Embedding into discretely absolutely star-Lindelöf spaces II

YAN-KUI SONG∗

ABSTRACT. A space $X$ is discretely absolutely star-Lindelöf if for every open cover $U$ of $X$ and every dense subset $D$ of $X$, there exists a countable subset $F$ of $D$ such that $F$ is discrete closed in $X$ and $St(F, U) = X$, where $St(F, U) = \bigcup\{U \in U : U \cap F \neq \emptyset\}$. We show that every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed $G_δ$-subspace.

Keywords: star-Lindelöf, absolutely star-Lindelöf, centered-Lindelöf

2000 AMS Classification: 54D20, 54G20

1. Introduction

By a space, we mean a topological space. A space $X$ is absolutely star-Lindelöf (see [1]) (discretely absolutely star-Lindelöf)(see [12, 13]) if for every open cover $U$ of $X$ and every dense subset $D$ of $X$, there exists a countable subset $F$ of $D$ such that $St(F, U) = X$ ($F$ is discrete closed in $X$ and $St(F, U) = X$, respectively), where $St(F, U) = \bigcup\{U \in U : U \cap F \neq \emptyset\}$.

A space $X$ is star-Lindelöf (see [4, 7] under different names) (discretely star-Lindelöf)(see [9, 16]) if for every open cover $U$ of $X$, there exists a countable discrete closed subset, respectively) $F$ of $X$ such that $St(F, U) = X$. It is clear that every separable space and every discretely star-Lindelöf space are star-Lindelöf as well as every space of countable extent(in particular, every countably compact space or every Lindelöf space).

A family of subsets is centered (linked) provided every finite subfamily (every two elements, respectively) has nonempty intersection and a family is called

∗The author acknowledges support from the NSF of China Grant 10571081 and Project supported by the National Science Foundation of Jiangsu Higher Education Institutions of China (Grant No 07KJB110055)
\(\sigma\)-centered (\(\sigma\)-linked) if it is the union of countably many centered subfamilies (linked subfamilies, respectively). A space \(X\) is centered-Lindelöf (linked-Lindelöf) (see [2, 3]) if for every open cover \(U\) of \(X\) has \(\sigma\)-centered (\(\sigma\)-linked) subcover.

From the above definitions, it is not difficult to see that every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf, every discretely absolutely star-Lindelöf space is discretely star-Lindelöf, every absolutely star-Lindelöf space is star-Lindelöf, every star-Lindelöf space is centered-Lindelöf, every centered-Lindelöf space is linked-Lindelöf.

Bonanzinga and Matveev [2] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed subspace in a Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. They asked if every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed \(G_\delta\)-subspace in a Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. The author [10] gave a positive answer to their question. The author [10] showed that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented as a closed \(G_\delta\)-subspace in a Hausdorff (regular, Tychonoff, respectively) absolutely star-Lindelöf space. The author [13] showed that every separable Hausdorff (regular, Tychonoff, normal) star-Lindelöf space can be represented in a Hausdorff (regular, Tychonoff, normal, respectively) discretely absolutely star-Lindelöf space as a closed \(G_\delta\)-subspace. The author [14] showed that every Hausdorff linked-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed subspace and asked the following question:

**Question 1.1.** Is it true that every Hausdorff (regular, Tychonoff) linked-Lindelöf-space can be represented a closed \(G_\delta\)-subspace in a Hausdorff (regular, Tychonoff, respectively) discretely absolutely star-Lindelöf space?

The purpose of this note is to give a construction showing every Hausdorff linked-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed \(G_\delta\)-subspace, which give a positive answer to the above question in the class of Hausdorff spaces.

Throughout this paper, the cardinality of a set \(A\) is denoted by \(|A|\). Let \(\omega\) denote the first infinite cardinal. For a cardinal \(\kappa\), let \(\kappa^+\) be the smallest cardinal greater than \(\kappa\). As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each pair of ordinals \(\alpha, \beta\) with \(\alpha < \beta\), we write \([\alpha, \beta]\) = \(\{\gamma : \alpha \leq \gamma \leq \beta\}\) and \((\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}\). Other terms and symbols that we do not define will be used as in [5].

