Topologies on function spaces and hyperspaces

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Abstract. Let $Y$ and $Z$ be two fixed topological spaces, $O(Z)$ the family of all open subsets of $Z$, $C(Y, Z)$ the set of all continuous maps from $Y$ to $Z$, and $O_Z(Y)$ the set $\{ f^{-1}(U) : f \in C(Y, Z) \text{ and } U \in O(Z) \}$. In this paper, we give and study new topologies on the sets $C(Y, Z)$ and $O_Z(Y)$ calling $(A, A_0)$-splitting and $(A, A_0)$-admissible, where $A$ and $A_0$ families of spaces.

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1. Preliminaries

Let $Y$ and $Z$ be two fixed topological spaces. By $C(Y, Z)$ we denote the set of all continuous maps from $Y$ to $Z$. If $t$ is a topology on the set $C(Y, Z)$, then the corresponding topological space is denoted by $C_t(Y, Z)$.

Let $X$ be a space. To each map $g : X \times Y \to Z$ which is continuous in $y \in Y$ for each fixed $x \in X$, we associate the map $g^* : X \to C(Y, Z)$ defined as follows: for every $x \in X$, $g^*(x)$ is the map from $Y$ to $Z$ such that $g^*(x)(y) = g(x, y)$, $y \in Y$. Obviously, for a given map $h : X \to C(Y, Z)$, the map $h^o : X \times Y \to Z$ defined by $h^o(x, y) = h(x)(y)$, $(x, y) \in X \times Y$, satisfies $(h^o)^* = h$ and is continuous in $y$ for each fixed $x \in X$. Thus, the above association (defined in [7]) between the mappings from $X \times Y$ to $Z$ that are continuous in $y$ for each fixed $x \in X$, and the mappings from $X$ to $C(Y, Z)$ is one-to-one.

In 1946 R. Arens [1] introduced the notion of an admissible topology: a topology $t$ on $C(Y, Z)$ is called admissible if the map $e : C_t(Y, Z) \times Y \to Z$, called evaluation map, defined by $e(f, y) = f(y)$, is continuous.

In 1951 R. Arens and J. Dugundji [2] introduced the notion of a splitting topology: a topology $t$ on $C(Y, Z)$ is called splitting if for every space $X$, the continuity of a map $g : X \times Y \to Z$ implies the continuity of the map $g^* : X \to C(Y, Z)$.

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\[ g^* : X \to C_t(Y, Z). \] On the set \( C(Y, Z) \) there exists the greatest splitting topology, denoted here by \( t_{gs} \) (see [2]). They also proved that a topology \( t \) on \( C(Y, Z) \) is admissible if and only if for every space \( X \), the continuity of a map \( h : X \to C_t(Y, Z) \) implies that of the map \( h^* : X \times Y \to Z \).

If in the above definitions it is assumed that the space \( X \) belongs to a fixed class \( A \) of topological spaces, then the topology \( t \) is called \( A \)-splitting or \( A \)-admissible, respectively (see [8]). In the case where \( A = \{ X \} \) we write \( X \)-splitting (respectively, \( X \)-admissible) instead of \( \{ X \} \)-splitting (respectively, \( \{ X \} \)-admissible).

Let \( X \) be a space. In what follows by \( \mathcal{O}(X) \) we denote the family of all open subsets of \( X \). Also, for two fixed topological spaces \( Y \) and \( Z \) we denote by \( \mathcal{O}_Z(Y) \) the set \( \{ f^{-1}(U) : f \in C(Y, Z) \text{ and } U \in \mathcal{O}(Z) \} \).

The Scott topology \( \Omega(Y) \) on \( \mathcal{O}(Y) \) (see, for example, [11]) is defined as follows: a subset \( \mathcal{H} \) of \( \mathcal{O}(Y) \) belongs to \( \Omega(Y) \) if:

\begin{itemize}
  \item[(a)] the conditions \( U \in \mathcal{H}, V \in \mathcal{O}(Y), \text{ and } U \subseteq V \) imply \( V \in \mathcal{H} \), and
  \item[(b)] for every collection of open sets of \( Y \), whose union belongs to \( \mathcal{H} \), there are finitely many elements of this collection whose union also belongs to \( \mathcal{H} \).
\end{itemize}

The strong Scott topology \( \Omega^s(Y) \) on \( \mathcal{O}(Y) \) (see [12]) is defined as follows: a subset \( \mathcal{H} \) of \( \mathcal{O}(Y) \) belongs to \( \Omega^s(Y) \) if:

\begin{itemize}
  \item[(a)] the conditions \( U \in \mathcal{H}, V \in \mathcal{O}(Y), \text{ and } U \subseteq V \) imply \( V \in \mathcal{H} \), and
  \item[(b)] for every open cover of \( Y \) there are finitely many elements of this cover whose union also belongs to \( \mathcal{H} \).
\end{itemize}

The Isbell topology \( t_{Is} \) (respectively, strong Isbell topology \( t_{sIs} \)) on \( C(Y, Z) \) (see, for example, [13] and [12]) is the topology, which has as a subbasis the family of all sets of the form:

\[ (\mathcal{H}, U) = \{ f \in C(Y, Z) : f^{-1}(U) \in \mathcal{H} \}, \]

where \( \mathcal{H} \in \Omega(Y) \) (respectively, \( \mathcal{H} \in \Omega^s(Y) \)) and \( U \in \mathcal{O}(Z) \).

