The $\varepsilon$-approximated complete invariance property

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Dedicated to my teacher and friend Prof. Dr. Gaspar Mora

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Abstract

In the present paper we introduce a generalization of the complete invariance property (CIP) for metric spaces, which we will call the $\varepsilon$-approximated complete invariance property ($\varepsilon$-ACIP). For our goals, we will use the so called degree of nondensifiability (DND) which, roughly speaking, measures (in the specified sense) the distance from a bounded metric space to its class of Peano continua. Our main result relates the $\varepsilon$-ACIP with the DND and, in particular, proves that a densifiable metric space has the $\varepsilon$-ACIP for each $\varepsilon > 0$. Also, some essentials differences between the CIP and the $\varepsilon$-ACIP are shown.

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1. Introduction

In 1973 Ward [20] introduced the following concept:

Definition 1.1. A topological space $X$ has the complete invariance property (CIP) if for every non-empty and closed $C \subset X$ there is a continuous mapping $f : X \rightarrow X$ such that $\text{Fix}(f) = C$, where $\text{Fix}(f)$ stands for the set of fixed points of $f$. 
As is mentioned in [8], some spaces known to have the CIP include n-cells, dendrites, convex subsets of Banach spaces, compact manifolds without boundary, and all compact triangulable manifolds with or without boundary.

It is convenient to recall that a Peano continuum is a compact, connected and locally connected metric space \((X, d)\), or equivalently, by the Hahn-Mazurkiewicz Theorem (see, for instance, [19, 21]), \(X\) is the continuous image of the unit interval \(I = [0, 1]\).

In [20] was asked the following:

**Has every Peano continuum the CIP?**

The answer is negative: in [8, 9] are given some examples of \(n\)-dimensional Peano continua, with \(n > 1\), that fail to have the CIP. However, for \(n = 1\) the situation is very different:

**Theorem 1.2** (Martin and Tymchatyn [10], 1980). *Every 1-dimensional Peano continuum has the CIP.*

Since the publication of the Ward’s paper, many others works have been devoted to the study and analysis of the CIP and other issues related with it, see [2, 5, 6, 7, 11, 12, 13, 22] and references therein. So, it seems that the study of the CIP problem, and its variants, is an interesting and actual topic.

On the other hand, the so called *degree of nondensifiability* (DND), explained in detail in Section 2, has been used to prove, under suitable conditions, the existence of fixed points of continuous self mappings defined into a non-empty, bounded, closed and convex subset of a Banach space (see [3] and references therein). In the present paper, for a given metric space \((X, d)\), we introduce the concept of \(\varepsilon\)-approximated complete invariance property (\(\varepsilon\)-ACIP), which generalizes the CIP one and, by using the DND, we relate in our main result (see Theorem 3.2) this novel concept with the DND of a bounded metric space. In particular, our main result proves that densifiable metric spaces (and therefore every Peano continuum) have the \(\varepsilon\)-ACIP for each \(\varepsilon > 0\).

Also, and as consequence of our main result, we derive some properties for the \(\varepsilon\)-ACIP which are not satisfied by the CIP, namely, that the \(\varepsilon\)-ACIP is preserved (in the specified sense) by the countable or finite products of bounded metric spaces or by the continuous image of a bounded metric space.

## 2. The degree of nondensifiability

In this section, and for a better comprehension of the manuscript, we recall the concepts of \(\alpha\)-dense curves and densifiable sets and also that of degree of nondensifiability. As in Section 1, \((X, d)\) will be a metric space and we denote by \(B(X)\) the class of non-empty and bounded subsets of \(X\).

In 1997 Cherruault and Mora introduced in [15] the following concepts:

**Definition 2.1.** Let \(\alpha \geq 0\) and \(B \in B(X)\). A continuous mapping \(\gamma : I \rightarrow (X, d)\) is said to be an \(\alpha\)-dense curve in \(B\) if it satisfies:

(i) \(\gamma(I) \subset B\).

(ii) For any \(x \in B\) there is \(y \in \gamma(I)\) such that \(d(x, y) \leq \alpha\).

