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Fixed point of Lipschitz type mappings

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Abstract

In this paper, we prove some fixed point theorems for Lipschitz type mappings in the setting of metric spaces. Our results open up the unexplored area of fixed points of Lipschitz type mappings for investigation.

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1. Introduction and Preliminaries

In 2021, Popescu [9] introduced a new class of Picard operators, namely, the Górnicki mappings, which extends the notion of enriched contractions [2].

The following result was obtained in [9]:

Theorem 1.1. *Suppose that (X, d) is a complete metric space and $T : X \to X$ is a mapping satisfying*

$$d(Tx, Ty) \le M[d(x, y) + d(x, Tx) + d(y, Ty)], \tag{1.1}$$

where $M \in [0, 1)$ and the following condition:
(***C***) *Assume that there exist real constants a, b with $a \in [0, 1)$ and $b > 0$ such that for arbitrary $x \in X$ there exists $u \in X$ satisfying*

(i) $d(u, Tu) \le ad(x, Tx)$;
(ii) $d(u, x) \le bd(x, Tx)$.

Then T has a fixed point.

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It is pertinent to mention here that Popescu augmented (1.1) with the condition (**C**) above and named it as "Górnicki mappings". Condition (**C**) was first appeared in Górnicki [5] in the study of fixed points of Lipschitz mappings.

The following example suggests that (1.1) may contain the class of non-expansive mappings:

Example 1.2. Let $X = [-1, 1]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in (-1, 1); \\ -x, & \text{if } x \in \{-1, 1\}. \end{cases}$$

Then T satisfies (1.1) for each M satisfying $1/2 \leq M < 1$. Also, T satisfies the following generalized non-expansive condition

$$d(Tx, Ty) \leq \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

However, T does not satisfy the contractive condition

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

In [3], the author showed the significance of the class of Górnicki mappings. It can easily be observed that condition (**C**) provides a natural setting for a sequence $\{x_n\} \subset X$ to be a Cauchy sequence. Setting $M = 1$ in (1.1), we get the triangle inequality, i.e., $d(Tx, Ty) \leq d(Tx, x) + d(x, y) + d(y, Ty)$. In this case, since the triangle inequality always holds in a metric space assumption (1.1) with $M = 1$ does not subject to mapping to any condition and, therefore, the existence of a fixed point is not guaranteed. However, a fixed point is guaranteed if we assume the mapping to be continuous [3] or some other weaker properties [7].

In the next section, we show that if we replace (1.1) by a weaker continuity assumption, then the existence of fixed point is still guaranteed. If T is a self-mapping of a metric space (X, d) then the set $O(x, T) = \{T^n x \mid n = 0, 1, 2, \ldots\}$ is called the orbit of T at x and T is called orbitally continuous [4] if $z = \lim_{i\to\infty} T^{m_i} x$ implies $Tz = \lim_{i\to\infty} TT^{m_i} x$. Every continuous self-mapping is orbitally continuous, but not conversely.

2. Main results

Theorem 2.1. *Suppose that (X, d) is a complete metric space and $T : X \to X$ is an orbitally continuous mapping satisfying (**C**). Then T has a fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point. Then following the arguments given in [9], one can easily show that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Suppose that T is orbital continuous. Orbital continuity of T implies that $\lim_{n\to\infty} Tx_n = Tz$. This yields $Tz = z$, that is, z is a fixed point of T. □

The following example illustrates Theorem 2.1.

Example 2.2. Let $X = [-2, 10]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = \begin{cases} -8x - 6, & \text{if } x \in [-2, -1]; \\ x/2, & \text{if } x \in (-1, 10]. \end{cases}$$

Then for any $x \in X$, there exists $y = 0$ such that (i) and (ii) of (**C**) hold for $a = 1/2$ and $b = 2$ [7]. Furthermore, since the orbit of T at $x = 0$ is constantly 0 and T is orbitally continuous at $x = 0$, we can conclude that T has a unique fixed point at $x = 0$.

The following classes of functions were studied in [7]. Let $f : X \to [0, \infty)$ be a function. Let $\mathcal{A}$ denote the class of functions $\alpha : X \times X \to [0, \infty)$ satisfying: for any sequence $\{x_n\} \subset X$, if $\{fx_n\}$ converges, then $\limsup_{n\to\infty} \alpha(x_n, x_{n+1}) < 1$. Let $\mathcal{K}$ denote the class of functions $\kappa : X \times X \to [0, \infty)$ satisfying: for any sequence $\{x_n\} \subset X$, if $\{fx_n\}$ is a non-increasing sequence converging to 0, then the sequence $\{\kappa(x_n, x_{n+1})\}$ is bounded. Let $\mathcal{L}$ denote the class of functions $l : X \times X \to [0, \infty)$ satisfying: for any sequence $\{x_n\} \subset X$, if $\{fx_n\}$ is a non-increasing sequence converging to 0, then the sequence $\{l(x_n, x_{n+1})\}$ is bounded below away from zero. Let $\chi : [0, \infty) \to [0, \infty)$ be a continuous function satisfying $\chi(t) < t$ for all $t > 0$.

2.1. **Fixed point results for a new class of contractive mappings.** Motivated by the applicability of Condition (**C'**) (see below [7]) in the diverse settings, we prove the following theorem which extends Theorem 4 of [9] (also Theorem 1.4 of [6]).

Theorem 2.3. *Suppose that (X, d) is a complete metric space and $T : X \to X$ is a mapping satisfying*

$$d(Tx, Ty) \le \chi([d(x, y) + d(x, Tx) + d(y, Ty)]), \tag{2.1}$$

and the following condition:
(***C'***) *Assume that there exist $\alpha \in \mathcal{A}(f), \kappa \in \mathcal{K}(f)$ and $l \in \mathcal{L}(f)$, such that for each $x \in X$, there exists $y \in X$ satisfying*

(i') $d(y, Ty) \le \alpha(x, y) d(x, Tx)$;
(ii') $d(x, y) \le \kappa(x, y)[d(x, Tx)]^{l(x,y)}$.