The compact open topology (see [7]) on \( C(Y, Z) \), denoted here by \( t_{co} \), is the topology for which the family of all sets of the form

\[ (K, U) = \{ f \in C(Y, Z) : f(K) \subseteq U \}, \]

where \( K \) is a compact subset of \( Y \) and \( U \) is an open subset of \( Z \), form a subbase. It is known that \( t_{co} \subseteq t_{Is} \) (see, for example, [13]).

A subset \( K \) of a space \( X \) is said to be bounded if every open cover of \( X \) has a finite subcover for \( K \) (see [12]).

A space \( X \) is called corecompact (see [11]) if for every \( x \in X \) and for every open neighborhood \( U \) of \( x \), there exists an open neighborhood \( V \) of \( x \) such that the subset \( V \) is bounded in the space \( U \) (see [11]).
Below, we give some well known results:

1. The Isbell topology and, hence, the compact open topology, and the point open topology (denoted here by $t_{po}$) on $C(Y, Z)$ are always splitting (see, for example, [2], [3], and [13]).

2. The compact open topology on $C(Y, Z)$ is admissible if $Y$ is a regular locally compact space. In this case the compact open topology is also the greatest splitting topology (see [2]).

3. The Isbell topology on $C(Y, Z)$ is admissible if $Y$ is a corecompact space. In this case the Isbell topology is also the greatest splitting topology (see, for example, [12] and [14]).

4. A topology larger than a admissible topology is also admissible (see [2]).

5. A topology smaller than a splitting topology is also splitting (see [2]).

6. The strong Isbell topology on $C(Y, Z)$ is admissible if $Y$ is a locally bounded space (see [12]).

For a summary of all the above results and some open problems on function spaces see [10]. Also, [4] and [5] are other papers related to this area.

In what follows if $\varphi : X \to Y$ is a map and $X_0 \subseteq X$, then by $\varphi|_{X_0} : X_0 \to Y$ we denote the restriction of the map $\varphi$ on the set $X_0$. Also, if $h : X \times Y \to Z$ is a map and $X_0 \subseteq X$, then by $h|_{X_0 \times Y}$ we denote the restriction of the map $h$ on the set $X_0 \times Y$.

In Sections 2 and 3 we give and study new topologies on the sets $C(Y, Z)$ and $OZ(Y)$ calling $(\mathcal{A}, \mathcal{A}_0)$-splitting and $(\mathcal{A}, \mathcal{A}_0)$-admissible, where $\mathcal{A}$ and $\mathcal{A}_0$ families of spaces.

2. $(\mathcal{A}, \mathcal{A}_0)$-splitting and $(\mathcal{A}, \mathcal{A}_0)$-admissible topologies on the set $C(Y, Z)$

**Note 1.** Let $\mathcal{A}$ be a family of topological spaces. For every $X \in \mathcal{A}$ we denote by $X_0$ a subspace of $X$ and by $\mathcal{A}_0$ the family of all such subspaces $X_0$. In all paper by $(\mathcal{A}, \mathcal{A}_0)$ we denote the family of all pairs $(X, X_0)$ such that $X \in \mathcal{A}$, $X_0 \in \mathcal{A}_0$, and $X_0$ is a subspace of $X$.

**Definition 2.1.** A topology $t$ on $C(Y, Z)$ is called $(\mathcal{A}, \mathcal{A}_0)$-splitting if for every pair $(X, X_0) \in (\mathcal{A}, \mathcal{A}_0)$, the continuity of a map $g : X \times Y \to Z$ implies the continuity of the map $g^*|_{X_0} : X_0 \to C_t(Y, Z)$, where $g^* : X \to C_t(Y, Z)$ the map which is defined in preliminaries.

A topology $t$ on $C(Y, Z)$ is called $(\mathcal{A}, \mathcal{A}_0)$-admissible if for every pair $(X, X_0) \in (\mathcal{A}, \mathcal{A}_0)$, the continuity of a map $h : X \to C_t(Y, Z)$ implies that of the map $h^*|_{X_0 \times Y} : X_0 \times Y \to Z$, where $h^* : X \times Y \to Z$ the map which is defined in preliminaries.

In the case where $\mathcal{A} = \{X\}$ and $\mathcal{A}_0 = \{X_0\}$, where $X_0$ is a subspace of $X$, we write $(X, X_0)$-splitting (respectively, $(X, X_0)$-admissible) instead of $(\{X\}, \{X_0\})$-splitting (respectively, $(\{X\}, \{X_0\})$-admissible).
Clearly, the following theorem is true.

**Theorem 2.2.** The following statements are true:

1. Every splitting (respectively, admissible) topology on $C(Y, Z)$ is $(A, A_0)$-splitting (respectively, $(A, A_0)$-admissible), where $A$ and $A_0$ are arbitrary families of spaces such that every element $X_0 \in A_0$ is a subspace of an element $X \in A$.

2. Every $A$-splitting (respectively, $A$-admissible) topology on $C(Y, Z)$ is $(A, A_0)$-splitting (respectively, $(A, A_0)$-admissible), where $A$ and $A_0$ are arbitrary families of spaces such that every element $X_0 \in A_0$ is a subspace of an element $X \in A$.

**Example 2.3.**

1. The point-open, the compact open, and the Isbell topologies are $(A, A_0)$-splitting, where $A$ and $A_0$ are arbitrary families of spaces such that every element $X_0 \in A_0$ is a subspace of an element $X \in A$.

