A note on a fixed point theorem for ray oriented maps

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Abstract. In this paper, we will prove a fixed point theorem for a ray-oriented map defined on a nonempty closed bounded convex subset of a Banach space.


Keywords: Fixed point, ray-oriented map.

Notations

Let $X$ be a Banach space and $K$ be a nonempty closed bounded convex subset of $X$. Let $T: K \to K$ be a mapping. Let $R_x$ be the ray passing through the segment $<x, Tx>$ and so $R_x := \{(1-\lambda)x + \lambda Tx : \lambda \in R\}$. Let $<x, y>$ be defined to be as $\{(1-\lambda)x + \lambda y : \lambda \in [0,1]\}$ and $(x, y) := \{(1-\lambda)x + \lambda y : \lambda \in (0,1)\}$. For any $x_1, x_2 \in R_x$, we say that $x_1 \leq x_2$ whenever $\lambda_1 \leq \lambda_2$ where $x_1 = (1-\lambda_1)x + \lambda_1 Tx$ and $x_2 = (1-\lambda_2)x + \lambda_2 Tx$ for some $\lambda_1, \lambda_2 \in R$.

1. Introduction

Let $X$ be a Banach space and $K$ be a nonempty closed bounded convex subset of $X$. Suppose $T: K \to K$ is a mapping satisfying the following conditions: (i). For some element $x_0$ of $K$, $R_{x_0} \cap K$ is invariant under $T$ and (ii). For each element $x \in R_{x_0} \cap K$, $<x, Tx> \cap K$ is continuous. Then, we will prove that there exists $y_0 \in R_{x_0} \cap K$ such that $<y_0, Ty_0> \subseteq R_{x_0} \cap K$ is invariant under $T$.

Moreover, the above theorem will be followed by a corollary as in the following: Suppose $T: [a, b] \to [a, b]$ is a mapping where $a, b \in \mathbb{R}$. If for each $x \in [a, b]$, the map $T$ restricted to the segment joining $x$ and $Tx$ is continuous. Then we will prove that there exists an invariant interval under $T$ and so it will have a fixed point in $[a, b]$. This result extends one dimensional Brouwer’s result for a larger class of mappings which need not be continuous. Also one can find some similar treatment for the convergence of fixed point in the real line by Beardon [1]. For further important fixed point results one can refer to [2].
2. Main Results

Theorem 2.1. Let $X$ be a normed linear space and let $K$ be a nonempty closed bounded convex subset of $X$. Suppose $T: K \rightarrow K$ is a mapping satisfying the following conditions:

(1) For some element $x_0$ of $K$, $R_{x_0} \cap K$ is invariant under $T$ and

(2) For each element $x \in R_{x_0} \cap K$, $|x, Tx| > 0 \cap K$ is continuous

Then, $T$ has a fixed point in $R_{x_0} \cap K$.

Note: When we say $T \mid < x, Tx >$ is continuous, we mean that $T$ is right continuous at $x$ and left continuous at $Tx$ if $x < Tx$.

Proof. Assume that the conclusion of the theorem is false. That is, $T$ does not have a fixed point in $R_{x_0} \cap K$. Therefore, for every $b \in R_{x_0} \cap K$, $< b, Tb >$ is not invariant under $T$.

Fix $y_0 \in R_{x_0} \cap K$ and let $x_0 \in < y_0, Ty_0 >$ such that $Tx_0 \notin < y_0, Ty_0 >$.

Let $G_{x_0} = R_{x_0} \cap K$. Now we can easily prove that $A = \{ \lambda \in R : (1 - \lambda)x_0 + \lambda T x_0 \in K \}$ is bounded.

Let $\alpha = \inf A$ and $\beta = \sup A$. Let $a = (1 - \alpha)x_0 + \alpha T x_0$ and $b = (1 - \beta)x_0 + \beta T x_0$. Therefore, there exists a sequence $\{ \alpha_n \} \in A$ such that $\{ \alpha_n \}$ converges to $\alpha$. Hence $(1 - \alpha_n)x_0 + \alpha_n T x_0$ converges to $(1 - \alpha)x_0 + \alpha T x_0$. Therefore it is easy to see that $a \in G_{x_0}$ and $b \in G_{x_0}$. Hence $G_{x_0} = \{ (1 - \lambda)a + \lambda b : 0 \leq \lambda \leq 1 \}$.

Now, define a map $g : G_{x_0} \rightarrow G_{x_0}$ by

$$g(z) = \begin{cases} x_0 & \text{if } z \leq x_0, \\ z & \text{if } z \in (x_0, Tx_0), \\ Tx_0 & \text{if } z \geq Tx_0. \end{cases}$$

Since $g$ and $T$ are continuous,

$$goT : < x_0, Tx_0 > \rightarrow < x_0, Tx_0 >$$

is also continuous.

Hence the map $goT$ has a fixed point $z_0 \in < x_0, Tx_0 >$.

Case 1: $z_0 = x_0$ Then

$$x_0 = z_0 = goT(z_0) = goT(x_0) = g(Tx_0) = Tx_0.$$ 

Hence $x_0 = Tx_0$, contradicting our assumption.

Case 2: $z_0 \in (x_0, Tx_0)$. If $Tx_0 \leq x_0$, then $z_0 = (goT)(z_0) = g(Tz_0) = x_0$, contradicting $z_0 \in (x_0, Tx_0]$.

If $Tx_0 \in < x_0, Tx_0 >$, then $z_0 = (goT)(z_0) = g(Tz_0) = Tx_0$, again contradicting our assumption, $< z_0, Tz_0 >$ is not invariant under $T$. Therefore,

$$Tz_0 \geq Tx_0$$
That is, 

\[(2.2) \quad z_0 = g\circ T(z_0) = g(Tz_0) = Tx_0\]

Substituting (2.2) in (2.1) we get

\[(2.3) \quad T^2x_0 \geq Tx_0\]

Now let us construct \(B = \{x \in R_{x_0} \cap K : x < Tx < T^2x\}\).

Moreover it is bounded above and so it must have a least upper bound. Therefore let \(u\) be the least upper bound of \(B\).

Then there exists \(x_n \in B\) such that \(x_n \to u\).

Suppose \(Tu < u\), then there exists a positive integer \(N\) such that for all \(n \geq N\), \(x_n \in < u, Tu >\). Then since \(T| < u, Tu >\) is continuous, \(Tx_n \to Tu\).

Since \(x_n < Tx_n, u \leq Tu\), a contradiction. Therefore, \(u \leq Tu\).

Since \(T| < u, Tu >\) is not invariant, by 2.3 we have \(T^2u \geq Tu\).

Therefore, \(u < Tu < T^2u\). Hence \(u \in B\).

But again, \(T^2u \geq T^2u\). Therefore, \(u < Tu < T^2u < T^3u\).

Hence \(Tu \in B\), which is a contradiction.

Therefore there exists a \(y_0 \in R_{x_0} \cap K\) such that \(T| < y_0, Ty_0 >\) is invariant.

Hence \(T\) has a fixed point in \(< y_0, Ty_0 >\). \(\square\)

**Corollary 2.2.** Suppose \(T : [a, b] \to [a, b]\) is a mapping where \(a, b \in \mathbb{R}\). For each element \(x \in [a, b]\), \(T| < x, Tx >\) is continuous. Then \(T\) has a fixed point in \([a, b]\).

**Remark 2.3.** There exists a discontinuous mapping \(T\) satisfying the conditions of corollary 2.2. \((T : [0, 1] \to [0, 1]\) by \(T(0) = 0\) and \(T(1) = 1\) for \(0 < x \leq 1\)).

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