

The Alexandroff property and the preservation of strong uniform continuity

GERALD BEER

ABSTRACT. In this paper we extend the theory of strong uniform continuity and strong uniform convergence, developed in the setting of metric spaces in [13, 14], to the uniform space setting, where again the notion of shields plays a key role. Further, we display appropriate bornological/variational modifications of classical properties of Alexandroff [1] and of Bartle [7] for nets of continuous functions, that combined with pointwise convergence, yield continuity of the limit for functions between metric spaces.

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1. INTRODUCTION

In any introductory analysis course, where one studies functions between metric spaces, one observes that the pointwise limit of a sequence of continuous functions need not be continuous, whereas uniform convergence - in fact uniform convergence on compact subsets - preserves continuity. On the other hand, it is easy to construct a sequence of piecewise linear continuous real-valued functions on $[0, 1]$ pointwise convergent but not uniformly convergent to the zero function, so uniform convergence on compacta while sufficient is hardly necessary. So one is led to ask, as Arzelà first formally did [2, 3], what precisely must be added to pointwise convergence to yield continuity of the limit? For a comprehensive guide to the literature on the preservation of continuity, the reader may consult [18]. Perhaps the most satisfying add-on has been given by P. Alexandroff [1], one of the founders of general topology.

Definition 1.1. Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces and let f, f_1, f_2, f_3, \dots be a sequence of functions from X to Y . Then $\langle f_n \rangle$ is said to have the Alexandroff property with respect to f if for each $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there exists a strictly increasing sequence $\langle n_k \rangle$ of integers such that $n_1 \geq n_0$ and a countable open cover $\{V_k : k \in \mathbb{N}\}$ of X such that $\forall k \in \mathbb{N}, \forall x \in V_k$, we have $\rho(f(x), f_{n_k}(x)) < \varepsilon$.

Alexandroff of course showed that for sequences of continuous functions, in the presence of pointwise convergence to the function f , the Alexandroff property is equivalent to continuity of f . In fact, his result is valid without the metrizable assumption on the domain [1, pg. 266]. On the other hand, the Alexandroff property alone does not guarantee pointwise convergence: if $f = f_2 = f_4 = f_6 = \dots$, then $\langle f_n \rangle$ has the Alexandroff property with respect to f no matter how $\{f_1, f_3, f_5, \dots\}$ are defined. Obviously if a sequence has the Alexandroff property with respect to f , this property is not in general inherited by its subsequences.

So perhaps a more appropriate question to ask in this setting is the following: is there a topology on the set of all functions Y^X from X to Y finer than the topology of pointwise convergence that is somehow intrinsic to the preservation of continuity? That is, is there a topology on Y^X for which the set of continuous functions $\mathcal{C}(X, Y)$ is a closed subset and for which pointwise convergence in $\mathcal{C}(X, Y)$ entails convergence in this topology? Discovered forty years ago by Bouleau [16], the answer to this question also falls out of a general theory of topologies of strong uniform convergence of functions with values in a metric target space $\langle Y, \rho \rangle$ with respect to a bornology \mathcal{B} of nonempty subsets of $\langle X, d \rangle$ [13, Corollary 6.8]. Recall that a *bornology* \mathcal{B} is a cover of X by nonempty subsets that is stable under taking finite unions and under taking nonempty subsets of members of the cover [12, 13, 20]. Evidently, the largest bornology is $\mathcal{P}_0(X)$, the family of all nonempty subsets of X , whereas the smallest is $\mathcal{F}_0(X)$, the family of nonempty finite subsets of X .

Strong uniform convergence of a net $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ of functions from X to Y to $f \in Y^X$ with respect to a particular bornology \mathcal{B} is described as follows: for each $\varepsilon > 0$ and each $B \in \mathcal{B}$, there exists an index λ_0 in the underlying directed set for the net $\langle \Lambda, \succeq \rangle$ such that for each $\lambda \succeq \lambda_0$ there exists $\delta_\lambda > 0$ such that whenever $d(x, B) < \delta_\lambda$, then $\rho(f_\lambda(x), f(x)) < \varepsilon$. This notion is fundamentally variational in nature: we insist not only on uniform convergence on members of \mathcal{B} but convergence around the edges of elements of \mathcal{B} in some almost-uniform sense. Notice also that since each bornology contains the singletons, we automatically get pointwise convergence, whatever the bornology may be.

In the special case of the bornology $\mathcal{F}_0(X)$, convergence in this sense of a net of continuous functions forces continuity of the limit, and conversely, if the limit is continuous, then this sort of convergence must ensue. For a general bornology \mathcal{B} on X , strong uniform convergence is characterized by the preservation of the variational notion of strong uniform continuity of functions on members of \mathcal{B} [13, Theorem 6.7], that reduces to ordinary pointwise continuity when \mathcal{B} is a bornology of relatively compact subsets (see Definition 2.2 *infra*).

The purpose of this article is to identify the appropriate bornological Alexandroff property that corresponds to strong uniform convergence of functions with respect to an arbitrary bornology on the domain. We choose to do so for nets of functions rather than just for sequences, and we work in the more general context of Hausdorff uniform spaces rather than metric spaces, developing in the process the rudiments of the theory of strong uniform continuity and strong uniform convergence in this setting. In particular, our results apply to locally convex spaces, where one might be interested say in bornologies of weakly relatively compact sets. Falling out of our analysis is a variational-bornological form of quasi-uniform convergence as defined by Bartle [7].

2. PRELIMINARIES

Let X be a Hausdorff topological space. If \mathcal{V} is a family of subsets of X we say \mathcal{V} is a *cover* of $A \subseteq X$ provided $A \subseteq \cup \mathcal{V}$. A second family of subsets \mathcal{U} is said to *refine* \mathcal{V} if $\forall U \in \mathcal{U}, \exists V \in \mathcal{V}$ with $U \subseteq V$. If $\langle X, d \rangle$ is a metric space, we write $S_d(x, \alpha)$ for the open ball with center $x \in X$ and radius $\alpha > 0$. If $A \subseteq \langle X, d \rangle$ we put $S_d(A, \alpha) := \cup_{a \in A} S_d(a, \alpha) = \{x : d(x, A) < \alpha\}$, where $d(x, \emptyset) = \infty$ is understood.

Rephrasing, a bornology on X is a family of nonempty subsets \mathcal{B} that contains the singletons, that is stable under finite unions, and whenever $B \in \mathcal{B}$ and $\emptyset \neq B_0 \subseteq B$, then $B_0 \in \mathcal{B}$. Other bornologies of note beyond those mentioned in the Introduction are (1) the bornology of relatively compact subsets of X ; (2) the subsets $\mathcal{B}(f)$ of X on which some function f with domain X and values in a metric space is bounded; (3) for a metric space $\langle X, d \rangle$, the separable subsets of X ; (4) for a metric space $\langle X, d \rangle$, the d -bounded subsets of X ; (5) for a metric space $\langle X, d \rangle$, the d -totally bounded subsets of X ; (6) for a locally convex topological vector space, the family of subsets that are absorbed by each neighborhood of the origin [19, pg. 51]. Bornologies on a metrizable space that are the bounded subsets with respect to some admissible metric have been characterized by Hu [21], whereas those that are the totally bounded subsets with respect to some admissible metric have been characterized by Beer, Costantini and Levi [10]. By a *base* for a bornology, we mean a subfamily of the bornology that is cofinal with respect to inclusion. Each of the bornologies listed above have closed bases, and the bornology of metrically bounded sets has a countable closed base. An example of a bornology that does not have a closed base is the family of countable subsets of \mathbb{R} .

