On pseudo-$k$-spaces

ANNAMARIA MIRANDA

ABSTRACT. In this note a new class of topological spaces generalizing $k$-spaces, the pseudo-$k$-spaces, is introduced and investigated. Particular attention is given to the study of products of such spaces, in analogy to what is already known about $k$-spaces and quasi-$k$-spaces.

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1. Introduction

The first example of two $k$-spaces whose cartesian product is not a $k$-space was given by Dowker (see [2]). So a natural question is when a $k$-space satisfies that its product with every $k$-space is also a $k$-space. In 1948 J.H.C. Whitehead proved that if $X$ is a locally compact Hausdorff space then the cartesian product $i_X \times g$, where $i_X$ stands for the identity map on $X$, is a quotient map for every quotient map $g$. Using this result D.E. Cohen proved that if $X$ is locally compact Hausdorff then $X \times Y$ is a $k$-space for every $k$-space $Y$ (see Theorem 3.2 in [1]). Later the question was solved by Michael who showed that a $k$-space has this property iff it is a locally compact space (see [5]).

A similar question, related to quasi-$k$-spaces, was answered by Sanchis (see [8]). Quasi-$k$-spaces were investigated by Nagata (see [7]) who showed that “a space $X$ is a quasi-$k$-space (resp. a $k$-space) if and only if $X$ is a quotient space of a regular (resp. paracompact) $M$-space (see [6]).

The study of quasi-$k$-spaces suggests to define a larger class of spaces simply replacing countable compactness with pseudocompactness in the definition.

This note begins with the study of general properties about pseudo-$k$-spaces which leads on results about products of pseudo-$k$-spaces, in analogy with those known about $k$-spaces and more generally about quasi-$k$-spaces.

For terminology and notations not explicitly given we refer to [3].
We consider pseudocompact spaces which are not necessarily Tychonoff. Recall that

**Definition 2.1.** A topological space $X$ is called pseudocompact if every continuous real-valued function defined on $X$ is bounded.

**Definition 2.2.** A topological space $X$ is called locally compact (resp. locally countably compact) if each point of $X$ has a compact (resp. countably compact) neighborhood.

In analogy with the definitions of locally compact (resp. locally countably compact) space we have the following

**Definition 2.3.** A topological space $X$ is called locally pseudocompact if each point of $X$ has a pseudocompact neighborhood.

Clearly a locally compact space is locally pseudocompact and we have

**Proposition 2.4.** The cartesian product of a locally pseudocompact space $X$ and a locally compact space $Y$ is locally pseudocompact.

*Proof.* It suffices to observe that Corollary 3.10.27 in [3] holds even if the pseudocompact factor is not necessarily Tychonoff. $\square$

**Proposition 2.5.** If all spaces $X_s$ are pseudocompact then the sum $\bigoplus_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is locally pseudocompact.

Now we are going to define a new class of spaces which is larger than the class of $k$-spaces.

**Definition 2.6.** A topological space $X$ is called a pseudo-$k$-space if $X$ is a Hausdorff space and $X$ is the image of a locally pseudocompact Hausdorff space under a quotient mapping.

In other words, pseudo-$k$-spaces are Hausdorff spaces that can be represented as quotient spaces of locally pseudocompact Hausdorff spaces. Clearly every locally pseudocompact Hausdorff space is a pseudo-$k$-space.

We can compare this kind of spaces with the one of quasi-$k$-spaces. To this aim recall that

**Definition 2.7.** A Hausdorff space $X$ is a quasi-$k$-space if, and only if, a subset $A \subseteq X$ is closed in $X$ whenever the intersection of $A$ with any countably compact subset $Z$ of $X$ is closed in $Z$. 

Condition (2) in Theorem 2.11 yields

**Proposition 2.8.** Every quasi-

quasi-

k-space is a pseudo-

k-space.

The following example will show that the class of quasi-

k-spaces is strictly contained in the class of pseudo-

k-spaces.

**Definition 2.9.** A Hausdorff space \( X \) is called \( H \)-closed if \( X \) is a closed sub-

space of every Hausdorff space in which it is contained.