2. If $Y$ is a regular locally compact space, then the compact-open topology is $(A, A_0)$-admissible, where $A$ and $A_0$ are arbitrary families of spaces such that every element $X_0 \in A_0$ is a subspace of an element $X \in A$.

3. If $Y$ is a corecompact space, then the Isbell topology is $(A, A_0)$-admissible, where $A$ and $A_0$ are arbitrary families of spaces such that every element $X_0 \in A_0$ is a subspace of an element $X \in A$.

4. If $Y$ is a locally bounded space, then the strong Isbell topology is $(A, A_0)$-admissible, where $A$ and $A_0$ are arbitrary families of spaces such that every element $X_0 \in A_0$ is a subspace of an element $X \in A$.

5. Let $X$ be a space, $x_0 \in X$, $X_0$ the subspace $\{x_0\}$ of $X$, and $t$ an arbitrary topology on $C(Y, Z)$ which it is not $X$-splitting. Then, the topology $t$ is $(X, X_0)$-splitting. It is clear that this topology is not splitting.

6. Let $X$ be a space, $x_0 \in X$, $X_0$ the subspace $\{x_0\}$ of $X$, and $t$ an arbitrary topology on $C(Y, Z)$ which it is not $X$-admissible. Then, the topology $t$ is $(X, X_0)$-admissible. It is clear that this topology is not admissible.

**Theorem 2.4.** The following statements are true:

1. A topology smaller than an $(A, A_0)$-splitting topology is also $(A, A_0)$-splitting.

2. A topology larger than an $(A, A_0)$-admissible topology is also $(A, A_0)$-admissible.

**Proof.** We prove only the statement (1). The proof of (2) is similar. Let $t_1$ be an $(A, A_0)$-splitting topology on $C(Y, Z)$ and $t_2$ a topology on $C(Y, Z)$ such that $t_2 \subseteq t_1$. We prove that the topology $t_2$ is a $(A, A_0)$-splitting topology. Indeed, let $(X, X_0) \in (A, A_0)$ and let $g : X \times Y \to Z$ be a continuous map. Since the topology $t_1$ is $(A, A_0)$-splitting, the map $g^*|_{X_0} : X_0 \to C_{t_1}(Y, Z)$ is continuous. Also, since $t_2 \subseteq t_1$, the identical map $id : C_{t_1}(Y, Z) \to C_{t_2}(Y, Z)$ is
continuous. So, the map $g^*|_{X_0} : X_0 \to C_t(Y, Z)$ is continuous as a composition of continuous maps. Thus, the topology $t_2$ is $(A, A_0)$-splitting. □

**Definition 2.5.** Let $(A_1^1, A_1^0)$ and $(A_2^1, A_2^0)$ two pairs of spaces, where $A_1^1$ (respectively, $A_2^1$) and $A_0^1$ (respectively, $A_0^0$) are arbitrary families of spaces such that every element $X_0 \in A_0^1$ (respectively, every element $X_0 \in A_0^0$) is a subspace of an element $X \in A_1^{1}$ (respectively, of an element $X \in A_2^{1}$). We say that the pairs $(A_1^1, A_1^0)$ and $(A_2^1, A_2^0)$ are equivalent if a topology $t$ on $C(Y, Z)$ is $(A_1^1, A_1^0)$-splitting if and only if $t$ is $(A_2^1, A_2^0)$-splitting, and $t$ is $(A_1^1, A_1^0)$-admissible if and only if $t$ is $(A_2^1, A_2^0)$-admissible. In this case we write $(A_1^1, A_1^0) \sim (A_2^1, A_2^0)$.

**Theorem 2.6.** For every pair $(A, A_0)$, where $A$ and $A_0$ are arbitrary families of spaces such that every element $X_0 \in A_0$ is a subspace of an element $X \in A$, there exists a pair $(X(A), X(A_0))$, where $X(A)$ is a space and $X(A_0)$ is a subspace of $X(A)$ such that

$$(A, A_0) \sim (X(A), X(A_0)).$$

**Proof.** Let $T_{sp}^c$ be the set of all topologies on $C(Y, Z)$ which are not $(A, A_0)$-splitting and let $T_{ad}^c$ be the set of all topologies on $C(Y, Z)$ which are not $(A, A_0)$-admissible. For each $t \in T_{sp}^c$ there exists in $(A, A_0)$ a pair $(X_t^{sp}, X_t^{ad})$ such that $t$ is not $(X_t^{sp}, X_t^{ad})$-splitting. Similarly, for each $t \in T_{ad}^c$ there exists in $(A, A_0)$ a pair $(X_t^{ad}, X_t^{ad})$ such that $t$ is not $(X_t^{ad}, X_t^{ad})$-admissible. Let

$$A' = \{X_t^{sp} : t \in T_{sp}^c\} \cup \{X_t^{ad} : t \in T_{ad}^c\}$$

and

$$A'_0 = \{X_t^{sp}_{1,0} : t \in T_{sp}^c\} \cup \{X_t^{ad}_{1,0} : t \in T_{ad}^c\}.$$ 

Of course, we can suppose that the spaces from $A'$ and $A'_0$ are pair-wise disjoint. Let $X(A)$ and $X(A_0)$ be the free union of all the spaces from $A'$ and $A'_0$, respectively. We prove that the pair $(X(A), X(A_0))$ is the required pair.