In what follows, letters in bold caps will denote diagonal uniformities. For facts and terminology about diagonal uniformities we will rely totally on the excellent general textbook by Willard [24]. If $\langle X, \mathbf{D} \rangle$ is a Hausdorff uniform space and $x_0 \in X$ and $D \in \mathbf{D}$, we write $D(x_0)$ for $\{x \in X : (x_0, x) \in D\}$. Of course, $\{D(x_0) : D \in \mathbf{D}\}$ forms a local base at x_0 for the induced topology. If $A \in \mathcal{P}_0(X)$, we call the uniform neighborhood $D(A) := \cup_{a \in A} D(a)$ an *enlargement* of A . Thus, in the metric context, each set of the form $S_d(A, \alpha)$ is an enlargement of A . Disjoint subsets A and B of X are called *asymptotic* if for each $D \in \mathbf{D}$, we have $D(A) \cap B \neq \emptyset$.

A bornology \mathcal{B} on a Hausdorff uniform space is said to be *stable under small enlargements* [12] if it contains an enlargement of each of its members. The d -bounded subsets of a metric space are always stable under small enlargements with respect to the metric uniformity. Evidently, the relatively compact sets are stable under small enlargements if and only if X is locally compact.

As is well-known, a base for a uniformity consists of its symmetric open entourages [24, pg. 241]. Another important fact for our purposes - that we regard as a folk-lemma - is now stated and proved for completeness. It generalizes the fact that each open cover of a compact metric space has a Lebesgue number (a property that actually is characteristic of the larger class of UC metric spaces [6, 8, 23]). It is the provenance of the idea of bornological uniform cover defined in Section 4.

Lemma 2.1. *Let K be a nonempty compact subset of a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$. Then if \mathcal{V} is an open cover of K , there exists an entourage D_0 such that $\{D_0(x) : x \in K\}$ refines \mathcal{V} .*

Proof. If this fails, then for each entourage D there exists $x_D \in K$ such that $D(x_D)$ is contained in no element of \mathcal{V} . Direct \mathbf{D} by reverse inclusion, i.e., $D_1 \preceq D_2$ provided $D_1 \supseteq D_2$. Then by compactness of K , the net $D \mapsto x_D$ has a cluster point p . Choose D and $V \in \mathcal{V}$ such that $D(p) \subseteq V$. Then with a symmetric D_1 chosen such that $D_1 \circ D_1 \subseteq D$, $\exists D_2 \subseteq D_1$ such that $x_{D_2} \in D_1(p)$. But then $D_2(x_{D_2}) \subseteq V$, and we have a contradiction. \square

The variational notion of strong uniform continuity of a function f on a nonempty subset of the domain, while exhaustively studied in the metric context by Beer and Levi [13, 14], appears earlier in a paper of Beer and DiConcilio [11] that characterized those metrics that give rise to the same Attouch-Wets topologies (see, e.g., [4, 5, 8, 12]) on the closed subsets of X . The definition has a straight-forward extension to the uniform setting.

Definition 2.2. *Let $\langle X, \mathbf{D} \rangle$ and $\langle Y, \mathbf{T} \rangle$ be Hausdorff uniform spaces and let $f : X \rightarrow Y$. We say that f is strongly uniformly continuous on $B \in \mathcal{P}_0(X)$ if for each entourage $T \in \mathbf{T}$, $\exists D \in \mathbf{D}$ such that whenever $b \in B$ and $x \in X$ satisfy $(b, x) \in D$, then $(f(b), f(x)) \in T$.*

Consistent with the terminology, the condition is stronger than uniform continuity of the restriction of f to B . While we rely exclusively on this formulation, the reader may prefer this more aesthetically pleasing equivalent: for each entourage $T \in \mathbf{T}$, there exists a symmetric entourage $D \in \mathbf{D}$ such that whenever $\{x, w\} \subseteq D(B)$ and $(x, w) \in D$, then $(f(x), f(w)) \in T$. From this perspective, it is easy to see that if f is strongly uniformly continuous on B , then it is strongly uniformly continuous on $\text{cl}(B)$.

We now run through some easily verified facts established in [13] in the metric context. Strong uniform continuity of f on $\{x_0\}$ is equivalent to the ordinary continuity of f at x_0 , while strong uniform continuity of f on X is equivalent to global uniform continuity. For each $f \in \mathcal{C}(X, Y)$, Lemma 2.1 easily yields the strong uniform continuity of f on each nonempty compact subset. In general, if

$f : X \rightarrow Y$, then $\mathcal{B}_f := \{B \in \mathcal{P}_0(X) : f \text{ is strongly uniformly continuous on } B\}$ while possibly empty is closed under taking nonempty subsets and finite unions and is thus a bornology if and only if $f \in \mathcal{C}(X, Y)$. Given a bornology \mathcal{B} we say $f : X \rightarrow Y$ is *strongly uniformly continuous on \mathcal{B}* provided $\mathcal{B} \subseteq \mathcal{B}_f$. We write $\mathcal{C}_{\mathcal{B}}^s(X, Y)$ for those functions in $\mathcal{C}(X, Y)$ that are strongly uniformly continuous on \mathcal{B} and $\mathcal{C}_{\mathcal{B}}(X, Y)$ for those functions in $\mathcal{C}(X, Y)$ whose restriction to each element of \mathcal{B} is uniformly continuous. The latter class usually properly contains the former (see Example 3.8 and Example 4.10 *infra*).

We will be looking at topologies of uniform convergence and strong uniform convergence on Y^X and their induced relative topologies on $\mathcal{C}(X, Y)$ where $\langle X, \mathbf{D} \rangle$ and $\langle Y, \mathbf{T} \rangle$ are Hausdorff uniform spaces. All are determined by uniformities on Y^X , and all are stronger than the topology of pointwise convergence and thus all are completely regular and Hausdorff. Let \mathcal{B} be a bornology on X . The classical *topology of uniform convergence* $\mathcal{T}_{\mathcal{B}}$ on Y^X [22] has as a base for its entourages

$$[B, T] := \{(f, g) : \forall x \in B, (f(x), g(x)) \in T\} \quad (B \in \mathcal{B}, T \in \mathbf{T}).$$

Of course, such a topology does not reference the uniformity \mathbf{D} at all and makes sense when the domain is not a uniform space. Topologies of uniform convergence for spaces of continuous linear transformations of course play a fundamental role in functional analysis, e.g., in the context of normed linear spaces, the norm topology, the weak* topology, and the bounded weak* topology all fit within this framework (see, e.g., [19]).

The *topology of strong uniform convergence* $\mathcal{T}_{\mathcal{B}}^s$ on Y^X has as a base for its entourages

$$[B, T]^s := \{(f, g) : \exists D \in \mathbf{D} \forall x \in D(B), (f(x), g(x)) \in T\} \quad (B \in \mathcal{B}, T \in \mathbf{T}).$$

These topologies are unchanged by replacing \mathcal{B} in their definitions by a base for the bornology or the given uniformities by bases for them. Often, we replace \mathbf{T} by a symmetric open base in our arguments. For $\mathcal{T}_{\mathcal{B}}^s$, we may assume without loss of generality that \mathcal{B} has a closed base, as strong uniform convergence of a net on \mathcal{B} implies strong uniform convergence on the bornology with base $\{\text{cl}(B) : B \in \mathcal{B}\}$. When restricting $\mathcal{T}_{\mathcal{B}}$ to $\mathbf{C}(X, Y)$, we may also assume that \mathcal{B} has a closed base.