For a Hausdorff space \( X \), this definition is equivalent to say that every open cover \( \{U_s\}_{s \in S} \) of \( X \) contains a finite subfamily \( \{U_{s_1}, U_{s_2}, \ldots, U_{s_k}\} \) such that \( \overline{U_{s_1}} \cup \overline{U_{s_2}} \cup \ldots \cup \overline{U_{s_k}} = X \).

**Example 2.10.** A \( H \)-closed space which is not a quasi-

k-space.

Let \( S \) be the family of all free ultrafilters on \( \mathbb{N} \), let \( k\mathbb{N} = \mathbb{N} \cup S \) be the Katětov extension of \( \mathbb{N} \). We have that

1. \( k\mathbb{N} \) is a \( H \)-closed space;
2. \( k\mathbb{N} \) is not a quasi-

k-space.

It is enough to show that all countably compact subsets of \( k\mathbb{N} \) have finite cardinality. Let \( Y \subset X = k\mathbb{N} \) be countably compact. \( S \) is closed and discrete in \( X \) so \( Y \cap S \) is closed and discrete in \( Y \), therefore \( Y \cap S = \{p_1, \ldots, p_n\} \).

Hence \( Y = S \cup \{p_1, \ldots, p_n\} \), where \( S \subset \mathbb{N} \).

Assume that \( S \) is infinite. Since \( p_1, \ldots, p_n \) are distinct ultrafilters, there exists \( S_1 \subset S \) such that \( |S_1| = \omega \), \( S_1 \subset p_1 \) and \( S_1 \notin p_i \) for every \( i \neq 1 \). In fact let \( H_i \in p_1 \) such that \( H_i \notin p_i \) for every \( i \neq 1 \), then \( S_1 = \bigcap_{i=1}^{n} H_i \in p_1 \) and \( S_1 \notin p_i \) for every \( i \neq 1 \), otherwise \( S_1 \subset p_i \) and \( S_1 \subset H_i \) implies \( H_i \in p_i \). Moreover \( S_1 \) is infinite. Indeed, if \( p \) is an ultrafilter, \( A = \{x_1, \ldots, x_n\} \) and \( A \in p \), then \( \{x_i\} \notin p \) implies that \( \mathbb{N}\setminus \{x_i\} \in p \) for every \( i \), so \( \bigcap_{i=1}^{n} \mathbb{N}\setminus \{x_i\} = \mathbb{N}\setminus A \in p \), a contradiction.

Now, let \( G \subset S_1 \) such that \( |G| = \omega \) and \( |S_1\setminus G| = \omega \). Then \( G \in p_1 \) or \( \mathbb{N}\setminus G \in p_1 \). Since \( S_1 \subset p_1 \) it follows that \( G \in p_1 \) or \( S_1\setminus G \in p_1 \). Let us suppose that \( S_1\setminus G \in p_1 \). Then \( G \notin p_1 \). Therefore \( G \notin p_i \) for every \( i \).

Since \( G \notin p_i \) \( \forall i \in \{1, \ldots, n\} \), it follows that for every \( i \) there exists \( A_i \subset p_i \) such that \( G \cap A_i = \emptyset \), so \( V_i = A_i \cup \{p_i\} \) is an open neighborhood of \( p_i \) such that \( V_i \cap G = \emptyset \), therefore \( p_i \notin \overline{G} \) for every \( i \), hence \( G \) is closed in \( Y \) and, since \( G \subset \mathbb{N} \), \( G \) is also discrete. So \( G \) is an infinite closed discrete subspace of the countably compact space \( Y \), a contradiction. Hence \( S \) is finite.

In conclusion, since any \( H \)-closed space is a pseudocompact space, \( k\mathbb{N} \) is a pseudo-

k-space which is not a quasi-

k-space.

Now we give two useful characterizations of pseudo-

k-spaces.
Theorem 2.11. Let $X$ be a Hausdorff space. The following conditions are equivalent:

1. $X$ is a pseudo-$k$-space.
2. For each $A \subset X$, the set $A$ is closed provided that the intersection of $A$ with any pseudocompact subspace $Z$ of $X$ is closed in $Z$.
3. $X$ is a quotient space of a topological sum of pseudocompact spaces.