Let $t$ be an $(A, A_0)$-splitting topology on $C(Y, Z)$. We prove that this topology is $(X(A), X(A_0))$-splitting. Indeed, let $g : X(A) \times X \to Z$ be a continuous map. It suffices to prove that the map $g^*|_{X(A)} : X(A) \to C_t(Y, Z)$ is continuous. Let $X \in A' \subseteq A$. Then, the restriction $g|_{X \times Y}$ of the map $g$ on $X \times Y \subseteq X(A) \times Y$ is also a continuous map and, therefore, since the topology $t$ is $(A, A_0)$-splitting we have that the map $(g|_{X \times Y})^*|_{X_0} : X_0 \to C_t(Y, Z)$ is continuous. Since $X(A_0)$ is the free union of all the spaces from $A'_0$ and $(g|_{X \times Y})^*|_{X_0} = (g^*|_{X(A_0)})|_{X_0}$, it follows that the map $g^*|_{X(A_0)} : X(A_0) \to C_t(Y, Z)$ is continuous. Thus, the topology $t$ on $C(Y, Z)$ is $(X(A), X(A_0))$-splitting.

Now, let $t$ be an $(X(A), X(A_0))$-splitting topology on $C(Y, Z)$. We prove that $t$ is $(A, A_0)$-splitting. We suppose that $t$ is not $(A, A_0)$-splitting. Then, $t \in T_{sp}^c$ and, therefore, $t$ is not $(X_t^{sp}, X_t^{sp})$-splitting for some pair $(X_t^{sp}, X_t^{sp}) \in (A, A_0)$. Thus, there exists a continuous map $g : X_t^{sp} \times Y \to Z$ such that the
map \( g^*|_{A_0^p} : X_{i,0}^p \to C_t(Y, Z) \) is not continuous. Since the space \( X(A) \) is the free union of all the spaces from the family \( A' \), the map \( g \) can be extended to a continuous map \( g_1 : X(A) \times Y \to Z \). Since the map \( g^*|_{X_{i,0}^p} \) is not continuous, \( X_{i,0}^p \in A_0^p \), and the space \( X(A_0) \) is the free union of all spaces from \( A_0^p \) we have that the map

\[
g^*|_{X(A_0)} : X(A_0) \to C_t(Y, Z)
\]

is not continuous, which contradicts our assumption that \( t \) is a \((X(A), X(A_0))\)-splitting topology. Thus, a topology \( t \) on \( C(Y, Z) \) is \((A, A_0)\)-splitting if and only if it is \((X(A), X(A_0))\)-splitting.

Similarly, a topology \( t \) on \( C(Y, Z) \) is \((A, A_0)\)-admissible if and only if is \((X(A), X(A_0))\)-admissible. Hence,

\[(A, A_0) \sim (X(A), X(A_0)).\]

\( \square \)

**Theorem 2.7.** There exists the greatest \((A, A_0)\)-splitting topology, where \( A \) and \( A_0 \) are arbitrary families of spaces such that every element \( X_0 \in A_0 \) is a subspace of an element \( X \in A \).

**Proof.** Let \( \{t_i : i \in I\} \) be the family of all \((A, A_0)\)-splitting topologies on \( C(Y, Z) \). We consider the topology \( t = \bigvee\{t_i : i \in I\} \). Clearly, \( t \) is \((A, A_0)\)-splitting and \( t_i \subseteq t \), for every \( i \in I \). Thus, \( t \) is the greatest \((A, A_0)\)-splitting topology.

\( \square \)

**Note 2.** In what follows we denote by \( t(A, A_0) \) the greatest \((A, A_0)\)-splitting topology on \( C(Y, Z) \).

**Theorem 2.8.** The following statements are true:

1. If \((A, A_0) = \cup\{(A^i, A_0^i) : i \in I\}\), then

   \[t(A, A_0) = \cap\{t(A^i, A_0^i) : i \in I\}.\]

2. If \((A, A_0) = \cap\{t(X, X_0) : (X, X_0) \in (A, A_0)\}\).

3. If \((A, A_0) = \cap\{(A^i, A_0^i) : i \in I\}\), then

   \[\bigvee\{t(A^i, A_0^i) : i \in I\} \subseteq t(A, A_0).\]

**Proof.** (1) Since \((A, A_0) = \cup\{(A^i, A_0^i) : i \in I\}\) we have that every topology which is \((A, A_0)\)-splitting is also \((A^i, A_0^i)\)-splitting, for every \( i \in I \). Thus, the topology \( t(A, A_0) \) is \((A^i, A_0^i)\)-splitting and, therefore,

\[t(A, A_0) \subseteq t(A^i, A_0^i),\]

for every \( i \in I \). So, we have

\[t(A, A_0) \subseteq \cap\{t(A^i, A_0^i) : i \in I\}.\]