3. COINCIDENCE OF FUNCTION SPACES TOPOLOGIES

Evidently, for each Hausdorff uniform codomain $\langle Y, \mathbf{T} \rangle$, we have $\mathcal{T}_{\mathcal{B}}^s$ finer than $\mathcal{T}_{\mathcal{B}}$ on Y^X (see Examples 3.7 and 3.8 *infra*). The next two results speak to when $\mathcal{T}_{\mathcal{B}}^s$ collapses to $\mathcal{T}_{\mathcal{B}}$ on Y^X and on $\mathcal{C}(X, Y)$. Our first result extends [13, Theorem 6.2] to uniform spaces.

Theorem 3.1. *Let \mathcal{B} be a bornology on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$.*

- (1) *If \mathcal{B} is stable under small enlargements, then for each Hausdorff uniform space $\langle Y, \mathbf{T} \rangle$, the standard uniformities for $\mathcal{T}_{\mathcal{B}}$ and for $\mathcal{T}_{\mathcal{B}}^s$ agree on Y^X ;*
- (2) *If $\mathcal{T}_{\mathcal{B}}^s = \mathcal{T}_{\mathcal{B}}$ on \mathbb{R}^X , then \mathcal{B} is stable under small enlargements.*

Proof. Statement (1) is obvious, for if $D(B) \in \mathcal{B}$ where $D \in \mathbf{D}$ and $B \in \mathcal{B}$, then $[D(B), T] \subseteq [B, T]^s$ for any $T \in \mathbf{T}$. For statement (2), suppose no enlargement of $B_0 \in \mathcal{B}$ again lies in \mathcal{B} . For each superset B of B_0 that lies in the bornology and each entourage D , pick $x_{D,B} \in D(B_0) \setminus B$, and let χ_B be the characteristic function of $\{x_{D,B} : D \in \mathbf{D}\}$, defined by

$$\chi_B(x) := \begin{cases} 1 & \text{if } x = x_{D,B} \text{ for some } D \in \mathbf{D} \\ 0 & \text{otherwise} \end{cases}.$$

Direct $\mathcal{B}_0 := \{B \in \mathcal{B} : B_0 \subseteq B\}$ by inclusion. Then $B \mapsto \chi_B$ is easily seen to be uniformly convergent to the zero function on elements of \mathcal{B} because whenever $B \in \mathcal{B}$ and $B_1 \supseteq B \cup B_0$, we have χ_{B_1} identically equal to zero on B . But strong uniform convergence fails, as each χ_B takes on the value 1 on each enlargement of B_0 . \square

We now come to coincidence of the function space topologies on $\mathcal{C}(X, Y)$. Our result is anticipated by recent work in the context of metric spaces [14], but different methods must be employed as sequences do not suffice. The key notion was introduced in [9] (see also [15]).

Definition 3.2. *Let \mathcal{B} be a bornology on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$. We say that \mathcal{B} is shielded from closed sets if $\forall B \in \mathcal{B}, \exists B_1 \in \mathcal{B}$ such that $B \subseteq B_1$ and each neighborhood of B_1 contains an enlargement of B .*

The superset B_1 of B in the definition is called a *shield* for B . The terminology we chose can be justified as follows: each nonempty closed set C disjoint from B_1 fails to intersect some enlargement of B , so that B_1 protects B from the closed set. Evidently, if B_1 is a shield for B , then B_1 contains $\text{cl}(B)$ and moreover is a shield for $\text{cl}(B)$. The bornology of relatively compact subsets of X is shielded from closed sets, as whenever B is relatively compact, $\text{cl}(B)$ serves as a shield for B . Evidently each bornology \mathcal{B} that is stable under small enlargements is shielded from closed sets. For a bornology shielded from closed sets that is of neither type, in $X = [0, \infty)$, for each $n \in \mathbb{N}$ with $n \geq 3$, put

$$B_n := \bigcup_{k=0}^{\infty} [k + \frac{1}{n}, k + 1 - \frac{1}{n}].$$

Our bornology \mathcal{B} will consist of all sets of the form $E \cup F$ where $F \in \mathcal{F}_0(X)$ and for some $n \geq 3, E \subseteq B_n$. Let $B = \{3\} \cup B_3$. No enlargement of B lies in the bornology and $\text{cl}(B) = B$ fails to shield itself from closed sets, as one can

easily construct a sequence in X that is asymptotic to B and does not cluster. Nevertheless, the bornology is shielded from closed sets, for if $E \cup F \in \mathcal{B}$ where $E \subseteq B_n$ and F is finite, then $B_{n+1} \cup F$ is a shield for $E \cup F$.

Theorem 3.3. *Let \mathcal{B} be a bornology having a closed base on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$.*

- (1) *If \mathcal{B} is shielded from closed sets, then for each Hausdorff uniform space $\langle Y, \mathbf{T} \rangle$, the standard uniformities for $\mathcal{T}_{\mathcal{B}}$ and for $\mathcal{T}_{\mathcal{B}}^s$ agree on $\mathcal{C}(X, Y)$;*
- (2) *If X is normal and $\mathcal{T}_{\mathcal{B}}^s = \mathcal{T}_{\mathcal{B}}$ on $\mathcal{C}(X, \mathbb{R})$, then \mathcal{B} is shielded from closed sets.*

Proof. For (1), fix $B \in \mathcal{B}$ and T an open entourage in \mathbf{T} . Let B_1 be a shield for B in \mathcal{B} . It suffices to show that $[B_1, T] \subseteq [B, T]^s$. To this end, let $(f, g) \in [B_1, T]$ be arbitrary and put $C = \{x \in X : (f(x), g(x)) \notin T\}$. If $C = \emptyset$, then $\forall x \in X, (f(x), g(x)) \in T$ and in particular $(f(x), g(x)) \in T$ for all x lying in any enlargement of B . Otherwise, since $x \mapsto (f(x), g(x))$ is continuous and $(Y \times Y) \setminus T$ is closed, C is nonempty, closed and disjoint from B_1 . Thus for some $D \in \mathbf{D}$, $C \cap D[B] = \emptyset$, which means that for each $x \in D[B]$, $(f(x), g(x)) \in T$ as required.

For (2), suppose $B_0 \in \mathcal{B}$ has no shield in \mathcal{B} . As any shield for $\text{cl}(B_0)$ is a shield for B_0 as well, we may assume that B_0 is closed. Direct $\mathcal{B}_0 := \{B \in \mathcal{B} : B_0 \subseteq B\}$ by inclusion. For each $B \in \mathcal{B}_0$, pick C_B closed and disjoint from B yet asymptotic to B_0 . Then by normality choose $f_B \in \mathcal{C}(X, [0, 1])$ with $f_B(B) = \{0\}$ and $f_B(C_B) = \{1\}$. Then if f is the zero function on X , the net $\langle f_B \rangle_{B \in \mathcal{B}_0}$ $\mathcal{T}_{\mathcal{B}}$ -converges to f , whereas $\mathcal{T}_{\mathcal{B}}^s$ -convergence fails: the uniform neighborhood

$$\begin{aligned}
 & [B_0, \{(\alpha, \beta) : |\alpha - \beta| < \frac{1}{2}\}]^s(f) \cap (\mathcal{C}(X, Y) \times \mathcal{C}(X, Y)) \\
 &= \{g \in \mathcal{C}(X, \mathbb{R}) : |f(x) - g(x)| < \frac{1}{2} \forall x \text{ in some enlargement of } B_0\}
 \end{aligned}$$

contains no f_B for $B \in \mathcal{B}_0$. □

We now characterize shielded from closed sets for a bornology with closed base in a general Hausdorff uniform space in terms of the validity of a Dini-type theorem [8, pg. 19]. This result has no antecedent in the metric context. Recall that $f : X \rightarrow \mathbb{R}$ is called *upper semicontinuous* if for each $\alpha \in \mathbb{R}$, $\{x \in X : f(x) \geq \alpha\}$ is a closed subset of X ; equivalently, its *hypograph* $\{(x, \alpha) : x \in X, \alpha \in \mathbb{R} \text{ and } \alpha \leq f(x)\}$ is a closed subset of $X \times \mathbb{R}$ [8, pg. 20].