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The class of $\alpha$-dense curves in $B$ is denoted by $\Gamma_{\alpha,B}$. The set $B$ is said to be densifiable if $\Gamma_{\alpha,B} \neq \emptyset$ for each $\alpha > 0$.

For a detailed exposition of the $\alpha$-dense curves and densifiable sets, see [1, 14, 17]. Some comments are necessary before to continue:

(I) Let us note that, given $B \in \mathcal{B}(X)$, $\Gamma_{\alpha,B} \neq \emptyset$ for each $\alpha \geq \text{Diam}(B)$, the diameter of $B$. Indeed, fixed $x_0 \in B$, the mapping $\gamma(t) = x_0$ is an $\alpha$-dense curve in $B$ for each $\alpha \geq \text{Diam}(B)$.

(II) If $B = I^n$ for some integer $n > 1$ then a 0-dense curve is, precisely, a space-filling curve (see [19]), i.e. a continuous mapping from $I$ onto $I^n$. So, we can say that the $\alpha$-dense curves are a generalization of the space-filling curves.

(III) By recalling that the Hausdorff distance between $B_1, B_2 \in \mathcal{B}(X)$ is given by

$$d_H(B_1, B_2) = \max \{ \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} d(b_1, b_2), \sup_{b_2 \in B_2} \inf_{b_1 \in B_1} d(b_1, b_2) \},$$

is clear that if $\gamma$ is an $\alpha$-dense curve in $B \in \mathcal{B}(X)$, then $d_H(B, \gamma(I)) \leq \alpha$. We also recall that $d_H$ is pseudometric, and is a metric if $X$ is complete, and a metric in the class of non-empty, bounded and closed subsets of $X$.

Next, we show some examples.

**Example 2.2** (A compact and connected but not densifiable set). Let, in the Euclidean plane, the set

$$B = \{(x, \sin(1/x)) : x \in [-1, 0) \cup (0, 1]\} \bigcup \{(0, y) : y \in [-1, 1]\}.$$

Then, given any continuous $\gamma : I \rightarrow \mathbb{R}^2$ with $\gamma(I) \subset B$, $\gamma(I)$ has to be contained is some of the three connected components of $B$. So, if $0 < \alpha < 1$, there is not an $\alpha$-dense curve in $B$, and consequently $B$ is not densifiable.

**Example 2.3** (A densifiable set without the CIP). Consider, in the Euclidean plane, the sets

$$B_1 = \{(x, \sin(1/x)) : x \in (0, 1]\}, \quad B_2 = \{(0, y) : y \in [-1, 1]\},$$

and let $B = B_1 \cup B_2$, often called the topologist’s sine. Then, is easy to prove that $B$ is densifiable. In the following lines we will show that $B$ has not the CIP.

Define the set

$$C = B \cap (I \times [-1, 0]),$$

and assume that there is a continuous $f : B \rightarrow B$ such that $f(C) = C$. As $C \cap B_1 = f(C \cap B_1) \subset f(B_1)$ and $f(B_1)$ is path-wise connected, $f(B_1) = B_1$. Hence, as $B$ is compact, we have $B = \overline{B_1} = \overline{f(B_1)} \subset \overline{f(B)} = f(B)$, where the bar stands for the closure. This means that $f$ is surjective and therefore $f(B_2) = B_2$.

So, there is a continuous surjection $\varphi : [-1, 1] \rightarrow [-1, 1]$ such that $f(0, y) = (0, \varphi(y))$ for all $y \in [-1, 1]$. Hence there exists $a \in [-1, 1]$ such that $\varphi(a) = 1$. 

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Set \( b = \varphi(1) \). As \([-1, 0]\) is the set of fixed points of \( \varphi \), we conclude that \( a \in (0, 1) \) and \( b \in [-1, 1) \).

Next, define \( \psi : [a, 1] \to [-1, 1]^2 \) as \( \psi(x) = (x, \varphi(x)) \) and denote by \( \Delta \) the diagonal of \([-1, 1]^2\). Let us note that \( \psi([a, 1]) \cap \Delta = \emptyset \) because \( \varphi \) does not have any fixed point in \([a, 1]\). Hence, \( \psi \) is a path (see the below definition) in \([-1, 1] \setminus \Delta\).