Proof. (1)$\Rightarrow$(2) Let $X$ be a pseudo-$k$-space and let $f : Y \to X$ be a quotient mapping of a locally pseudocompact Hausdorff space $Y$ onto $X$. Suppose that the intersection of a set $A$ with any pseudocompact subspace $P$ of $X$ is closed in $P$. Take a point $y \in f^{-1}(A)$ and a neighborhood $U \subset Y$ of the point $y$ such that $U$ is pseudocompact. Since the space $f(U)$ is pseudocompact (see Theorem 3.10.24 [3] which holds even if the range space $Y$ is not Tychonoff), the set $A \cap f(U)$ is closed in $f(U)$.

Now, if $y \notin f^{-1}(A)$ then $f(y) \notin A \cap f(U)$ so there exists an open set $T$ in $X$ containing $f(y)$ such that $T \cap (A \cap f(U)) = \emptyset$. It follows that $f^{-1}(T) \cap f^{-1}(A) \cap U = \emptyset$ where the set $f^{-1}(T) \cap U$ represents a neighborhood of $y$ disjoint from $f^{-1}(A)$. This is a contradiction. Then $y \in f^{-1}(A)$.

(2)$\Rightarrow$(3) Now consider a Hausdorff space $X$ and denote by $P(X)$ the family of non-empty pseudocompact subspaces of $X$. Let $\hat{X} = \bigoplus_{P \in P(X)} P$, where $i_P$ is the embedding of the subspace $P$ in the space $X$, is continuous (see Proposition 2.1.11 [3]). Suppose now that $A$ is closed in $\hat{X}$, this means $A \cap P$ closed in $P$, for every pseudocompact subset $P$ of $X$. Then, by (2), $A$ is closed in $X$. It follows that $f$ is a quotient map.

(3)$\Rightarrow$(1) If $X$ is a quotient space of a topological sum of pseudocompact spaces then $X$ is a pseudo-$k$-space, by Proposition 2.5. □

Corollary 2.12. A Hausdorff space $X$ is a pseudo-$k$-space if, and only if, a subset $A \subset X$ is open in $X$ whenever the intersection of $A$ with any pseudocompact subset $P$ of $X$ is open in $P$.

Regarding the continuity of a mapping whose domain is a pseudo-$k$-space we have the following

Theorem 2.13. A mapping $f$ of a pseudo-$k$-space $X$ to a topological space $Y$ is continuous if and only if for every pseudocompact subspace $P \subset X$ the restriction $f|_P : P \to Y$ is continuous.

From the definition of a pseudo-$k$-space we obtain

Theorem 2.14. If there exists a quotient mapping $f : X \to Y$ of a pseudo-$k$-space $X$ onto a Hausdorff space $Y$, then $Y$ is a pseudo-$k$-space.
Theorem 2.11 yields

**Theorem 2.15.** The sum $\oplus_{s \in S} X_s$ is a pseudo-$k$-space if and only if all spaces are pseudo-$k$-spaces.

### 3. On products of pseudo-$k$-spaces

The cartesian product of two pseudo-$k$-spaces need not be a pseudo-$k$-space. So, when a pseudo-$k$-space satisfies that its product with every pseudo-$k$-space is also a pseudo-$k$-space?

Proposition 2.4 states that the cartesian product of a locally compact space and a locally pseudocompact Hausdorff space is a locally pseudocompact space.

This result, together with Definition 2.6, yields

**Theorem 3.1.** *The cartesian product* $X \times Y$ *of a locally compact Hausdorff space* $X$ *and a pseudo-$k$-space* $Y$ *is a pseudo-$k$-space.*

**Proof.** Let $g : Z \rightarrow Y$ be a quotient mapping of a locally pseudocompact Hausdorff space $Z$ onto a pseudo-$k$-space $Y$. The cartesian product $f : id_X \times g : X \times Z \rightarrow X \times Y$ is a quotient mapping, by virtue of the Whitehead Theorem (see Lemma 4 in [9], or Theorem 3.3.17 in [3]). Now, since, by Proposition 2.4, $X \times Z$ is a locally pseudocompact Hausdorff space, it follows that $X \times Y$ is a pseudo-$k$-space. □

The previous Theorem gives a sufficient condition to obtain that the cartesian product of two pseudo-$k$-spaces is a pseudo-$k$-space. This condition, for regular spaces, is also necessary, as we will see in Theorem 3.4.