Theorem 3.4. *Let \mathcal{B} be a bornology with closed base on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$. The following conditions are equivalent:*

- (1) *The bornology \mathcal{B} is shielded from closed sets;*
- (2) *Whenever $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is a net of upper semicontinuous functions $\mathcal{T}_{\mathcal{B}}$ -convergent from above to $f \in \mathcal{C}(X, \mathbb{R})$, then the net is $\mathcal{T}_{\mathcal{B}}^s$ -convergent to f .*

Proof. (1) \Rightarrow (2). Fix $B \in \mathcal{B}$ and let $\varepsilon > 0$. Let B_1 be a shield for B in \mathcal{B} . It suffices to show that if $\exists \lambda \in \Lambda$ such that $\forall x \in B_1$, we have $f_\lambda(x) < f(x) + \varepsilon$, then for some $D \in \mathbf{D}$ and all $x \in D(B)$, we have $f_\lambda(x) < f(x) + \varepsilon$. Put $C := \{x \in X : f_\lambda(x) \geq f(x) + \varepsilon\}$. If C is empty, we are done. Otherwise by the upper semicontinuity of $f_\lambda - f$, C is closed, nonempty and disjoint from B_1 , so for some $D \in \mathbf{D}$ we have $D(B) \cap C = \emptyset$ as required.

(2) \Rightarrow (1). Suppose $B_0 \in \mathcal{B}$ has no shield in \mathcal{B} . Without loss of generality we may assume B_0 is closed. Letting \mathcal{B}_0 and for each $B \in \mathcal{B}_0$, C_B be exactly as in the proof of Theorem 3.3, denote the characteristic function of C_B by χ_B . Since C_B is closed, χ_B is upper semicontinuous, and it is verified exactly as in the proof of the last theorem that the net $\langle \chi_B \rangle_{B_0 \subseteq B}$ is $\mathcal{T}_{\mathcal{B}}$ -convergent (from above) to the zero function but is not $\mathcal{T}_{\mathcal{B}}^s$ -convergent. \square

As another application of shields, we show that if \mathcal{B} is a bornology on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$ that is shielded from closed sets, then each function on $\langle X, \mathbf{D} \rangle$ with values in a Hausdorff uniform space $\langle Y, \mathbf{T} \rangle$ that is uniformly continuous when restricted to elements of \mathcal{B} must lie in $\mathcal{C}_{\mathcal{B}}^s(X, Y)$. Our proof bears no resemblance to the one given in the metric context [14].

Theorem 3.5. *Let $\langle X, \mathbf{D} \rangle$ and $\langle Y, \mathbf{T} \rangle$ be Hausdorff uniform spaces and let \mathcal{B} be a bornology on X that is shielded from closed sets. Suppose $f \in \mathcal{C}(X, Y)$ is uniformly continuous when restricted to each element of the bornology. Then f is strongly uniformly continuous on \mathcal{B} .*

Proof. Fix $B \in \mathcal{B}$ and let $B_1 \in \mathcal{B}$ be a shield for B . Given $T \in \mathbf{T}$, we must produce an entourage $D \in \mathbf{D}$ so that whenever $b \in B$, $x \in X$ and $x \in D(b)$, then $(f(b), f(x)) \in T$. Choose a symmetric T_1 with $T_1 \circ T_1 \subseteq T$, and then by uniform continuity of f on B_1 a symmetric entourage \overline{D} such that $\{b_1, b_2\} \subseteq B_1$ and $(b_1, b_2) \in \overline{D} \Rightarrow (f(b_1), f(b_2)) \in T_1$. Let \widehat{D} satisfy $\widehat{D} \circ \widehat{D} \subseteq \overline{D}$. Since $f \in \mathcal{C}(X, Y)$ for each $b_1 \in B_1$, \exists an open neighborhood V_{b_1} of b_1 contained in $\widehat{D}(b_1)$ such that $\forall x \in V_{b_1}$ we have $((f(x), f(b_1)) \in T_1$. Put $V = \cup_{b_1 \in B_1} V_{b_1}$. Now suppose $x \in V$ and $b \in B$ satisfy $(x, b) \in \widehat{D}$. Choosing $b_1 \in B_1$ with $x \in V_{b_1}$, we have $(b, b_1) \in \overline{D}$ so that $(f(b), f(b_1)) \in T_1$. It follows that $(f(b), f(x)) \in T$. But since B_1 is a shield for B there exists a symmetric entourage $D \subseteq \widehat{D}$ for which $D(B) \subseteq V$. This choice of D does the job. \square

Clearly, in the statement of the last theorem, we must include continuity of f as an assumption (consider the bornology of finite subsets). We also note that the converse of the last result fails in the metric context [14].

There is a uniform topology on Y^X intermediate in strength between $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}}^s$ that we wish to introduce, that makes sense when the domain X is only a topological space. With \mathcal{T} denoting the topology of X , a base for its entourages consists of these subsets of $Y^X \times Y^X$:

$$[B, T]^\square := \{(f, g) : \exists U \in \mathcal{T} \text{ such that } B \subseteq U \text{ and } \forall x \in U, (f(x), g(x)) \in T\},$$

where B runs over the bornology \mathcal{B} and T runs over \mathbf{T} . It is left to the reader to verify that the standard conditions for a base for a uniformity are satisfied [24]. We denote the induced topology by $\mathcal{T}_{\mathcal{B}}^{\square}$ in the sequel. Since $[B, T]^{\square} \subseteq [B, T]$, clearly, $\mathcal{T}_{\mathcal{B}}$ is coarser than $\mathcal{T}_{\mathcal{B}}^{\square}$. When the domain is equipped with a diagonal uniformity, we have for each entourage T and each $B \in \mathcal{B}$, $[B, T]^s \subseteq [B, T]^{\square}$, and so $\mathcal{T}_{\mathcal{B}}^{\square}$ is coarser than $\mathcal{T}_{\mathcal{B}}^s$.

The intermediate topology on Y^X for the bornology $\mathcal{F}_0(X)$ was studied by Bouleau under the name *sticking topology* [16, 17]. The reason that this intermediate topology has attracted no interest whatsoever when restricted to continuous functions is the following.