But, the set \([-1, 1] \setminus \Delta\) has two components \( \Omega_1 \) and \( \Omega_2 \) which are above and below of \( \Delta \), respectively. Then, \( \psi(a) = (a, 1) \in \Omega_1 \) and \( \psi(1) = (1, b) \in \Omega_2 \), which is contradictory. So, \( B \) does not have the CIP as claimed.

Following Willard [21], we recall that a topological space \( Y \) is said to be path-wise connected (resp. arc-wise connected) if for any \( x, y \in B \) there is a continuous (resp. a one-to-one continuous) \( f : I \to B \), often called path (resp. arc) such that \( f(0) = x \) and \( f(1) = y \). However, if \( Y \) is a Hausdorff space (and, in particular, a metric space), both concepts are equivalents (see [21, Corollary 31.6]). Here, as we work with metric spaces, for our goals is more convenient to use the term arc-wise connected.

At this point, we can state the following result (see [17]):

**Proposition 2.4.** Let \( B \in \mathcal{B}(X) \) be arc-wise connected. Then \( B \) is densifiable if, and only if, it is precompact.

Although, by the Hahn-Mazurkiewicz Theorem, \( I^n \) is a Peano continuum and in particular densifiable, the above result also demonstrates us that \( I^n \) is densifiable. Moreover, we can give an explicit expression of an \( \alpha \)-dense curve in \( I^n \), \( \gamma \), for an arbitrarily small \( \alpha > 0 \), such that \( \gamma(I) \) is also a 1-dimensional Peano continuum:

**Example 2.5** (1-dimensional Peano continua densifying \( I^n \)). Fixed \( n > 1 \), for a given integer \( k \geq 1 \) define \( \gamma_k : I \to \mathbb{R}^n \) as

\[
\gamma_k(t) = \left( t, \frac{1}{2}(1 - \cos(\pi mt)), \ldots, \frac{1}{2}(1 - \cos(\pi m^{k-1}t)) \right),
\]

for all \( t \in I \). Then, \( \gamma_k \) is a \( \frac{\sqrt{n-1}}{k} \)-dense curve in \( I^n \) (see [1, Proposition 9.5.4])

**Remark 2.6.** Other examples of \( \alpha \)-dense curves in more general subsets of \( \mathbb{R}^n \) than \( I^n \) can be found in [18].

From the concepts of \( \alpha \)-dense curves, we can define the so called degree of nondensifiability, which was introduced by Mora and Mira in [16] and analyzed in [4]:

**Definition 2.7.** Given \( B \in \mathcal{B}(X) \), we define the degree of nondensifiability, DND, of \( B \) as

\[ \phi_d(B) = \inf \{ \alpha \geq 0 : \Gamma_{\alpha,B} \neq \emptyset \}. \]

As we have pointed out above, \( \Gamma_{\alpha,B} \neq \emptyset \) for each \( \alpha \geq \text{Diam}(B) \) and therefore the DND is well defined. Also, let us note that, for a given \( B \in \mathcal{B}(X) \), \( \phi_d(B) \)
measures (in the specified sense) the distance from $B$ to the class of its Peano continua.

**Example 2.8** (see [16]). Let $B$ be the closed unit ball of a Banach space $V$, and $d$ the distance in $V$ induced by its norm. Then,

$$\phi_d(B) = \begin{cases} 0, & \text{if } V \text{ is finite dimensional} \\ 1, & \text{if } V \text{ is infinite dimensional} \end{cases}.$$ 

Some properties of the DND are given in the next result. (see [4, 16]).

**Proposition 2.9.** The DND satisfies the following:

1. If $\phi_d(B) = 0$, then $B$ is precompact. Moreover, if $B$ is precompact and arc-wise connected then $\phi_d(B) = 0$.
2. $\phi_d(B) = \phi_d(\overline{B})$, for each $B \in \mathcal{B}(X)$ where, as usual, the bar stands for the closure.