Now, we prove the converse relation, that is

\[\cap\{t(A^i, A_0^i) : i \in I\} \subseteq t(A, A_0).\]
For the above relation it suffices to prove that the topology \( \cap \{ t(A^i, \mathcal{A}^i_0) : i \in I \} \) is \((\mathcal{A}, \mathcal{A}_0)\)-splitting. Let \((X, X_0) \in (\mathcal{A}, \mathcal{A}_0)\) and let \( g : X \times Y \to Z \) be a continuous map. We prove that the map

\[
g^*|_{X_0} : X_0 \to C_{t(\cap \{ t(A^i, \mathcal{A}^i_0) : i \in I \})}(Y, Z)
\]

is continuous. Since \((X, X_0) \in (\mathcal{A}, \mathcal{A}_0)\), there exists \( i \in I \) such that \((X, X_0) \in (A^i, \mathcal{A}^i_0)\). This means that the map

\[
g^*|_{X_0} : X_0 \to C_{t(A^i, \mathcal{A}^i_0)}(Y, Z)
\]

is continuous. Also, since \( \cap \{ t(A^i, \mathcal{A}^i_0) : i \in I \} \subseteq t(A^i, \mathcal{A}^i_0)\), the identical map

\[
id : C_{t(\cap \{ t(A^i, \mathcal{A}^i_0) : i \in I \})}(Y, Z) \to C_{t(A^i, \mathcal{A}^i_0)}(Y, Z)
\]

is continuous. So, the map

\[
g^*|_{X_0} : X_0 \to C_{\cap \{ t(A^i, \mathcal{A}^i_0) : i \in I \}}(Y, Z)
\]

is continuous as a composition of continuous maps. Thus, the topology

\[
\cap \{ t(A^i, \mathcal{A}^i_0) : i \in I \}
\]

is \((\mathcal{A}, \mathcal{A}_0)\)-splitting.

(2) The proof of this is a corollary of the statement (1).

(3) The proof of this follows by the fact that the topology

\[
\bigvee \{ t(A^i, \mathcal{A}^i_0) : i \in I \}
\]

is \((\mathcal{A}, \mathcal{A}_0)\)-splitting. □

**Theorem 2.9.** Let \( t \) be an \((\mathcal{A}, \mathcal{A}_0)\)-admissible topology on \( C(Y, Z) \). If

\[
(C_t(Y, Z), C_t(Y, Z)) \in (\mathcal{A}, \mathcal{A}_0),
\]

then \( t \) is admissible and \( t(\mathcal{A}, \mathcal{A}_0) \subseteq t \).

**Proof.** Let \( \text{id} \equiv h : C_t(Y, Z) \to C_t(Y, Z) \) be the identical map. Clearly, this map is continuous. Since

\[
(C_t(Y, Z), C_t(Y, Z)) \in (\mathcal{A}, \mathcal{A}_0)
\]

and \( t \) is \((\mathcal{A}, \mathcal{A}_0)\)-admissible, the map \( h^\circ |_{C_t(Y, Z)} \equiv h^\circ : C_t(Y, Z) \times Y \to Z \) is continuous. Hence, the topology \( t \) is admissible.

Now, since the map \( h^\circ \equiv g : C_t(Y, Z) \times Y \to Z \) is continuous,

\[
(C_t(Y, Z), C_t(Y, Z)) \in (\mathcal{A}, \mathcal{A}_0),
\]

and the topology \( t(\mathcal{A}, \mathcal{A}_0) \) is \((\mathcal{A}, \mathcal{A}_0)\)-splitting, the map

\[
g^*|_{C_t(Y, Z)} = \text{id} : C_t(Y, Z) \to C_t(\mathcal{A}, \mathcal{A}_0)(Y, Z)
\]

is also continuous. Thus, \( t(\mathcal{A}, \mathcal{A}_0) \subseteq t \). □

**Corollary 2.10.** Let \( t \) be an \((\mathcal{A}, \mathcal{A}_0)\)-splitting and \((\mathcal{A}, \mathcal{A}_0)\)-admissible topology on \( C(Y, Z) \). If \( (C_t(Y, Z), C_t(Y, Z)) \in (\mathcal{A}, \mathcal{A}_0) \), then \( t(\mathcal{A}, \mathcal{A}_0) = t \).

**Proof.** By Theorem 2.9, \( t(\mathcal{A}, \mathcal{A}_0) \subseteq t \). Also, since the topology \( t \) is \((\mathcal{A}, \mathcal{A}_0)\)-splitting, \( t \subseteq t(\mathcal{A}, \mathcal{A}_0) \). Thus, \( t(\mathcal{A}, \mathcal{A}_0) = t \). □
Theorem 2.11. Let $Y$ be a regular locally compact space, $\mathcal{A}$ the family of all $T_1$-spaces, $i = 0, 1, 2, 3, 3\frac{1}{2}$, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{fs}$.

Proof. Since $Y$ is a regular locally compact space, the compact open topology coincides with the Isbell topology on $C(Y, Z)$ and it is admissible. Hence, $t_{co}$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible. Also, the topology $t_{co}$ is splitting and, therefore, $t_{co}$ is $(\mathcal{A}, \mathcal{A}_0)$-splitting. Since $Z \in \mathcal{A}$, we have that $C_{t_{co}}(Y, Z) \in \mathcal{A}$ (see preliminaries) and, therefore, $(C_{t_{co}}(Y, Z), C_{t_{co}}(Z)) \in (\mathcal{A}, \mathcal{A}_0)$. Thus, by Corollary 2.10 we have that $t(\mathcal{A}, \mathcal{A}_0) = t_{co}$. □

Theorem 2.12. Let $Y$ be a regular locally compact space, $\mathcal{A}$ the family of all topological spaces whose weight is not greater than a certain fixed infinite cardinal, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{fs}$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.11 and follows by Corollary 2.10 and Theorem 3.4.16 of [6]. □

Theorem 2.13. Let $Y$ be a regular second-countable locally compact space, $\mathcal{A}$ the family of all metrizable spaces, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{fs}$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.11 and follows by Corollary 2.10 and Exercises 4.2.H and 3.4.E(c) of [6]. □

Theorem 2.14. Let $Y$ be a regular locally compact $\kappa$-Lindelöf space, $\mathcal{A}$ the family of all completely metrizable spaces, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{fs}$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.11 and follows by Corollary 2.10 and Exercise 4.3.F(a) of [6]. □

Theorem 2.15. Let $Y$ be a corecompact space, $\mathcal{A}$ the family of all $T_i$-spaces, where $i = 0, 1, 2$, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{fs}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{fs}$.