2. **Embedding into discretely absolutely star-Lindelöf spaces as a closed \(G_\delta\)-subspaces**

First, we show that every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed \(G_\delta\)-subspace.
Recall the Alexandorff duplicate $A(X)$ of a space $X$. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $(x, 0) \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{(x, 1)\})$, where $U$ is a neighborhood of $x$ in $X$. It is well-known that $A(X)$ is Hausdorff (regular, Tychonoff, normal) iff $X$ is, $A(X)$ is compact iff $X$ is and $A(X)$ is Lindelöf iff $X$ is.

Recall from [6] that a space $X$ is absolutely countably compact (=acc) if for every open cover $U$ of $X$ and every dense subset $D$ of $X$, there exists a finite subset $F$ of $D$ such that $St(F, U) = X$. It is not difficult to show that every Hausdorff acc space is countably compact (see [6]). In our construction, we use the following lemma.

**Lemma 2.1** ([8, 15]). If $X$ is countably compact, then $A(X)$ is acc. Moreover, for any open cover $U$ of $A(X)$, there exists a finite subset $F$ of $X \times \{1\}$ such that $A(X) \setminus St(F, U) \subseteq X \times \{0\}$ is a finite subset consisting of isolated points of $X \times \{0\}$.

**Theorem 2.2.** Every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed $G_δ$-subspace.

**Proof.** If $|X| \leq \omega$, then $X$ is separable. The author [13] showed that every separable Hausdorff (regular, Tychonoff, normal) space can be represented in Hausdorff (regular, Tychonoff, normal, respectively) discretely absolutely star-Lindelöf space as a closed $G_δ$-subspace.

Let $X$ be a star-Lindelöf space with $|X| > \omega$ and let $T$ be $X$ with the discrete topology and let $\kappa$ be the one-point Lindelöfication of $T$. Pick a cardinal $\kappa$ with $\kappa \geq |X|$. Define $S(X, \kappa) = X \cup (Y \times \kappa^+)$. We topologize $S(X, \kappa)$ as follows: $Y \times \kappa^+$ has the usual product topology and is an open subspace of $S(X, \kappa)$, and a basic neighborhood of a point $x$ of $X$ takes the form $G(U, \alpha) = U \cup (U \times (\alpha, \kappa^+))$, where $U$ is a neighborhood of $x$ in $X$ and $\alpha < \kappa^+$. Then, it is easy to see that $X$ is a closed subset of $S(X, \kappa)$ and $S(X, \kappa)$ is Hausdorff if $X$ is Hausdorff.

Let $R(X) = A(S(X, \kappa)) \setminus (X \times \{1\})$. Then, $R(X)$ is Hausdorff if $X$ is Hausdorff.

Let $P(R(X)) = ((X \times \{0\}) \times \omega) \cup (R(X) \times \omega)$ be the subspace of the product of $R(X) \times (\omega + 1)$. For each $n \in \omega$, let $X_n = (X \times \{0\}) \times \{\omega\}$ and $X_n = R(X) \times \{n\}$ for each $n \in \omega$. Then, $P(R(X)) = X_\omega \cup \cup_{n \in \omega} X_n$. 


From the construction of the topology of $\mathcal{P}(\mathcal{R}(X))$, it is not difficult to see that $X$ can be represented in $\mathcal{P}(\mathcal{R}(X))$ as a closed $G_\delta$-subspace, since $X$ is homeomorphic to $X_\kappa$, and $\mathcal{P}(\mathcal{R}(X))$ is Hausdorff if $X$ is Hausdorff.