On the other hand, for our main result we will use Theorem 1.2 and the DND. So, we will need that the $\alpha$-dense curves used in the definition of the DND be 1-dimensional Peano continua. Note that, a priori, an $\alpha$-dense curve is not necessarily a 1-dimensional Peano continua: for instance, a $n$-dimensional Peano continua or, in particular, the space-filling curves in $I^n$ given in [19]. However, in the next result, we prove that the DND can be defined by means of $\alpha$-curves such that the image of $I$ under these curves be a 1-dimensional Peano continua.

**Theorem 2.10.** Given $B \in \mathcal{B}(X)$ and $\alpha > 0$, let $\Gamma^{(1)}_{\alpha,B} \subset \Gamma_{\alpha,B}$ be the class of $\alpha$-dense curves in $B$ such that $\gamma^{(1)}(I)$ is a 1-dimensional Peano continuum for all $\gamma^{(1)} \in \Gamma^{(1)}_{\alpha,B}$. By putting

$$\phi_d^{(1)}(B) = \inf \{ \alpha \geq 0 : \Gamma^{(1)}_{\alpha,B} \neq \emptyset \},$$

we have $\phi_d(B) = \phi_d^{(1)}(B)$.

**Proof.** Let $\alpha$ be such that $\alpha > \phi_d(B)$ and $\gamma : I \longrightarrow (X, d)$ an $\alpha$-dense curve in $B$. So, by the compactness of $\gamma(I)$, given any $\varepsilon > 0$ there exists a finite set \( \{y_1, \ldots, y_n\} \subset \gamma(I) \) (without loss of generality we assume $n > 1$) such that

$$B \subset \bigcup_{i=1}^{n} \overline{B}_d(y_i, \alpha + \varepsilon),$$

$\overline{B}_d(y_i, \alpha + \varepsilon)$ being the closed ball centered at $y_i$ of radius $\alpha + \varepsilon$.

As $\gamma(I)$ is a Peano continuum it is arc-wise connected (see, for instance, [21, Theorem 31.2]), for each $i = 1, \ldots, n - 1$ there exists a one-to-one continuous $h_i : I \longrightarrow \gamma(I)$ with $h_i(0) = y_i$ and $h_i(1) = y_{i+1}$. In particular, each $h_i(I)$ is a 1-dimensional Peano continuum, for $i = 1, \ldots, n$. Define, for each $i = 1, \ldots, n-1,$
the one-to-one continuous \( \tau_i : I \rightarrow \left[ \frac{i-1}{n-1}, \frac{i}{n} \right] \) as \( \tau_i(t) = \frac{i+1+t}{n-1} \) for all \( t \in I \).

Then, the mapping \( \gamma^{(1)} : I \rightarrow (X, d) \) given by

\[
\gamma^{(1)}(t) = h_i(\tau_i(t)), \quad \text{for } t \in \left[ \frac{i-1}{n-1}, \frac{i}{n} \right], \quad i = 1, \ldots, n-1,
\]

is continuous, \( \gamma^{(1)}(I) \subset \gamma(I) \subset B \) and \( \gamma^{(1)}(I) \) is a 1-dimensional Peano continuum because it is the finite union of 1-dimensional Peano continua. Also, from (2.1) we have \( \gamma^{(1)} \in \Gamma^{(1)}_{\alpha+\varepsilon, B} \). By the arbitrariness of \( \varepsilon > 0 \), we conclude that \( \phi^{(1)}_d(B) \leq \alpha \) and by the arbitrariness of \( \alpha > \phi_d(B) \), the inequality \( \phi^{(1)}_d(B) \leq \phi_d(B) \) holds.

On the other hand, if \( \gamma \in \Gamma^{(1)}_{\alpha, B} \), from the inclusion \( \Gamma^{(1)}_{\alpha, B} \subset \Gamma_{\alpha, B} \), we have \( \gamma \in \Gamma_{\alpha, B} \). Thus, \( \phi_d(B) \leq \phi^{(1)}_d(B) \) and the proof is now complete.

\[\Box\]

To conclude this section, we give a result for the DND of the product of bounded metric spaces.