Proof. Since $Y$ is corecompact, the Isbell topology $t_{fs}$ on $C(Y, Z)$ is admissible. Hence the topology $t_{fs}$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible. Also, the topology $t_{fs}$ is splitting and, therefore, $t_{fs}$ is $(\mathcal{A}, \mathcal{A}_0)$-splitting. Since $Z \in \mathcal{A}$, we have that $C_{t_{fs}}(Y, Z) \in \mathcal{A}$ (see preliminaries) and, therefore, $(C_{t_{fs}}(Y, Z), C_{t_{fs}}(Z)) \in (\mathcal{A}, \mathcal{A}_0)$. Thus, by Corollary 2.10 we have that $t(\mathcal{A}, \mathcal{A}_0) = t_{fs}$. □

Theorem 2.16. Let $Y$ be a corecompact space, $\mathcal{A}$ the family of all second-countable spaces, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{fs}}(Y, Z) \in \mathcal{A}_0$, and $Y, Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{fs}$. 

Proof. The proof of this theorem is similar to the proof of Theorem 2.15 and follows by Corollary 2.10 and the fact that $C_{1t}(Y, Z) \in A$ (see [12]). \qed

3. ON DUAL TOPOLOGIES

Note 3. Let $Y$ and $Z$ be two fixed topological spaces. By $O_Z(Y)$ we denote the set

$$\{f^{-1}(U) : f \in C(Y, Z) \text{ and } U \in O(Z)\}.$$ 

Let $\mathcal{H} \subseteq O_Z(Y)$, $\mathcal{H} \subseteq C(Y, Z)$, and $U \in O(Z)$. We set

$$(\mathcal{H}, U) = \{f \in C(Y, Z) : f^{-1}(U) \in \mathcal{H}\}$$

and

$$(\mathcal{H}, U) = \{f^{-1}(U) : f \in \mathcal{H}\}.$$ 

Definition 3.1. (See [9]) Let $\tau$ be a topology on $O_Z(Y)$. The topology on $C(Y, Z)$, for which the set

$$\{(\mathcal{H}, U) : \mathcal{H} \in \tau, U \in O(Z)\}$$

is a subbasis, is called dual to $\tau$ and is denoted by $t(\tau)$.

Now, let $t$ be a topology on $C(Y, Z)$. The topology on $O_Z(Y)$, for which the set

$$\{(\mathcal{H}, U) : \mathcal{H} \in t, U \in O(Z)\}$$

is a subbasis, is called dual to $t$ and is denoted by $\tau(t)$.

We observe that if $\tau$ is a topology on $O_Z(Y)$ and $\sigma$ a subbasis for $\tau$, then the set $\{(\mathcal{H}, U) : \mathcal{H} \in \sigma, U \in O(Z)\}$ is a subbasis for $t(\tau)$ (see Lemma 2.5 in [9]). Also, if $t$ is a topology on $C(Y, Z)$ and $s$ a subbasis for $t$, then the set $\{(\mathcal{H}, U) : \mathcal{H} \in s, U \in O(Z)\}$ is a subbasis for $\tau(t)$ (see Lemma 2.6 in [9]).

Note 4. Let $X$ be a space and $g : X \times Y \to Z$ a continuous map. If $g_x : Y \to Z$ is the map for which $g_x(y) = g(x, y)$, for every $y \in Y$, then by $\overline{\tau}$ we denote the map of $X \times O(Z)$ into $O_Z(Y)$, for which $\overline{\tau}(x, U) = g_x^{-1}(U)$ for every $x \in X$ and $U \in O(Z)$.

Now, let $h : X \to C(Y, Z)$ be a map. By $\overline{h}$ we denote the map of $X \times O(Z)$ into $O_Z(Y)$, for which $\overline{h}(x, U) = (h(x))^{-1}(U)$ for every $x \in X$ and $U \in O(Z)$.

Definition 3.2. Let $\tau$ be a topology on $O_Z(Y)$. We say that a map $M : X \times O(Z) \to O_Z(Y)$ is continuous with respect to the first variable if for every fixed element $U$ of $O(Z)$, the map $M_U : X \to (O_Z(Y), \tau)$, for which $M_U(x) = M(x, U)$ for every $x \in X$, is continuous.

Definition 3.3. A topology $\tau$ on $O_Z(Y)$ is called $(A, A_0)$-splitting if for every $(X, X_0) \in (A, A_0)$ the continuity of a map $g : X \times Y \to Z$ implies the continuity with respect to the first variable of the map $\overline{\tau}_{X_0 \times O(Z)} : X_0 \times O(Z) \to (O_Z(Y), \tau)$.

A topology $\tau$ on $O_Z(Y)$ is called $(A, A_0)$-admissible if for every $(X, X_0) \in (A, A_0)$ and for every map $h : X \to C(Y, Z)$ the continuity with respect to the first variable of the map $\overline{h} : X \times O(Z) \to (O_Z(Y), \tau)$ implies the continuity of
A topology \( \tau \) on \( \mathcal{O}_Z(Y) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting if and only if the topology \( t(\tau) \) on \( C(Y,Z) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting.