Proposition 3.6. *Let $\langle X, \mathcal{T} \rangle$ be a Hausdorff space and let $\langle Y, \mathbf{T} \rangle$ be a Hausdorff uniform space. Suppose \mathcal{B} is a bornology on X . Then the standard uniformities for $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}_{\mathcal{B}}^{\square}$ when restricted to $\mathcal{C}(X, Y)$ agree.*

Proof. Let $B \in \mathcal{B}$ and $T \in \mathbf{T}$ be given. We intend to show that

$$[B, T_0] \cap (\mathcal{C}(X, Y) \times \mathcal{C}(X, Y)) \subseteq [B, T]^{\square} \cap (\mathcal{C}(X, Y) \times \mathcal{C}(X, Y)),$$

where $T_0 \in \mathbf{T}$ is a symmetric entourage chosen such that $T_0^3 \subseteq T$. Let f and g be continuous functions with $(f, g) \in [B, T_0]$. Choosing for each $b \in B, U_b \in \mathcal{T}$ such that $\forall x \in U_b$ both $(f(b), f(x)) \in T_0$ and $(g(b), g(x)) \in T_0$, then with $U := \cup_{b \in B} U_b$, we have $\forall x \in U, (f(x), g(x)) \in T$. \square

Example 3.7. We present a pointwise $= \mathcal{T}_{\mathcal{F}_0(X)}$ -convergent sequence of real-valued continuous functions on $[0, \infty)$ that fails to be $\mathcal{T}_{\mathcal{F}_0(X)}^{\square}$ -convergent. For each $n \in \mathbb{N}$ define $g_n : [0, \infty) \rightarrow \mathbb{R}$ by

$$g_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x \leq 1/n \\ 0 & \text{otherwise} \end{cases}.$$

While $\langle g_n \rangle$ converges pointwise to the characteristic function of the origin $\chi_{\{0\}}$, the uniform distance between each g_n and the limit is one on each neighborhood of $\{0\}$.

In the next example, we show that $\mathcal{T}_{\mathcal{B}}^s$ can be properly finer than $\mathcal{T}_{\mathcal{B}}^{\square}$ even on $\mathcal{C}(X, \mathbb{R})$.

Example 3.8. In the plane \mathbb{R}^2 equipped with the usual metric, let X be the metric subspace $\{(n, 0) : n \in \mathbb{N}\} \cup \{(n, \frac{1}{n}) : n \in \mathbb{N}\}$. Since the topology of X is discrete, each real function defined on X is continuous. Let us put $E := \{(n, 0) : n \in \mathbb{N}\}$, and consider the bornology \mathcal{B} on X consisting of all subsets of the form $A \cup F$ where A is a (possibly empty) subset of E and $F \in \mathcal{F}_0(X)$. For each $n \in \mathbb{N}$, define $f_n : X \rightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x = (j, \frac{1}{j}) \text{ for some } j \leq n \\ 1 & \text{otherwise} \end{cases}.$$

Evidently, the sequence $\langle f_n \rangle$ is uniformly convergent to the characteristic function χ_E of E on elements of the bornology, and since each $B \in \mathcal{B}$ is open in X , we have $\mathcal{T}_{\mathcal{B}}^{\square}$ -convergence. But each f_n has uniform distance one from χ_E on any enlargement of E .

By Theorem 3.3, the topologies $\mathcal{T}_{\mathcal{B}}^s$ and $\mathcal{T}_{\mathcal{B}}^{\square}$ agree on $C(X, Y)$ provided the bornology is shielded from closed sets. Actually, they agree on Y^X under this assumption.

Proposition 3.9. *Let \mathcal{B} be a bornology on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$. The following conditions are equivalent:*

- (1) \mathcal{B} is shielded from closed sets;
- (2) for each Hausdorff uniform space $\langle Y, \mathbf{T} \rangle$, the natural uniformities on $\mathcal{T}_{\mathcal{B}}^{\square}$ and $\mathcal{T}_{\mathcal{B}}^s$ agree on Y^X ;
- (3) $\mathcal{T}_{\mathcal{B}}^{\square}$ and $\mathcal{T}_{\mathcal{B}}^s$ agree on \mathbb{R}^X .

Proof. (1) \Rightarrow (2). Let $B \in \mathcal{B}$ and $T \in \mathbf{T}$ be given. Choosing a shield $B_1 \in \mathcal{B}$ for B , it is evident that $[B_1, T]^{\square} \subseteq [B, T]^s$.

(2) \Rightarrow (3). This is trivial.

(3) \Rightarrow (1). We prove the contrapositive. If (1) fails, there exists $B_0 \in \mathcal{B}$ such that each superset $B \in \mathcal{B}$ has an open neighborhood V_B that contains no enlargement of B_0 . Put $\mathcal{B}_0 := \{B \in \mathcal{B} : B_0 \subseteq B\}$, and $\forall B \in \mathcal{B}_0, \forall D \in \mathbf{D}$ pick $x_{(B,D)}$ lying in $D(B_0) \setminus V_B$. Next for each $B \in \mathcal{B}_0$ write $E(B) = \{x_{(B,D)} : D \in \mathbf{D}\}$. Directing \mathcal{B}_0 by inclusion, we claim that the net $(\chi_{E(B)})_{B \in \mathcal{B}_0}$ converges in $\mathcal{T}_{\mathcal{B}}^{\square}$ to the zero function, which we denote by f .

To see this, fix $\widehat{B} \in \mathcal{B}$ and set $\widehat{B}_1 = \widehat{B} \cup B_0 \in \mathcal{B}_0$. Whenever $\widehat{B}_1 \subseteq B, \chi_{E(B)}$ is zero on V_B and hence is zero on an open neighborhood of \widehat{B} , establishing the claim.

On the other hand, the net fails to be $\mathcal{T}_{\mathcal{B}}^s$ -convergent to f , as the uniform distance between each $\chi_{E(B)}$ and f is one when the characteristic functions are restricted to any enlargement of B_0 . \square

4. THE BORNOLICAL ALEXANDROFF PROPERTY

In this section we introduce the bornological Alexandroff property, that combined with $\mathcal{T}_{\mathcal{B}}$ -convergence of nets of functions strongly uniformly continuous on \mathcal{B} , is at once necessary and sufficient for $\mathcal{T}_{\mathcal{B}}^s$ convergence, and for strong uniform continuity of the limit. Recall that an open cover of a uniform space $\langle X, \mathbf{D} \rangle$ is called a *uniform cover* [24] if for some $D \in \mathbf{D}$, the cover is refined by $\{D(x) : x \in X\}$. This naturally leads to the following definition that mixes large and small, the characteristic feature of bornological analysis.

Definition 4.1. *Let \mathcal{B} be a bornology on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$. An open cover \mathcal{V} of X is called a *bornological uniform cover* or a \mathcal{B} -*uniform cover* of X if for each $B \in \mathcal{B}$ there exists an entourage D such that $\{D(b) : b \in B\}$ refines \mathcal{V} . If in each case, D can be chosen so that $\{D(b) : b \in B\}$ refines some finite subfamily of \mathcal{V} , then the cover is called a \mathcal{B} -*finitely uniform cover**

Each uniform cover of X is a bornological uniform cover with respect to each bornology on X , and when $\mathcal{B} = \mathcal{P}_0(X)$, each bornological uniform cover is a uniform cover because $X \in \mathcal{B}$. Given a family \mathcal{A} of nonempty subsets of X , we could define the notion of \mathcal{A} -uniform cover exactly as in the above definition. But there is no loss of generality in assuming that \mathcal{A} is already a bornology - in fact a bornology with closed base - for if \mathcal{V} is a \mathcal{A} -uniform cover, then it is a \mathcal{B} -uniform cover with respect to the smallest bornology containing $\{\text{cl}(A) : A \in \mathcal{A}\}$. Similar remarks apply to bornological finitely uniform covers.

The next proposition motivates our looking at bornological uniform covers in the present context.