We show that $\mathcal{P}(\mathcal{R}(X))$ is discretely absolutely star-Lindelöf. To this end, let $U$ be an open cover of $\mathcal{P}(\mathcal{R}(X))$. Without loss of generality, we assume that $U$ consists of basic open sets of $\mathcal{P}(\mathcal{R}(X))$. Let $S$ be the set of all isolated points of $\kappa^+$ and let

$$D_{n_1} = (((T \times S) \times \{0\}) \times \{n\}) \cup (((T \times \kappa^+) \times \{1\}) \times \{n\}),$$

$$D_{n_2} = ((\{\infty\} \times \kappa^+) \times \{1\}) \times \{n\} \text{ and } D_n = D_{n_1} \cup D_{n_2} \text{ for each } n \in \omega.$$ 

If we put $D = \cup_{n \in \omega} D_n$. Then, every element of $D$ is isolated in $\mathcal{P}(\mathcal{R}(X))$, and every dense subset of $\mathcal{P}(\mathcal{R}(X))$ contains $D$. Thus, it is sufficient to show that there exists a countable subset $F$ of $D$ such that

$$F \text{ is discrete closed in } \mathcal{P}(\mathcal{R}(X)) \text{ and } St(F, U) = \mathcal{P}(\mathcal{R}(X)).$$

For each $x \in X$, there exists a $U_x \in U$ such that $\langle (x, 0), \omega \rangle \in U_x$. Hence there exist $\alpha_x < \kappa^+$, $n_x \in \omega$ and an open neighborhood $V_x$ of $x$ in $X$ such that

$$\langle V_x \times \{0\} \rangle \cup \{\alpha_x, n_x\} \cup \{(\alpha_x, \kappa^+)\} \times [n_x, \omega) \subseteq U_x.$$ 

If we put $V = \{V_x : x \in X\}$, then $V$ is an open cover of $X$. For each $n \in \omega$, let $X'_n = \cup\{x : n_x = n\}$, then $X = \cup_{n \in \omega} X'_n$. For each $x' \in X \setminus X'_n$, there exists a $U_{x'} \in U$ such that $\langle (x', 0), n \rangle \in U_{x'}$. Hence, there exist $\alpha_{x'} < \kappa^+$ and an open neighborhood $V_{x'}$ of $x'$ in $X$ such that

$$\langle V_{x'} \times \{0\} \rangle \cup \{(\alpha_{x'}, \kappa^+)\} \times \{n\} \subseteq U_{x'}.$$ 

If we put $V_n = \{V_x : x \in X'_n\} \cup \{V_{x'} : x' \in X \setminus X'_n\}$.

Then, $V_n$ is an open cover of $X$. Hence, there exists a countable subset $F'_n$ of $X$ such that $X = St(F'_n, U)$, since $X$ is star-Lindelöf. If we pick $\alpha_{n_0} > \sup\{\alpha_x : x \in X'_n\}$, then $\alpha_{n_0} < \kappa^+$, since $|X| \leq \kappa$.

Let $X_{n_1} = ((X \times \{0\}) \times \{n\}) \cup (A(T \times [\alpha_{n_0}, \kappa^+) \times \{n\})$;

$X_{n_2} = A(T \times [0, \alpha_{n_0}]) \times \{n\}$ and $X_{n_3} = A(\{\infty\} \times \kappa^+) \times \{n\}$.

Then, $X_n = X_{n_1} \cup X_{n_2} \cup X_{n_3}$.

Let $F_{n_1} = (F'_n \times \{\alpha_{n_0}\}) \times \{1\} \times \{n\}$.

Then, $F_{n_1}$ is a countable subset of $D_{n_1}$ and

$$((X_n' \times \{0\}) \times \{\omega\}) \cup X_{n_1} \subseteq St(F_{n_1}, U),$$

since $U_x \cap F_{n_1} \neq \emptyset$ for each $x \in X'_n$ and $U_{x'} \cap F_{n_1} \neq \emptyset$ for each $x' \in X \setminus X'_n$. Since $F_{n_1} \subseteq D_{n_1}$ and $F_{n_1}$ is countable. Then, $F_{n_1}$ is closed in $X_n$ by the
construction of the topology of $X_n$. Hence, $F_{n1}$ is closed in $\mathcal{P}(\mathcal{R}(X))$, since $X_n$ is open and closed in $\mathcal{P}(\mathcal{R}(X))$.