Now, starting from a regular space $X$ which is not locally compact, we define, following a construction introduced by Michael in [5], a normal pseudo-$k$-space $Y(X)$ such that the product $X \times Y(X)$ is not a pseudo-$k$-space. This enable us not only to give examples of two pseudo-$k$-spaces whose product is not a pseudo-$k$-space, but also to show Theorem 3.4.

Suppose that $X$ is a regular space which is not locally compact at some $x_0 \in X$. Let $\{U_\alpha\}_{\alpha \in A}$ be a local base of non-compact closed sets at $x_0$. For every $\alpha \in A$ let $\lambda(\alpha)$ be a limit ordinal and $\{F_{\lambda}\}_{\lambda < \lambda(\alpha)}$ be a well-ordered family of non-empty closed subsets of $U_\alpha$ whose intersection is empty.

Each $\lambda(\alpha) + 1$, equipped with the order topology, is a compact Hausdorff space. Therefore $\lambda(\alpha) + 1$ is a normal pseudo-$k$-space.

Then, by Theorem 2.15 jointly with Theorem 2.27 in [3], the topological sum $\Lambda = \oplus \{\lambda(\alpha) + 1 : \alpha \in A\}$ is a normal pseudo-$k$-space.

Now, let us denote by $Y(X)$ the quotient space obtained by identifying all the final points $\lambda(\alpha) \in \lambda(\alpha) + 1$ to a single points $y_0$.\[\]
We have the following

**Theorem 3.2.** The space \( Y(X) \) is a normal pseudo-k-space. Moreover, if \( P \) is a pseudocompact subset of \( Y(X) \), then \(|\{\alpha \in A : P \cap \lambda(\alpha) \neq \emptyset\}| < \omega\).

**Proof.** Let us denote by \( g : \Lambda \rightarrow Y(X) \) the canonical projection defining \( Y(X) \). It is easy to verify that \( g \) is a closed mapping. So, since the normality preserves under closed mappings, it follows that \( Y(X) \) is normal. Moreover, since \( g \) is a continuous surjective closed map, then \( g \) is a quotient mapping. Then, by Theorem 2.14, the space \( Y(X) \) is a pseudo-k-space.

Now, suppose that there exists \( B \subset A, |B| \geq \omega \), such that a pseudocompact subset \( P \) of \( Y(X) \) meets each element of the family \( \{\lambda(\alpha) : \alpha \in B\} \). Observe that for every \( \alpha \in A \), since \( \lambda(\alpha) \) is open in \( Y(X) \), the set \( \lambda(\alpha) \cap P \) is open in \( P \). Then the set \( \{\lambda(\alpha) \cap P : \alpha \in B\} \) is a locally finite family of non-empty open subsets of \( P \). Since \( P \) is a Tychonoff space, this is equivalent to say that \( P \) is not pseudocompact (see Theorem 3.10.22 in [3]), a contradiction. \( \square \)

**Theorem 3.3.** Let \( X \) be a regular space which is not locally compact at a point \( x_0 \). The cartesian product \( X \times Y(X) \) is not a pseudo-k-space.

**Proof.** Let \( X \) be a regular space which is not locally compact at a point \( x_0 \). Let us show that the cartesian product \( X \times Y(X) \) is not a pseudo-k-space. It suffices to find a subset \( H \) of \( X \times Y(X) \), which is not closed even if the intersection of \( H \) with any pseudocompact subspace \( P \) of the space \( X \times Y(X) \) is closed in \( P \).

Recall that, in the definition of \( Y(X) \), the set \( A \) denotes an index set and to each \( \alpha \in A \) is associated a limit ordinal \( \lambda(\alpha) \) such that \( \bigcap_{\lambda < \lambda(\alpha)} F_\lambda \) is empty.

Now fix \( \alpha \in A \) and \( \lambda \in \lambda(\alpha) + 1 \) and define \( E_\lambda = \bigcap_{\mu < \lambda} F_\mu \). Then \( E_\lambda = \emptyset \).

Moreover the set \( S_\alpha = \bigcup \{E_\lambda \times \{\lambda\} : \lambda \in \lambda(\alpha) + 1\} \) is closed in \( X \times (\lambda(\alpha) + 1) \), which implies that it is closed in \( X \times \Lambda \).