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Then following the arguments given in [7], one can easily show that $d(x_n, Tx_n) \to 0$ and $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Using (2.1), we get

$$d(Tx_n, Tz) \le \chi([d(x_n, z) + d(x_n, Tx_n) + d(z, Tz)]).$$

On letting $n \to \infty$, we obtain $z = Tz$ and z is a fixed point of T. Uniqueness of the fixed point follows from (2.1). □

Replacing $\chi(t), \alpha(x,y), \kappa(x,y)$ and $l(x,y)$ in Theorem 2.3 by suitable functions, we get the following corollary:

Corollary 2.4. *Suppose that (X,d) is a complete metric space and $T : X \to X$ is a mapping satisfying*

$$d(Tx,Ty) \leq M[d(x,y) + d(x,Tx) + d(y,Ty)], 0 \leq M < 1. \tag{2.2}$$

Further assume that there exist $a \in (0,1), b > 0$ and $l > 0$ such that for each $x \in X$, there exists $y \in X$ satisfying $d(y,Ty) \leq ad(x,Tx)$ and $d(x,y) \leq b[d(x,Tx)]^l$. Then T has a unique fixed point.

On the other hand, Nguyen and Tram [7] followed a different approach. They established the existence of fixed points of a mapping T satisfying Condition **(C')** in a complete metric (X,d) under the assumption that the function $x \to d(x,Tx)$ has property (L). It is important to note that property (L) is weaker than lower semi-continuity property [7].

The following property [1] may be used as an alternate to the notion of orbital continuity.

$$\text{If } \; x_n \to z, \text{then } \; d(z,Tz) = 0; \text{here} \lim_{n\to\infty} d(x_n,Tx_n) = 0. \tag{2.3}$$

The following result is also valid if we replace (2.1) of Theorem 2.3 by (2.3).

Theorem 2.5. *Suppose that (X,d) is a complete metric space and $T : X \to X$ is a mapping satisfying Condition (**C'**) and (2.3). Then T has a fixed point.*

Remark 2.6. Let $f(y) = d(y,Ty)$ and assume f is lower semicontinuous. Then (2.3) holds since

$$0 \leq d(z,Tz) \leq \liminf d(x_n,Tx_n) = 0,$$

which implies that $d(z,Tz) = 0$ (since $x_n \to z$) [1].

2.2. **Fixed point results for Lipschitz mappings.** Lipschitz type mappings in fixed point theory constitutes a very important class of mapping and include contraction mappings, contractive mappings and nonexpansive mappings as its subclasses. In metric fixed point theory, there is no general method for the study of fixed points of Lipschitz type mappings and this area of study has largely remained unexplored. A self-mapping T of a metric space (X,d) is said to satisfy Lipschitz condition if $d(Tx,Ty) \leq kd(x,y)$ for some $k > 0$ and T is called a generalized Lipschitz type mapping if T satisfies a condition of the form (2.4) or some other condition of similar form [8]. In this paper, our results open up an unexplored area for the investigation of fixed points of Lipschitz type mappings.

In the next result, we consider the following classes of functions: Let $\psi : [0,\infty) \to [0,\infty)$ be continuous and $\psi(0) = 0$. Let $\chi : [0,\infty) \to [0,\infty)$ be continuous satisfying $\chi(t) < t$ for all $t > 0$.

Theorem 2.7. *Suppose that (X, d) is a complete metric space and $T : X \to X$ is a mapping satisfying Condition (**C'**) and*

$$d(Tx, Ty) \leq \psi(d(x, y)) + \chi(d(x, Tx) + d(y, Ty)). \tag{2.4}$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Then following the arguments given in [7], one can easily show that $d(x_n, Tx_n) \to 0$ and $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Using (2.4), we get

$$d(Tx_n, Tz) \leq \psi(d(x_n, z)) + \chi(d(x_n, Tx_n) + d(z, Tz)).$$

On letting $n \to \infty$, we get

$$d(z, Tz) \leq \psi(0) + \chi(d(z, Tz)) < d(z, Tz),$$

a contradiction. Hence $z = Tz$ and z is a fixed point of T. □

If we take $\psi(t) = Mt$, $\chi(t) = Kt, \alpha(x, y) = a, \kappa(x, y) = b$ and $l(x, y) = l$ in Theorem 2.7, then we get the following:

Corollary 2.8. *Suppose that (X, d) is a complete metric space and $T : X \to X$ is a mapping satisfying*

$$d(Tx, Ty) \leq Md(x, y) + K[d(x, Tx) + d(y, Ty)]. \tag{2.5}$$

Further assume that there exist $a, K \in (0, 1), M > 0, b > 0$ and $l > 0$ such that for each $x \in X$, there exists $y \in X$ satisfying $d(y, Ty) \leq ad(x, Tx)$ and $d(x, y) \leq b[d(x, Tx)]^l$. Then T has a fixed point.

We now give a fixed point theorem for a generalized non-expansive mapping.

Theorem 2.9. *Suppose that (X, d) is a complete metric space and $T : X \to X$ is a mapping satisfying Condition (**C'**) and*

$$d(Tx, Ty) \leq \max\{d(x, y), \frac{K'}{2}[d(x, fx) + d(y, fy)]\}, \tag{2.6}$$

where $0 \leq K' < 2$. Then T has a fixed point.

Proof. The proof is similar to Theorem 2.1. □

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