**Proof.** Suppose that the topology \( \tau \) on \( \mathcal{O}_Z(Y) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting, that is, for every pair \((X,X_0)\) in \((\mathcal{A},\mathcal{A}_0)\) the continuity of a map \( g : X \times Y \to Z \) implies the continuity with respect to the first variable of the map

\[
\mathcal{g}|_{X_0 \times \mathcal{O}(Z)} : X_0 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau).
\]

We prove that the topology \( t(\tau) \) on \( C(Y,Z) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting. Let \((X,X_0)\) in \((\mathcal{A},\mathcal{A}_0)\) and \( g : X \times Y \to Z \) be a continuous map. We need to prove that \( g^*|_{X_0} : X_0 \to C_t(\tau)(Y,Z) \) is a continuous map.

Let \( x \in X_0 \) and \((\mathcal{H},U)\) be an open neighborhood of \((g^*|_{X_0})(x)\) in \( C_t(\tau)(Y,Z) \). We must find an open neighborhood \( V \) of \( x \) in \( X_0 \) such that \((g^*|_{X_0})(V) \subseteq (\mathcal{H},U) \). We have that \((g^*|_{X_0})(x))^{-1}(U) \in \mathcal{H} \). Since \((g^*|_{X_0})(x) = g_x \), we have \( g_x^{-1}(U) \in \mathcal{H} \). Since the map \( g^*|_{X_0} \) is continuous.

Conversely, suppose that \( t(\tau) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting. We prove that \( \tau \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting. Let \((X,X_0)\) be an element of \((\mathcal{A},\mathcal{A}_0)\) and \( g : X \times Y \to Z \) a continuous map. It is sufficient to prove that \( \mathcal{g}|_{X_0 \times \mathcal{O}(Z)} : X_0 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau) \) is continuous with respect to the first variable.

Let \( U \) be a fixed element of \( \mathcal{O}(Z) \). Consider the map \( \mathcal{g}|_{X_0 \times \mathcal{O}(Z)} : X_0 \to (\mathcal{O}_Z(Y), \tau) \). Let \( x \in X_0 \), \( \mathcal{H} \in \tau \), and \( \mathcal{g}|_{X_0 \times \mathcal{O}(Z)}(x) = g_x^{-1}(U) \in \mathcal{H} \). We need to find an open neighborhood \( V \) of \( x \) in \( X_0 \) such that \( \mathcal{g}|_{X_0 \times \mathcal{O}(Z)}(V) \subseteq \mathcal{H} \).

Consider the open set \((\mathcal{H},U)\) of the space \( C_t(\tau)(Y,Z) \). Since

\[
(\mathcal{g}|_{X_0 \times \mathcal{O}(Z)})(x) = g_x^{-1}(U) \in \mathcal{H},
\]

we have \( g_x \in (\mathcal{H},U) \). Since \( t(\tau) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting, the map \( g^*|_{X_0} : X_0 \to C_t(\tau)(Y,Z) \) is continuous. Hence, there exists an open neighborhood \( V \) of \( x \) in \( X_0 \) such that \((g^*|_{X_0})(V) \subseteq (\mathcal{H},U) \). Let \( x' \in V \). Then, \((g^*|_{X_0})(x') = g_{x'} \in (\mathcal{H},U) \), that is, \( g_{x'}^{-1}(U) \in \mathcal{H} \) or \((\mathcal{g}|_{X_0 \times \mathcal{O}(Z)})(x') \in \mathcal{H} \). Thus, \( \mathcal{g}|_{X_0 \times \mathcal{O}(Z)}(V) \subseteq \mathcal{H} \), which means that the map \( \mathcal{g}|_{X_0 \times \mathcal{O}(Z)} \) is continuous. □

**Theorem 3.5.** A topology \( t \) on \( C(Y,Z) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting if and only if the topology \( \tau(t) \) on \( \mathcal{O}_Z(Y) \) is \((\mathcal{A},\mathcal{A}_0)\)-splitting.

**Proof.** The proof of this theorem is similar to the proof of Theorem 3.4. □
Example 3.6.

1. The topologies \( \tau(t_{co}) \) and \( \tau(t_{ts}) \) are \((A, A_0)\)-splitting for every pair \((A, A_0)\). This follows by the fact that the topologies \( t_{co} \) and \( t_{ts} \) are splitting and, therefore, \((A, A_0)\)-splitting.

2. Let \( Z \) be the Sierpinski space, \( \Omega(Y) \) the Scott topology, and \( \Omega_Z(Y) \) the relative topology of \( \Omega(Y) \) on \( \Omega(Z)(Y) \). Then, the topology \( \tau(\Omega_Z(Y)) \) coincides with the Isbell topology on \( C(Y, Z) \). Hence, the topology \( \tau(\Omega_Z(Y)) \) is splitting and, therefore, \((A, A_0)\)-splitting. Thus, the topology \( \tau(t(\Omega_Z(Y))) \) on \( \Omega(Z)(Y) \) is \((A, A_0)\)-splitting.

Theorem 3.7. A topology \( \tau \) on \( \Omega(Z)(Y) \) is \((A, A_0)\)-admissible if and only if the topology \( t(\tau) \) on \( C(Y, Z) \) is \((A, A_0)\)-admissible.