Denote by \( g \) the canonical projection \( g : \Lambda \rightarrow Y(X) \) and by \( h \) the function \( id_X \times g \), and define the set \( H = \bigcup_{\alpha \in A} h(S_\alpha) \subset X \times Y(X) \).

We shall show that \( H \) is the set we are searching for.

First let us prove that the intersection of \( H \) with any pseudocompact subset \( P \) of \( X \times Y(X) \) is closed in \( P \). The projection \( p_\mu(P) \) is a pseudocompact subset in \( Y(X) \) so, by virtue of Theorem 3.2, we have

\[ |\{\alpha \in A : p_\mu(P) \cap \lambda(\alpha) \neq \emptyset\}| < \omega \]

Then \( P \) meets finitely many \( X \times g(\lambda(\alpha) + 1) = X \times (\lambda(\alpha) \cup \{y_0\}) \supset h(S_\alpha) \).

Now, since \( h(S_\alpha) \) is closed in \( X \times Y(X) \) for each \( \alpha \in A \), it follows that the set \( H \cap P = \bigcup_{\alpha \in A} (h(S_\alpha) \cap P) \) is closed in \( P \).
Now let us show that \( H \) is not closed in \( X \times Y(X) \). The point \((x_0, y_0) \in X \times Y(X)\) belongs to \( \overline{H} \) but does not belong to \( H \). Take a neighborhood \( U \times V \) of \((x_0, y_0)\), \(U\) open in \( X \), \(V\) open in \( Y(X) \), and let \( U_\beta \) a closed non-compact neighborhood \( U_\beta \subset U \), for some \( \beta \in A \). Now, consider the canonical projection \( g : \Lambda \to Y(X) \), and fix \( \lambda \in g^{-1}(V) \cap \lambda(\beta) \). The set \( h(E_\lambda \times \{\lambda\}) \neq \emptyset \) is contained in \( (U \times V) \cap H \). Therefore \((x_0, y_0) \in \overline{H} \).

Suppose that \((x_0, y_0) \in H\), then \((x_0, y_0) \in h(S_\alpha)\) for some \( \alpha \in A \). This is a contradiction. \( \square \)

Theorems 3.1 and 3.3 provide the following characterization for locally compact spaces.

**Theorem 3.4.** Let \( X \) be a regular space. The following conditions are equivalent:

1. \( X \) is locally compact.
2. \( X \times Y \) is a pseudo-k-space, for each pseudo-k-space \( Y \).

**Proof.** (1) \( \Rightarrow \) (2) It follows from Theorem 3.1.

(2) \( \Rightarrow \) (1) Let \( X \) be a regular space which is not locally compact at a point \( x_0 \). Then, by virtue of Theorems 3.2 and 3.3, the space \( Y(X) \) is a pseudo-k-space such that \( X \times Y(X) \) is not a pseudo-k-space. \( \square \)

In terms of products of mappings we have

**Theorem 3.5.** Let \( X \) be a regular space. The following conditions are equivalent:

1. \( X \) is locally compact.
2. \( \text{id}_X \times g \) is a quotient map with domain a locally pseudocompact Hausdorff space, for every quotient map \( g \) with domain a locally pseudocompact Hausdorff space \( Y \).

**Proof.** (1) \( \Rightarrow \) (2) It comes directly from Whitehead Theorem (see Theorem 3.3.17 in [3]) and Proposition 2.4.

(2) \( \Rightarrow \) (1) If \( X \) is not locally compact then we can consider \( Y(X) \), defined as before, and the projection map \( g : \Lambda \to Y(X) \), which is a quotient map with domain the locally pseudocompact Hausdorff space \( \Lambda \). It is easy to show that \( h = \text{id}_X \times g \) is not a quotient map with domain a locally pseudocompact Hausdorff space. Indeed if \( h \) was a quotient map with domain a locally pseudocompact Hausdorff space then \( X \times Y(X) \) should be a pseudo-k-space, but \( X \times Y(X) \) is not a pseudo-k-space by virtue of Theorem 3.3. \( \square \)
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A. Miranda (amiranda@unisa.it)
Dip. di Matematica e Informatica, Università di Salerno, Via Ponte Don Melillo, 84084 Fisciano (Salerno), Italy