Proposition 4.2 (cf. [24, Theorem 36.8]). *Let \mathcal{B} be a bornology on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$ and let $\langle Y, \mathbf{T} \rangle$ be a second Hausdorff uniform space. Then $f \in \mathcal{C}(X, Y)$ is strongly uniformly continuous on \mathcal{B} if and only if for each open uniform cover \mathcal{V} of Y , $\{f^{-1}(V) : V \in \mathcal{V}\}$ is a \mathcal{B} -uniform cover of X .*

Proof. For sufficiency, let $B \in \mathcal{B}$ and $T \in \mathbf{T}$ be arbitrary, and choose a symmetric open entourage T_0 such that $T_0 \circ T_0 \subseteq T$. As $\{T_0(y) : y \in Y\}$ is a uniform open cover of Y , there exists $D \in \mathbf{D}$ such that $\{D(b) : b \in B\}$ refines $\{f^{-1}(T_0(y)) : y \in Y\}$. Now fix $b \in B$ and suppose $(b, x) \in D$. Choosing $y \in Y$ with $D(b) \subseteq f^{-1}(T_0(y))$, we have both $(f(b), y) \in T_0$ and $(y, f(x)) \in T_0$, and so $(f(b), f(x)) \in T$ as required.

For necessity let \mathcal{V} be an open uniform cover of Y , and choose an open symmetric entourage T such that $\{T(y) : y \in Y\}$ refines \mathcal{V} . Fix B in the bornology and choose by strong uniform continuity a symmetric entourage D such that whenever $(b, x) \in (B \times X) \cap D$, we have $(f(b), f(x)) \in T$. Choosing $V \in \mathcal{V}$ with $T(f(b)) \subseteq V$, we have $D(b) \subseteq f^{-1}(V)$ as required. \square

The following result is an immediate consequence of Lemma 2.1.

Proposition 4.3. *Let \mathcal{B} be a bornology on a uniform space $\langle X, \mathbf{D} \rangle$ consisting of a family of relatively compact sets. Then each open cover \mathcal{V} of X is a \mathcal{B} -finitely uniform cover.*

Before we state an appropriate modification of the Alexandroff property with respect to strong uniform convergence on bornologies, for the record, we restate the classical Alexandroff property for nets of functions defined on a Hausdorff space with values in a Hausdorff uniform space.

Definition 4.4. *Let $\langle X, \mathcal{T} \rangle$ be a Hausdorff space and let $\langle Y, \mathbf{T} \rangle$ be a Hausdorff uniform space. Let $f : X \rightarrow Y$ and let $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ be a net in Y^X . Then $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is said to have the Alexandroff property with respect to f provided for each $\lambda_0 \in \Lambda$ and $T \in \mathbf{T}$, there exists a cofinal subset Λ_0 of $\{\lambda \in \Lambda : \lambda \succeq \lambda_0\}$ and an open cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of X such that for each $\lambda \in \Lambda_0$, $\forall x \in V_\lambda$, we have $(f_\lambda(x), f(x)) \in T$.*

Definition 4.5. *Let \mathcal{B} be a bornology on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$ and let $\langle Y, \mathbf{T} \rangle$ be a second Hausdorff uniform space. Let $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ be a net in Y^X . We say that $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ has the bornological Alexandroff property with respect*

to $f : X \rightarrow Y$ and \mathcal{B} provided for each $\lambda_0 \in \Lambda$ and $T \in \mathbf{T}$, there exists a cofinal subset Λ_0 of $\{\lambda \in \Lambda : \lambda \succeq \lambda_0\}$ and a \mathcal{B} -finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ such that for each $\lambda \in \Lambda_0$, $\forall x \in V_\lambda$, we have $(f_\lambda(x), f(x)) \in T$.

In view of Proposition 4.3, we may immediately state

Proposition 4.6. *Let \mathcal{B} be a bornology of relatively compact subsets on a Hausdorff uniform space $\langle X, \mathbf{D} \rangle$ and let $\langle Y, \mathbf{T} \rangle$ be a second Hausdorff uniform space. Let $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ be a net in Y^X . Then the net has the Alexandroff property with respect to $f \in Y^X$ if and only if the net has the bornological Alexandroff property with respect to f and \mathcal{B} .*

We now come to our main theorem.

Theorem 4.7. *Let $\langle X, \mathbf{D} \rangle$ and $\langle Y, \mathbf{T} \rangle$ be Hausdorff uniform spaces. Let \mathcal{B} be a bornology on X and let $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ be a net in $\mathcal{C}_{\mathcal{B}}^s(X, Y)$ that is $\mathcal{T}_{\mathcal{B}}$ -convergent to $f : X \rightarrow Y$. The following conditions are equivalent:*

- (1) $f \in \mathcal{C}_{\mathcal{B}}^s(X, Y)$;
- (2) For each $B \in \mathcal{B}, T_0 \in \mathbf{T}$ and $\lambda_0 \in \Lambda$, there exists a finite set of indices $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ such that $\forall j \leq n, \lambda_j \succeq \lambda_0$ and an entourage $\tilde{D} \in \mathbf{D}$ such that $\forall x \in \tilde{D}(B), \exists j \in \{1, 2, \dots, n\}$ such that $(f(x), f_{\lambda_j}(x)) \in T_0$;
- (3) $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ has the bornological Alexandroff property with respect to $f : X \rightarrow Y$ and \mathcal{B} ;
- (4) $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is $\mathcal{T}_{\mathcal{B}}^s$ -convergent to f .

Proof. (1) \Rightarrow (4). Fix $T \in \mathbf{T}$ and $B \in \mathcal{B}$, and let T_0 be a symmetric entourage with $T_0^3 \subseteq T$. Fix $B \in \mathcal{B}$; by uniform convergence on B , choose $\lambda_B \in \Lambda$ such that $\forall \lambda \succeq \lambda_B, \forall b \in B$, we have $(f_\lambda(b), f(b)) \in T_0$. By strong uniform continuity of each f_λ and f on B , there exists a symmetric entourage $D_{B,\lambda} \in \mathbf{D}$ such that if $b \in B$ and $x \in X$ and $(x, b) \in D_{B,\lambda}$, we have both $(f_\lambda(x), f_\lambda(b))$ and $(f(x), f(b))$ in T_0 . It follows that

$$\forall \lambda \succeq \lambda_B, \forall x \in D_{B,\lambda}[B], (f_\lambda(x), f(x)) \in T,$$

which means that $(f_\lambda, f) \in [B, T]^s$ whenever $\lambda \succeq \lambda_B$.

(4) \Rightarrow (3). Fix $\lambda_0 \in \Lambda$ and $T \in \mathbf{T}$. We will actually produce a residual subset Λ_0 of $\{\lambda : \lambda \succeq \lambda_0\}$ and an open cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of X such that $\forall \lambda \in \Lambda_0, f_\lambda$ restricted to V_λ is T -close to f and for each $B \in \mathcal{B}$, a single index $\lambda \in \Lambda_0$ and an entourage D such that $\{D(b) : b \in B\}$ refines V_λ .

To this end, for each $B \in \mathcal{B}$, choose by strong uniform convergence an index $\lambda_B \succeq \lambda_0$ and for each $\lambda \succeq \lambda_B$ an open symmetric entourage $D_{B,\lambda}$ such that $\forall x \in D_{B,\lambda}(B), (f_\lambda(x), f(x)) \in T$. Now fix $x_0 \in X$ and for each $\lambda \succeq \lambda_{\{x_0\}}$, put

$$\mathcal{A}_\lambda := \{B \in \mathcal{B} : \lambda \succeq \lambda_B\}.$$

Note that $\forall \lambda \succeq \lambda_{\{x_0\}}$, we have $\{x_0\} \in \mathcal{A}_\lambda$ and the families \mathcal{A}_λ increase with λ . For each $\lambda \succeq \lambda_{\{x_0\}}$, put $V_\lambda := \cup_{B \in \mathcal{A}_\lambda} D_{B,\lambda}(B)$. Then

- $\{V_\lambda : \lambda \succeq \lambda_{\{x_0\}}\}$ is an open cover of X ;
- $\forall \lambda \succeq \lambda_{\{x_0\}}, \forall x \in V_\lambda, (f_\lambda(x), f(x)) \in T$;
- $\forall B \in \mathcal{B}$, we can choose an index λ such that $\lambda \succeq \lambda_{\{x_0\}}$ and $\lambda \succeq \lambda_B$, and by construction $\forall b \in B, D_{B,\lambda}(b) \subseteq V_\lambda$.

