \( \sigma = \sigma' \).

Proof. By Theorem 3.1, \( \alpha \subseteq \alpha' \). To see that \( \sigma \subseteq \sigma' \) it suffices to note that every uniformly bounded in \( \mathcal{B} \) family of subsets of \( X \) is non-expanding with respect to \( \sigma \).

4. Does closeness determine a ballean?

Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be balleans with common support \( X \), \( \sigma_1 \) and \( \sigma_2 \) be closeness on \( \mathcal{P}(X) \) defined by \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \). Is \( \mathcal{B}_1 = \mathcal{B}_2 \) provided that \( \sigma_1 = \sigma_2 \)?

We give a negative answer to this general question, but prove one partial statement (Theorem 4.2) in positive direction.

Example 4.1. Let \( X \) be a countable set. We consider two families \( \varphi_1, \varphi_2 \) of coverings of \( X \).

A family \( \varphi_1 \) is defined by the rule: \( F \in \varphi_1 \) if and only if every subset \( F \in \mathcal{F} \) is finite, and the set \( \{ F \in \mathcal{F} : x \in F \} \) is finite for every \( x \in X \).

A family \( \varphi_2 \) is defined by the rule: \( F \in \varphi_2 \) if and only if there exists a natural number \( n \) such that \( |F| \leq n \) for every \( F \in \mathcal{F} \), and there exists a natural number \( m \) such that \( |\{ F \in \mathcal{F} : x \in F \}| \leq m \) for every \( x \in X \).

Clearly, the families \( \varphi_1 \) and \( \varphi_2 \) are star-stable. Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be balleans on \( X \) determined by \( \varphi_1 \) and \( \varphi_2 \). Using arguments from Example 3.2, it is easy to see that \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) define the same closeness \( \sigma \): \( Y \sigma Z \) if and only if either \( Y, Z \) are finite, or \( Y, Z \) are infinite. Then we take a partition \( \{ F_n : n \in \omega \} \) of \( X \) such that \( |F_n| = n \) for every \( n \in \omega \). Clearly, \( \mathcal{F} \) is uniformly bounded in \( \mathcal{B}_1 \), but \( \mathcal{F} \) is not uniformly bounded in \( \mathcal{B}_2 \). It follows that \( \mathcal{B}_1 \) is stronger than \( \mathcal{B}_2 \).

It is worth to mark that Example 4.1 gives a ballean \( \mathcal{B} \) with the closeness \( \sigma \) such that \( \mathcal{B} \neq \mathcal{B}(\sigma) \). To see this, we put \( \mathcal{B} = \mathcal{B}_2 \) and note that \( \mathcal{B}(\sigma) = \mathcal{B}_1 \).

Theorem 4.2. Let \( \mathcal{B}_1 = (X_1, P_1, B_1) \) and \( \mathcal{B}_2 = (X_2, P_2, B_2) \) be ordinal balleans with common support and the same closeness. Then \( \mathcal{B}_1 = \mathcal{B}_2 \).

Proof. We assume on the contrary that, say, \( \mathcal{B}_2 \prec \mathcal{B}_1 \) does not hold, and choose \( \beta \in P_2 \) such that, for every \( \alpha \in P_1 \), there exists \( x(\alpha) \in X \) such that \( B_2(x(\alpha), \beta) \not\subseteq B_1(x(\alpha), \alpha) \). We may suppose that \( P_1 \) is well-ordered. In the proof of Theorem 2.1 from [3] we constructed inductively a subset

\[ Y = \{ y(\alpha) : \alpha \in P_1 \} \]

of \( X \) such that the family \( \{ B_1(y(\alpha), \alpha) : \alpha \in P_1 \} \) is disjoint and, for every \( \alpha' \in P \),

\[ B_2(y(\alpha'), \beta) \not\subseteq \bigcup \{ B_1(y(\alpha), \alpha) : \alpha \in P_1 \} \].

We put \( Z = B_2(Y, \beta) \). Then \( Y, Z \) are close in \( \mathcal{B}_2 \), but \( Y, Z \) are not close in \( \mathcal{B}_1 \), whence a contradiction. 
\( \square \)
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