Proof. Suppose that the topology \( \tau \) on \( \Omega(Z)(Y) \) is \((A, A_0)\)-admissible, that is, for every space \((X, X_0) \in (A, A_0)\) and for every map \( h : X \rightarrow C(Y, Z) \) the continuity with respect to the first variable of the map \( \overline{h} : X \times \Omega(Z) \rightarrow (C(Y, Z), \tau) \) implies the continuity of the map \( h^z_{X_0 \times Y} : X_0 \times Y \rightarrow Z \). We prove that \( t(\tau) \) is \((A, A_0)\)-admissible. Let \((X, X_0) \in (A, A_0)\) and \( h : X \rightarrow C(t(\tau))(Y, Z) \) be a continuous map. It is sufficient to prove that the map \( h^z_{X_0 \times Y} \) is continuous. Clearly, it suffices to prove that the map \( \overline{h} : X \times \Omega(Z) \rightarrow (C(Y, Z), \tau) \) is continuous with respect to the first variable.

Let \( x \in X, U \in \Omega(Z) \) and \( \mathcal{H} \in \tau \) such that \( \overline{h}_U(x) = \overline{h}(x, U) = (h(x))^{-1}(U) \in \mathcal{H} \). We prove that there exists an open neighborhood \( V \) of \( x \) in \( X \) such that \( \overline{h}_V(x) \subseteq \mathcal{H} \). Consider the open set \( (\mathcal{H}, U) \) of the space \( C(t(\tau))(Y, Z) \). Then, \( h(x) \in (\mathcal{H}, U) \).

Since the map \( h : X \rightarrow C(t(\tau))(Y, Z) \) is continuous, there exists an open neighborhood \( V \) of \( x \) in \( X \) such that \( h(V) \subseteq (\mathcal{H}, U) \).

Let \( x' \in V \). Then \( h(x') \in (\mathcal{H}, U) \), that is \( (h(x'))^{-1}(U) \in \mathcal{H} \) or \( h(x') \subseteq \mathcal{H} \). Thus, \( \overline{h}_V(x') \subseteq \mathcal{H} \), which means that \( \overline{h}_V \) is continuous.

Conversely, suppose that the topology \( t(\tau) \) is \((A, A_0)\)-admissible. We prove that the topology \( \tau \) is \((A, A_0)\)-admissible. Let \((X, X_0) \) be a pair of \((A, A_0)\) and \( h : X \rightarrow C(Y, Z) \) a map such that \( \overline{h} : X \times \Omega(Z) \rightarrow (C(Y, Z), \tau) \) is continuous with respect to the first variable. We need to prove that the map \( h^z_{X_0 \times Y} : X_0 \times Y \rightarrow Z \) is continuous.

Since \( t(\tau) \) is \((A, A_0)\)-admissible, it is sufficient to prove that the map \( h : X \rightarrow C(t(\tau))(Y, Z) \) is continuous.

Let \( x \in X, U \in \Omega(Z) \), and \( \mathcal{H} \in \tau \) such that \( h(x) \in (\mathcal{H}, U) \). Then, \( (h(x))^{-1}(U) \in \mathcal{H} \). Since the map \( \overline{h}_U : X \rightarrow (C(Y, Z), \tau) \) is continuous, there exists an open neighborhood \( V \) of \( x \) in \( X \) such that \( \overline{h}_U(V) \subseteq \mathcal{H} \).

Let \( x' \in V \). Then, \( h_U(x') = (h(x'))^{-1}(U) \in \mathcal{H} \) or \( h(x') \in (\mathcal{H}, U) \). Thus, \( h(V) \subseteq (\mathcal{H}, U) \), which means that the map \( h \) is continuous.

Theorem 3.8. A topology \( t \) on \( C(Y, Z) \) is \((A, A_0)\)-admissible if and only if the topology \( \tau(t) \) on \( \Omega(Z)(Y) \) is \((A, A_0)\)-admissible.

Proof. The proof of this theorem is similar to the proof of Theorem 3.7.
Example 3.9.

1. If $Y$ is a regular locally compact space, then the topology $\tau(t_{co})$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible for every pair $(\mathcal{A}, \mathcal{A}_0)$.

2. If $Y$ is a corecompact space, then the topology $\tau(t_{Is})$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible for every pair $(\mathcal{A}, \mathcal{A}_0)$.

3. If $Y$ is a locally bounded space, then the topology $\tau(t_{Is})$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible for every pair $(\mathcal{A}, \mathcal{A}_0)$.

4. Let $\Omega(Y)$ be the Scott topology on $O(Y)$. By $\Omega(Z)$ we denote the relative topology of $\Omega(Y)$ on $\Omega(Z)$. If $Y$ is corecompact, then the topology $\Omega(Z)$ is admissible (see Corollary 3.12 of [9]) and, therefore, it is $(\mathcal{A}, \mathcal{A}_0)$-admissible. Thus, the topology $t(\Omega(Z))$ on $C(Y, Z)$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible.

Theorem 3.10. Let $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$. Then in the set $O(Z)$ there exists the greatest $(\mathcal{A}, \mathcal{A}_0)$-splitting topology.

Proof. Let $\{\tau_i : i \in I\}$ be the set of all $(\mathcal{A}, \mathcal{A}_0)$-splitting topologies on $O(Z)$. We consider the topology $\tau = \vee \{\tau_i : i \in I\}$.

It is not difficult to prove that this topology is $(\mathcal{A}, \mathcal{A}_0)$-splitting. By this fact we have that this topology is the required greatest $(\mathcal{A}, \mathcal{A}_0)$-splitting topology. □

References


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