(3) \Rightarrow (2). Fix $B \in \mathcal{B}, T \in \mathbf{T}$, and $\lambda_0 \in \Lambda$. Apply the bornological Alexandroff property with respect to λ_0 and T to obtain a cofinal subset Λ_0 of $\{\lambda \in \Lambda : \lambda \succeq \lambda_0\}$ and a \mathcal{B} -finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of X such that

$$\forall \lambda \in \Lambda_0, \forall x \in V_\lambda, (f_\lambda(x), f(x)) \in T.$$

By the definition of bornological finitely uniform cover, there exists an entourage \tilde{D} and a finite set of indices $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ in Λ_0 such that $\{\tilde{D}(b) : b \in B\}$ refines $\{V_{\lambda_j} : 1 \leq j \leq n\}$. Then for each $x \in \tilde{D}(B)$ we see that x lies in some V_{λ_j} so that $(f_{\lambda_j}(x), f(x)) \in T$.

(2) \Rightarrow (1). Fix $B \in \mathcal{B}$ and $T \in \mathbf{T}$. Choose a symmetric entourage T_0 such that $T_0^3 \subseteq T$ and then by $\mathcal{T}_{\mathcal{B}}$ -convergence, an index $\lambda_0 \in \Lambda$ such that

$$\lambda \succeq \lambda_0 \Rightarrow \forall b \in B, (f_\lambda(b), f(b)) \in T_0.$$

Choose relative to B, T_0 and λ_0 the indices $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and the entourage \tilde{D} guaranteed by condition (2). By the strong uniform continuity of each f_{λ_j} on B , we can choose a symmetric entourage $D \in \mathbf{D}$ such that $D \subseteq \tilde{D}$ and whenever $b \in B$ and $(x, b) \in D$ we have

$$(f_{\lambda_j}(x), f_{\lambda_j}(b)) \in T_0 \text{ for } j = 1, 2, 3, \dots, n.$$

We intend to show that whenever $b \in B$ and $x \in X$ and $(x, b) \in D$, we have $(f(x), f(b)) \in T$. For such an x and b , choose $j \in \{1, 2, \dots, n\}$ such that $(f(x), f_{\lambda_j}(x)) \in T_0$. By construction we have these properties:

- $(f(x), f_{\lambda_j}(x)) \in T_0$;
- $(f_{\lambda_j}(x), f_{\lambda_j}(b)) \in T_0$ by strong uniform continuity of f_{λ_j} on B and the choice of D ;
- $(f_{\lambda_j}(b), f(b)) \in T_0$ by the choice of λ_0 and uniform convergence on B .

It follows that $(f(x), f(b)) \in T$ as required for the strong uniform continuity of f on B . \square

Corollary 4.8. *Let $\langle X, \mathbf{D} \rangle$ and $\langle Y, \mathbf{T} \rangle$ be Hausdorff uniform spaces. Then on $\mathcal{C}_{\mathcal{B}}^s(X, Y)$, the topology $\mathcal{T}_{\mathcal{B}}^s$ reduces to $\mathcal{T}_{\mathcal{B}}$.*

Corollary 4.9. *Let $\langle X, d \rangle$ and $\langle Y, \rho \rangle$ be metric spaces and let $\langle f_n \rangle$ be a sequence in $\mathcal{C}_{\mathcal{B}}^s(X, Y)$ $\mathcal{T}_{\mathcal{B}}$ -convergent to $f : X \rightarrow Y$. Then f is strongly uniformly continuous on \mathcal{B} if and only if each $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there exists a strictly increasing sequence $\langle n_k \rangle$ of integers such that $n_1 \geq n_0$ and a countable open cover $\{V_k : k \in \mathbb{N}\}$ of X such that $\forall k \in \mathbb{N}, \forall x \in V_k$, we have $\rho(f(x), f_{n_k}(x)) < \varepsilon$, and for each $B \in \mathcal{B}, \exists \delta > 0$ such that $\{S_d(b, \delta) : b \in B\}$ refines some finite subfamily of $\{V_k : k \in \mathbb{N}\}$.*

In [7, Definition 2.2], Bartle introduced a property that, combined with pointwise convergence of a net of continuous functions with values in a metric space $\langle Y, \rho \rangle$ to a function f , is necessary and sufficient for continuity of the limit, provided the domain is a compact Hausdorff space: $\forall \lambda_0 \in \Lambda, \forall \varepsilon > 0$, there exists a finite set of indices $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ such that $\forall j \leq n, \lambda_j \succeq \lambda_0$ and for each $x \in X, \exists j \leq n$ with $\rho(f(x), f_{\lambda_j}(x)) < \varepsilon$. Bartle called pointwise convergence plus this property *quasi-uniform convergence* which, unfortunately, is exactly what Alexandroff called pointwise convergence plus the Alexandroff property [1, pg. 265]. Condition (2) of Theorem 4.7 may be regarded as a variational-bornological version of Bartle's property, in that it requires not only that the property hold on each $B \in \mathcal{B}$ but that it almost holds around the edge of each B as well.

In the following example, we show that the implication (3) \Rightarrow (1) in Theorem 4.7 fails if in the definition of the Alexandroff property, we replace " \mathcal{B} -finitely uniform cover" by " \mathcal{B} -uniform cover". It suffices to work in the metric context and with sequences of real-valued continuous functions.

Example 4.10. We revisit the construction in Example 3.8. The reader can easily verify that each f_n is strongly uniformly continuous on \mathcal{B} , while the characteristic function χ_E of E fails to be strongly uniformly continuous on E as each enlargement of E contains infinitely many points of $X \setminus E$. For each $n \in \mathbb{N}$ put $V_n := E \cup \{(j, \frac{1}{j}) : j \leq n\}$. By construction, each f_n agrees with χ_E when restricted to V_n . By the discreteness of X , for each $n_0 \in \mathbb{N}, \mathcal{V}_{n_0} := \{V_n : n \geq n_0\}$ is an open cover of X . Notice that each \mathcal{V}_{n_0} is actually a uniform cover of X , as it is refined by all open balls of radius $\frac{1}{2}$. In particular, this makes \mathcal{V}_{n_0} a \mathcal{B} -uniform cover of X . Further, each $B \in \mathcal{B}$ has a finite subcover from \mathcal{V}_{n_0} ; in fact B will be contained in V_n for all n sufficiently large. But for each $\varepsilon > 0, \{S_d(x, \varepsilon) : x \in E\}$ fails to refine any finite subfamily of \mathcal{V}_{n_0} .

Given a net of functions $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ defined on a set X - perhaps without further structure - with values in a metric or uniform space Y that is pointwise convergent to some $f : X \rightarrow Y$, we can consider the family of nonempty subsets B of X for which $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ restricted to B has the Bartle property with respect to f . As is easily checked, the family of such sets forms a bornology. The last example shows that uniform convergence on elements of the bornology need not preserve strong uniform continuity on the bornology, and so there is no

hope that quasi-uniform convergence in the sense of Bartle on elements of the bornology can either.