On the other hand, since $Y$ is Lindelöf and $[0, \alpha_n]$ is compact, then $Y \times [0, \alpha_n]$ is Lindelöf, hence $X_{n2} = A(Y \times [0, \alpha_n]) \times \{n\}$ is Lindelöf. For each $\alpha \leq \alpha_n$, there exists a $U_{\alpha} \in \mathcal{U}$ such that

$$(\langle(\infty, \alpha), 0\rangle, n) \in U_{\alpha}.$$

Hence, there exists an open neighborhood $V_{\alpha}$ of $\alpha$ in $\kappa^+$ and an open neighborhood $V'_{\alpha}$ of $\alpha$ in $Y$ such that

$$(A(V'_n \times V_{\alpha}) \times \{n\}) \setminus (\langle(\infty, \alpha), 1\rangle, n)) \subseteq U_{\alpha}.$$

Let $\mathcal{V}_n' = \{V_{\alpha} : \alpha \leq \alpha_n\}$. Then, $\mathcal{V}_n'$ is an open cover of $[0, \alpha_n]$. Hence, there exists a finite subcover $\mathcal{V}_{n1}, \mathcal{V}_{n2}, \ldots, \mathcal{V}_{n_m}$, since $[0, \alpha_n]$ is compact. Let $T_n = \cup\{T \setminus V_{\alpha} : i \leq m\}$. Then, $T_n$ is a countable subset of $T$. For each $i \leq m$, we pick $x_i \in D_n \cap U_{\alpha_i}$. Let $F'_{n2} = \{x_i : i \leq m\}$. Then, $F'_{n2}$ is a finite subset of $D_n$ and

$$(\{\times [0, \alpha_n] \times \{0\}) \cup (A(T \setminus T_n) \times [0, \alpha_n]) \times \{n\}) \subseteq St(F'_{n2}, \mathcal{U}).$$

For each $t \in T_n$, since $\{t\} \times [0, \alpha_n]$ is compact, then $A(\{t\} \times [0, \alpha_n]) \times \{n\}$ is compact, hence there exists a finite subset $F_t$ of $D_n$ such that

$$A(\{t\} \times [0, \alpha_n]) \times \{n\} \subseteq St(F_t, \mathcal{U}).$$

Let $F_{n2} = F'_{n2} \cup F''_{n2}$. Then, $F_{n2}$ is a countable subset of $D_n$ and $F''_{n2}$ is closed in $\mathcal{P}(\mathcal{R}(X))$, since $X_n$ is open closed in $\mathcal{P}(\mathcal{R}(X))$. By the definition of $F''_{n2}$, we have

$$A(T_n \times [0, \alpha_n]) \times \{n\} \subseteq St(F''_{n2}, \mathcal{U}).$$

If we put $F_{n2} = F''_{n2}$, then $F_{n2}$ is a countable subset of $D_n$ and $F_{n2}$ is closed in $\mathcal{P}(\mathcal{R}(X))$, since $F'_{n2}$ is finite and and $F''_{n2}$ is closed in $\mathcal{P}(\mathcal{R}(X))$. By the definition of $F_{n2}$, we have

$$X_{n2} \cup (\{\times [0, \alpha_n] \times \{0\}) \times \{n\}) \subseteq St(F_{n2}, \mathcal{U}).$$

Finally, we show that there exists a finite subset $F_n$ of $D_n$ such that $X_{n3} \subseteq St(F_{n3}, \mathcal{U})$. Since $\{\infty \times \kappa^+\}$ is countably compact, then, By Lemma 2.1, $A(\{\infty \times \kappa^+\}) \times \{n\}$ is acc and there exists a finite subset $F''_{n3} \subseteq D_n$ such that

$$E_n = X_{n3} \setminus St(F''_{n3}, \mathcal{U}) \subseteq (\{\infty \times \kappa^+\}) \times \{n\}$$

and each point of $E_n$ is an isolated point of $(\{\infty \times \kappa^+\}) \times \{n\}$. For each point $x \in E_n$, there exists a $U_x \in \mathcal{U}$ such that $x \in U_x$. For each point $x \in E_n$, pick $d_x \in D_n \cap U_x$. Let $F_{n3} = \{d_x : x \in E\}$, then $F''_{n3}$ is a finite subset of $D_n$ and $E \subseteq St(F''_{n3}, \mathcal{U})$. If we put $F_{n3} = F''_{n3} \cup F''_{n3}$, then $F_{n3}$ is a finite subset of $D_n$ and

$$X_{n3} \subseteq St(F_{n3}, \mathcal{U}).$$
If we put $F_n = F_{n1} \cup F_{n2} \cup F_{n3}$, then $F_n$ is a countable subset of $D_n$ such that

$$((X_n' \times \{0\}) \times \{\omega\}) \cup X_n \subseteq St(F_n, U).$$

Since $F_{n1}$ and $F_{n2}$ are closed in $P(R(X))$, $F_{n3}$ is finite and each point of $F_n$ is isolated, then $F_n$ is discrete closed in $P(R(X))$.