**Proposition 2.11.** Let \( \Lambda \) be a finite set or \( \Lambda = \mathbb{N} \), and \((X_\lambda, d_\lambda)_{\lambda \in \Lambda}\) a family of metric spaces such that \( \text{Diam}(X_\lambda) \leq M \) for certain \( M > 0 \) and all \( \lambda \in \Lambda \). Put \( \phi^* = \sup \{ \phi_{d_\lambda}(X_\lambda) : \lambda \in \Lambda \} \), \( X^* = \prod_{\lambda \in \Lambda} X_\lambda \) and \( d^*(x, y) = \max \{ d_\lambda(x, y) : \lambda \in \Lambda \} \) if \( \Lambda \) is finite or \( d^*(x, y) = \sum_{k \geq 1} 2^{-k} d_k(x, y) \) if \( \Lambda = \mathbb{N} \), for all \( x, y \in X^* \). Then,

\[
\phi_{d^*}(X^*) \leq \phi^*.
\]

Moreover if \( \Lambda \) is finite, then the equality holds.

**Proof.** Firstly, note that \((X^*, d^*)\) is, effectively, a bounded metric space and therefore \( \phi_{d^*}(X^*) \) is well defined (in fact, \( \phi_{d^*}(X^*) \leq M \)).

Assume, \( \Lambda = \mathbb{N} \) and let \( \alpha > \phi^* \). Let, for each \( k \geq 1 \), \( \gamma_k : I \rightarrow X_k \) an \( \alpha \)-dense curve in \( X_k \). So, for each \( k \geq 1 \), given \( x_k \in X_k \) there is \( t_k \in I \) such that

\[
(2.2) \quad d_k(x_k, \gamma_k(t_k)) \leq \alpha.
\]

Let \( \omega = (\omega_k)_{k \geq 1} : I \rightarrow I^N \) be a space-filling curve (see [19, Section 7.5]). That is, \( \omega \) (and hence each coordinate function \( \omega_k \)) is continuous and \( \omega(I) = I^N \). Define \( \gamma : I \rightarrow X^* \) as

\[
\gamma(t) = (\gamma_k(\omega_k(t)))_{k \geq 1}, \quad \text{for all } t \in I.
\]

It is clear that \( \gamma \) is continuous and \( \gamma(I) \subset X^* \). Also, given \((x_k)_{k \geq 1} \subset X^* \) take \((t_k)_{k \geq 1} \subset I \) satisfying (2.2) and \( t \in I \) such that \( \omega(t) = (t_k)_{k \geq 1} \). So, we have

\[
d^*((x_k)_{k \geq 1}, \gamma(t)) = \sum_{k \geq 1} \frac{d_k(x_k, \gamma_k(\omega_k(t)))}{2^k} = \sum_{k \geq 1} \frac{d_k(x_k, \gamma_k(t_k))}{2^k} \leq \alpha.
\]

and consequently \( \gamma \) is an \( \alpha \)-dense curve in \( X^* \). Then, \( \phi_{d^*}(X^*) \leq \alpha \) and letting \( \alpha \rightarrow \phi^* \), we conclude \( \phi_{d^*}(X^*) \leq \phi^* \).
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If $\Lambda$ is finite, without loss of generality we assume $\Lambda = \{1, \ldots, n\}$ for some $n > 1$, we take $\omega = (\omega_1, \ldots, \omega_n) : I \to I^n$ a space-filling curve (again, [19]) and the proof follows in a totally analogous way that above.