When \mathcal{B} is a bornology on X that is shielded from closed sets, we have already seen that $\mathcal{C}_{\mathcal{B}}^s(X, Y) = \mathcal{C}_{\mathcal{B}}(X, Y)$, and further, $\mathcal{T}_{\mathcal{B}}^s$ reduces to $\mathcal{T}_{\mathcal{B}}$ on $\mathcal{C}(X, Y)$. Thus, if we restrict our attention to continuous functions defined on bornologies that are shielded from closed sets, we are reduced to the classical setting. For this reason, the classical function spaces based say on bornologies with a compact base or on the bornology of metrically bounded subsets [22] seem adequate when in fact they more generally are not, as described in some detail in [13].

In the particular case that the bornology is one with a compact base, then as we have noted in Section 2, the strongly uniformly continuous functions reduce to $\mathcal{C}(X, Y)$. Since the Alexandroff property for a net of continuous functions coupled with pointwise convergence gives continuity of the limit without assuming any uniform structure on the domain, one would guess that a version of Theorem 4.7 can be stated without any uniform structure on the domain. This expectation is of course enhanced by Proposition 4.6.

The proof of the following result in this direction, obtained by modifying the proof of Theorem 4.7, is left to the reader (we suggest the circuit (1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1), noting that (4) \Rightarrow (3) only requires $\mathcal{T}_{\mathcal{F}_0(X)}^{\square}$ -convergence).

Theorem 4.11 (cf. [18, Theorem 2.10]). *Let $\langle X, \mathcal{T} \rangle$ be a Hausdorff space and $\langle Y, \mathbf{T} \rangle$ be a Hausdorff uniform space. Suppose \mathcal{B} be a bornology on X with compact base, and let $\langle f_{\lambda} \rangle_{\lambda \in \Lambda}$ be a net in $\mathcal{C}(X, Y)$ $\mathcal{T}_{\mathcal{B}}$ -convergent to $f : X \rightarrow Y$. The following conditions are equivalent:*

- (1) $f \in \mathcal{C}(X, Y)$;
- (2) *For each nonempty compact subset C of X , $T_0 \in \mathbf{T}$ and $\lambda_0 \in \Lambda$, there exists a finite set of indices $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ such that $\forall j \leq n$, $\lambda_j \succeq \lambda_0$ and a neighborhood U of C such that $\forall x \in U$, $\exists j \in \{1, 2, \dots, n\}$ such that $(f(x), f_{\lambda_j}(x)) \in T$;*
- (3) $\langle f_{\lambda} \rangle_{\lambda \in \Lambda}$ has the classical Alexandroff property with respect to f ;
- (4) $\langle f_{\lambda} \rangle_{\lambda \in \Lambda}$ is $\mathcal{T}_{\mathcal{B}}^{\square}$ -convergent to f .

When $\mathcal{B} = \mathcal{F}_0(X)$ in Theorem 4.11, we see that a pointwise limit of a net of continuous functions is continuous if and only if we have $\mathcal{T}_{\mathcal{F}_0(X)}^{\square}$ -convergence [16, 17]. While we know of no reference for the equivalence of condition (2) with the preservation of continuity under pointwise convergence, it would be remarkable if this has not appeared in the literature.

By Proposition 3.6, the intermediate topology collapses to the topology of pointwise convergence for continuous functions, so it is in some sense the weakest topology finer than the topology of pointwise convergence preserving continuity. Again, when the domain is a uniform space, in view of Proposition 3.9, on Y^X , we get $\mathcal{T}_{\mathcal{F}_0(X)}^{\square} = \mathcal{T}_{\mathcal{F}_0(X)}^s$ because the bornology has a compact base and so is shielded from closed sets. This was the form in which the equivalence

of conditions (1) and (4) of Theorem 4.11 was given by Beer and Levi [13, Corollary 6.8] in the setting of metric spaces as a corollary to a general result involving the preservation of strong uniform continuity on bornologies.

REFERENCES

1. P. Alexandroff, *Einführung in die Mengenlehre und die theorie der reellen Funktionen*, Deutscher Verlag der Wissenschaften, Berlin, 1964
2. C. Arzelà, *Intorno alla continuità della somma di infinità di funzioni continue*, Rend. R. Accad. Sci. Bologna (1883-84), 79-84.
3. C. Arzelà, *Sulle serie di funzioni*, Mem. R. Accad. Sci. Ist. Bologna, serie 5 (8) (1899-1900), 131-186 and 701-744.
4. H. Attouch, R. Lucchetti, and R. Wets, *The topology of the ρ -Hausdorff distance*, Ann. Mat. Pura Appl. **160** (1991), 303-320.
5. H. Attouch and R. Wets, *Quantitative stability of variational systems: I. The epigraphical distance*, Trans. Amer. Math. Soc. **328** (1991), 695-730.
6. M. Atsugi, *Uniform continuity of continuous functions of metric spaces*, Pacific J. Math. **8** (1958), 11-16.
7. R. Bartle, *On compactness in functional analysis*, Trans. Amer. Math. Soc. **79** (1955), 35-57.
8. G. Beer, *Topologies on closed and closed convex sets*, Kluwer Acad. Publ., Dordrecht, 1993.
9. G. Beer, C. Costantini, and S. Levi, *Bornological convergence and shields*, preprint.
10. G. Beer, C. Costantini, and S. Levi, *Total boundedness in metrizable spaces*, Houston J. Math., to appear.
11. G. Beer and A. Di Concilio, *Uniform continuity on bounded sets and the Attouch-Wets topology*, Proc. Amer. Math. Soc. **112** (1991), 235-243.
12. G. Beer and S. Levi, *Pseudometrizable bornological convergence is Attouch-Wets convergence*, J. Convex Anal. **15** (2008), 439-453.
13. G. Beer and S. Levi, *Strong uniform continuity*, J. Math. Anal. Appl. **350** (2009), 568-589.
14. G. Beer and S. Levi, *Uniform continuity, uniform convergence, and shields*, Set-Valued and Variational Anal. **18** (2010), 251-275.
15. G. Beer and M. Segura, *Well-posedness, bornologies, and the structure of metric spaces*, Appl. Gen. Top. **10** (2009), 131-157.
16. N. Bouleau, *Une structure uniforme sur un espace $F(E, F)$* , Cahiers Topologie Géom. Diff., **11** (1969), 207-214.
17. N. Bouleau, *On the coarsest topology preserving continuity*, preprint, 2006.
18. A. Caserta, G. Di Maio and L. Holá, *Arzelà's theorem and strong uniform convergence on bornologies*, J. Math. Anal. Appl. **371** (2010), 384-392.
19. N. Dunford and J. Schwartz, *Linear operators part I*, Wiley Interscience, New York, 1988
20. H. Hogbe-Nlend, *Bornologies and functional analysis*, North-Holland, Amsterdam, 1977.
21. S.-T. Hu, *Boundedness in a topological space*, J. Math Pures Appl. **228** (1949), 287-320.
22. R. McCoy and I. Ntantu, *Topological properties of spaces of continuous functions*, Springer Verlag, Berlin, 1988.
23. J. Rainwater, *Spaces whose finest uniformity is metric*, Pacific J. Math **9** (1959), 567-570.
24. S. Willard, *General topology*, Addison-Wesley, Reading, MA, 1970.

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GERALD BEER (gbeer@cslanet.calstatela.edu)
Department of Mathematics, California State University Los Angeles, 5151
State University Drive, Los Angeles, California 90032, USA.