Let $F = \bigcup_{n \in \omega} F_n$. Then, $F$ is a countable subset of $D$ and

$$St(F, U) = \bigcup_{n \in \omega} St(F_n, U) \supseteq \bigcup_{n \in \omega} (((X_n' \times \{0\}) \times \{\omega\}) \cup X_n) = P(R(X)).$$

Since each point of $F$ is isolated, then $F$ is discrete in $P(R(X))$. Since $F_n$ is discrete closed in $X_n$ and $X_n$ is open closed in $P(R(X))$ for each $n \in \omega$, then $F$ has not accumulation points in $R(X) \times \omega$. On the other hand, since $F$ is countable and $\kappa \geq |X| > \omega$, then every point of $X_\omega$ is not accumulation point of $F$ by the construction of the topology of $P(R(X))$. This shows that $F$ is closed in $P(R(X))$, which completes the proof. \(\square\)

Since every discretely absolutely star-Lindelöf space is discretely star-Lindelöf, the next corollary follows from Theorem 2.2.

**Corollary 2.3.** Every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely star-Lindelöf space as a closed $G_\delta$-subspace.

Since every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf, the next corollary follows from Theorem 2.2.

**Corollary 2.4.** Every Hausdorff star-Lindelöf space can be represented in a Hausdorff absolutely star-Lindelöf space as a closed $G_\delta$-subspace.

The author [10] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented a closed $G_\delta$-subspace in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. Thus, we have the next corollary.

**Corollary 2.5.** Every Hausdorff linked-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed $G_\delta$-subspace.

On the separation of Theorem 2.2, Song [14] showed that $R(X)$ is Tychonoff if $X$ is a locally-countable (i.e., each point of $X$ has a neighborhood $U$ with $|U| \leq \omega$) Tychonoff space. Thus, we have the following proposition by the construction of the topology of $P(R(X))$.

**Proposition 2.6.** If $X$ is a locally countable Tychonoff space, then $P(R(X))$ is Tychonoff.

By Theorem 2.2 and Proposition 2.6, we have the next corollary.

**Corollary 2.7.** Every locally-countable, star-Lindelöf Tychonoff space can be represented in a discretely absolutely star-Lindelöf Tychonoff space as a closed $G_\delta$-subspace.

The author [10] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented a closed $G_\delta$-subspace in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. Thus, we have the following corollary by Corollary 2.7.
Corollary 2.8. Every locally-countable, linked-Lindelöf Tychonoff space can be represented in a discretely absolutely star-Lindelöf Tychonoff space as a closed $G_δ$-subspace.

Remark 2.9. In Theorem 2.2, even if $X$ is locally-countable normal, $\mathcal{R}(X)$ need not be normal (hence, $\mathcal{P}(\mathcal{R}(X))$ need not be normal). Indeed, $X \times \{0\}$ and $A(\{\infty\} \times \kappa^+)$ are disjoint closed subsets of $\mathcal{R}(X)$ that can not be separated by disjoint open subsets of $\mathcal{R}(X)$. Thus, the author does not know if every locally countable, normal star-Lindelöf space can be represented in a normal discretely absolutely star-Lindelöf space as a closed $G_δ$-subspace.

Remark 2.10. The author does not know if every regular (Tychonoff, normal) star-Lindelöf space can be represented in a regular (Tychonoff, normal, respectively) discretely absolutely star-Lindelöf space as a closed subspace or as a closed $G_δ$-subspace.

References

YAN-KUI SONG (songyankuinjnu.edu.cn)
Institute of Mathematics, School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing, 210097, P. R. China