Assume $\phi_d(X^*) < \phi^*$ and take $\phi_d(X^*) < \alpha < \phi^*$ and an $\alpha$-dense curve in $X^*$, put $\gamma = (\gamma_1, \ldots, \gamma_n) : I \to (X^*, d^*).$ Then, fixed $1 \leq k \leq n,$ the mapping $\gamma_k : I \to (X_k, d_k)$ is continuous and one can check straightforwardly that it is an $\alpha$-dense curve in $X_k$. But, this is not possible as $\alpha < \phi^* \leq \phi_d(X_k).$

$\square$

3. The main result

We start this section with the following definition:

**Definition 3.1.** Given $\varepsilon \geq 0$, we will say that a metric space $(X,d)$ has the $\varepsilon$-approximated complete invariance property ($\varepsilon$-ACIP) if for each non-empty and closed $C \subset X$ there is a continuous $f_\varepsilon : X \to X$ such that $d_H(C, \text{Fix}(f_\varepsilon)) \leq \varepsilon.$

The following facts are clear from the definitions:

(I) If $(X,d)$ is bounded, then $(X,d)$ has $\varepsilon$-ACIP for every $\varepsilon \geq \text{Diam}(X)$.

(II) The 0-ACIP is, precisely, the CIP. Also, the CIP implies the $\varepsilon$-ACIP for each $\varepsilon > 0$, but as we will see below, the inverse implication does not hold in general. That is to say, there are metric spaces with the $\varepsilon$-ACIP for all $\varepsilon > 0$, but such metric spaces do not have the CIP.

Now, we are ready to state and prove our main result:

**Theorem 3.2.** Let $(X,d)$ a bounded metric space. Then, $(X,d)$ has the $\varepsilon$-ACIP for each $\varepsilon > \phi_d(X)$. In particular, if $X$ is densifiable then it has the $\varepsilon$-ACIP for each $\varepsilon > 0$.

**Proof.** Let $\varepsilon$ be such that $\varepsilon > \phi(X)$. Let any $C \subset X$ non-empty and closed, and $\gamma_\varepsilon : I \to (X,d)$ and $\varepsilon$-dense curve such that $\gamma_\varepsilon(I)$ is a 1-dimensional Peano continuum. Such $\varepsilon$-dense curve exists by virtue of Theorem 2.10.

Define the set

$$G_C = \{x \in \gamma_\varepsilon(I) : d(x,c) \leq \varepsilon, \text{ for some } c \in C\} \subset X.$$

It is clear that the set $G_C$ is non-empty and closed. Thus, by Theorem 1.2, there is $f_\varepsilon : X \to X$ with $\text{Fix}(f_\varepsilon) = G_C$.

Now, let $c \in C$. As $\gamma_\varepsilon$ is an $\varepsilon$-dense curve in $X$, there is $x \in \gamma_\varepsilon(I)$ with $d(x,c) \leq \varepsilon$. Then, $x \in G_C$ and therefore $x = f_\varepsilon(x)$. So, we have $\inf_{x \in \text{Fix}(f_\varepsilon)} d(c,x) \leq \varepsilon$ and from the arbitrariness of $c \in C$, we infer

$$\sup_{c \in C} \inf_{x \in \text{Fix}(f_\varepsilon)} d(c,x) \leq \varepsilon. \tag{3.1}$$

Likewise for a given $x \in \text{Fix}(f_\varepsilon)$, as $x \in G_C$, $d(c,x) \leq \varepsilon$ for some $c \in C$. Consequently, $\inf_{c \in C} d(c,x) \leq \varepsilon$ and noticing the arbitrariness of $x \in \text{Fix}(f_\varepsilon)$

$$\sup_{x \in \text{Fix}(f_\varepsilon)} \inf_{c \in C} d(c,x) \leq \varepsilon. \tag{3.2}$$
So, from (3.1) and (3.2), we have $d_H(C, \text{Fix}(f_\varepsilon)) \leq \varepsilon$.

If $X$ is densifiable then, by the definition of the DND, $\phi_d(X) = 0$ and therefore has the $\varepsilon$-ACIP for each $\varepsilon > 0$.

An immediate consequence of the above result is the following:

**Corollary 3.3.** Every Peano continuum has the $\varepsilon$-ACIP for each $\varepsilon > 0$.

As we have said above, in general, the $\varepsilon$-ACIP for each $\varepsilon > 0$ does not imply the CIP. We illustrate this fact in the following examples.

**Example 3.4.** Let $X$ be the topologist’s sine of Example 2.3. Then, $X$ is densifiable but does not have the CIP. However, by Theorem 3.2 $X$ has the $\varepsilon$-ACIP for each $\varepsilon > 0$.

**Example 3.5.** Let $X$ be a $n$-dimensional Peano continuum without the CIP (see [8, 9]). Then, by Corollary 3.3, $X$ has the $\varepsilon$-ACIP for each $\varepsilon > 0$.

We have remarked above that if $(X, d)$ is bounded, then $(X, d)$ has $\varepsilon$-ACIP for every $\varepsilon \geq \text{Diam}(X)$. This bound can be improved by Theorem 3.2:

**Example 3.6.** Let $X$ be the set given in Example 2.2. Then, $\text{Diam}(X) = 2$ and $\phi_d(X) = 1$. So, by Theorem 3.2, $X$ has $\varepsilon$-ACIP for every $\varepsilon > 1$.

As was proved in [6], the CIP need not be preserved by self-products. However, bearing in mind Proposition 2.11 and Theorem 3.2, we have the following result for the product of bounded metric spaces:

**Corollary 3.7.** With the notation of Proposition 2.11, $(X^*, d^*)$ has the $\varepsilon$-ACIP for each $\varepsilon > \phi^*$. In particular, the finite or countable product of Peano continua has the $\varepsilon$-ACIP for each $\varepsilon > 0$.

**Example 3.8.** Let $(X, d)$ be the 1-dimensional Peano continuum given in [6, Theorem 2.2]. Then, $X \times X$ does not have the CIP. However, by Corollary 3.7, $X \times X$ has the $\varepsilon$-ACIP for each $\varepsilon > 0$.

Also, the CIP need not to be preserved by continuous mappings. Indeed, take any metric space $(X, d)$ that does not have the CIP and $\tau$ the discrete topology on $X$. Then, $(X, \tau)$ has the CIP and the identity mapping $g : (X, \tau) \rightarrow (X, d)$ is continuous. However, for the $\varepsilon$-ACIP we have the following:
Corollary 3.9. Let \((X, d)\) and \((Y, d')\) be bounded metric spaces and \(g : (X, d) \rightarrow (Y, d')\) continuous. Then \((Y, d')\) has the \(\varepsilon\)-ACIP for each \(\varepsilon > \omega_{\phi_d(X)}(g)\), where
\[
\omega_r(g) = \sup \left\{ d'(f(x), f(y)) : x, y \in X, d(x, y) \leq r \right\},
\]
is the modulus of continuity of \(g\) of order \(r\), for \(r \geq 0\).

Proof. It is immediate to check that if \(\gamma : I \rightarrow (X, d)\) is an \(\alpha\)-dense curve in \((X, d)\), then \(g \circ \gamma : I \rightarrow (Y, d')\) is a \(\omega_\alpha(g)\)-dense curve in \((Y, d')\). Therefore, we infer that \(\phi_{d'}(Y) \leq \omega_{\phi_d(X)}(g)\) and by Theorem 3.2, \((Y, d')\) has the \(\varepsilon\)-ACIP for each \(\varepsilon > \omega_{\phi_d(X)}(g)\).

On the other hand, it is important to stress that the reciprocal of Theorem 3.2 is not true in general: there are metric spaces with the \(\varepsilon\)-ACIP for all \(\varepsilon > 0\) (in fact, with the CIP) that are not densifiable:

Example 3.10. Let \(X\) be the closed unit ball of an infinite dimensional Banach space. From the comments of Section 1, \(X\) has the CIP and therefore the \(\varepsilon\)-ACIP for all \(\varepsilon > 0\). However, from Example 2.8, \(\phi_d(X) = 1\) and noticing Proposition 2.9 \(X\) is not densifiable.

So, we conclude our exposition with the following question:

If \((X, d)\) is a bounded metric space having the \(\varepsilon\)-ACIP, for some \(\varepsilon > 0\), under what conditions can we relate, in some way, \(\varepsilon\) and \(\phi_d(X)\)?

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REFERENCES