Density of $\kappa$-Box-Products and the existence of generalized independent families

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Abstract

In this paper we will prove a slight generalisation of the Hewitt-Marczewski-Pondiczery theorem (theorem 2.3 below) concerning the density of $\kappa$-box-products. With this result we will prove the existence of generalized independent families of big cardinality (corollary 2.5 below) which were introduced by Wanjun Hu.

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1. Introduction

Let $d(X)$ denote the density and $w(X)$ the weight of the topological space $X$.

Definition 1.1. Let $\mu, \kappa$ be two cardinals with $\aleph_0 \leq \kappa \leq \mu$ and $\{X_i\}_{i \in \mu}$ be a family of topological spaces.

\(\bigcap_{\kappa \leq \mu} X_i\) denotes the $\kappa$-box-product which is induced on the full cartesian product $\prod_{i \in \mu} X_i$ by the canonical base

\[ \mathfrak{B} = \left\{ \bigcap_{i \in I} \pi_i^{-1}(U_i); I \in P_{<\kappa}(\mu) \text{ and } U_i \text{ is open in } X_i \right\} \]

where $P_{<\kappa}(\mu) := \{ I \subseteq \mu; |I| < \kappa \}$.

For $\kappa = \aleph_0$ the $\kappa$-box-product is the usual Tychonoff-product [8] and for $\kappa^+ = \mu$ the $\kappa$-box-product is the full box-product mentioned by Kelley [5] and Bourbaki [1].
In this paper we will discuss the density of $\kappa$-box-products and the connection with infinite combinatorics. The classical Hewitt-Marczewski-Pondiczery theorem states:

$$d\left(\bigbox_{i \in \kappa} X_i\right) \leq \mu \text{ for all spaces } X_i \text{ with } d(X_i) \leq \mu$$

This has been proven for separable spaces by E. Marczewski [6] in 1941. In 1944 E. S. Pondiczery [7] proved a slightly weaker version for Hausdorff spaces and in 1947 E. Hewitt [3] proved the general version as stated above. In theorem 2.4 we will prove:

$$d\left(\bigbox_{i \in \kappa} X_i\right) \leq \mu^{\kappa} \text{ for all spaces } X_i \text{ with } d(X_i) \leq \mu$$

2. DENSITY OF $\kappa$-BOX-PRODUCTS

In this section we will prove a generalisation of Theorem 1 in [2]. To do so we start with the following definition and proposition:

**Definition 2.1.** Let $\kappa, \mu$ be two infinite cardinals with $\mu \geq \kappa$, $\{X_i\}_{i \in I}$ a family of topological spaces and for all $i \in I$ let $\mathcal{B}_i$ be a base of the topology on $X_i$. $W \subseteq \prod_{i \in I} X_i$ is called a $\mu$-cube if for every $i \in I$ there exists $\mathcal{W}_i \subseteq \mathcal{B}_i$ with $W = \prod_{i \in I} (\mathcal{W}_i)$.

**Proposition 2.2.** Let $X$ be a set, $\mu \geq \kappa$ two infinite cardinals, $\{X_i\}_{i \in I}$ a family of topological spaces, $\{f_i : X \rightarrow X_i\}_{i \in I}$ a family of functions and let $W$ be a subset of $\prod_{i \in I} X_i$ which is a union of $\mu$-cubes.

For every cardinal $\lambda < \kappa$ and every tuple $\langle \{x_i\}_{i \in \lambda} : \{J_i\}_{i \in \lambda} \rangle$ of families $\{x_i\}_{i \in \lambda} \subseteq X$ and $\{J_i\}_{i \in \lambda} \subseteq P(I)$, where all $J_i$ are pairwise disjoint and not empty, there exists a subset $Q \subseteq W$ of cardinality less or equal to $\mu^{\kappa}$ so that for all families $\{J_i : x_i \in J_i\}_{i \in \lambda}$ the following holds:

$$\left(W \cap \bigcap_{i \in \lambda} pr_{J_i}^{-1}(f_j(x_{J_i})) \neq \emptyset\right) \Rightarrow \left(Q \cap \bigcap_{i \in \lambda} pr_{J_i}^{-1}(f_j(x_{J_i})) \neq \emptyset\right).$$

**Proof.** For every tuple $\langle \{x_i\}_{i \in \lambda} : \{J_i\}_{i \in \lambda} \rangle$ with $|\{i \in \lambda : |J_i| > 1\}| = 0$ the claim is pretty obvious.

So we assume that the proposition is valid for cardinals less than $\nu$ and let $\langle \{x_i\}_{i \in \lambda} : \{J_i\}_{i \in \lambda} \rangle$ be a tuple with $|\{i \in \lambda : |J_i| > 1\}| = \nu$.

Without loss of generality we may assume that $|J_i| > 1$ for all $i \in \nu$ and $|J_i| = 1$ for all other $i \geq \nu$ and that there exists at least one family $\{J_i : x_i \in J_i\}_{i \in \lambda}$ with $W \cap \bigcap_{i \in \lambda} pr_{J_i}^{-1}(f_j(x_{J_i})) \neq \emptyset$.

Let $p \in W$ be an point so that $pr_j(p) \in f_j(x_i)$ for all $\nu \leq i \leq \lambda$.

Then there exists an $J \in P_{\leq \nu}(I)$ with

$$\left\{ q \in \prod_{i \in I} X_i \cap J : \forall j \in J : pr_j(q) = pr_j(p) \right\} \subseteq W.$$

We choose for all $i \in \nu$ and $j_i \in (J_i - J)$ a point $q_{j_i} \in f_j(x_i)$ and we define a point $q \in W$ as follows:
\[ \text{pr}_i(q) := \begin{cases} \text{pr}_i(p), & \text{if } i \in \{I - \bigcup_{j \in \nu}(J_i - J)\} \\ q_j, & \text{if } i = j_i \text{ and } j_i \in (J_i - J) \end{cases} \]

By the definition of \(q\) we have \(q \in (W \cap \bigcap_{i \in \lambda} \text{pr}^{-1}_i(f_i(x_i)))\) for every family \(\{j_i; j_i \in J_i\}_{i \in \lambda}\) such that for all \(i \in \nu\): \(j_i \in (J_i - J)\).

Now we have to consider families \(\{j_i; j_i \in J_i\}_{i \in \lambda}\) with \(j_i \in (J_i \cap J)\) for at least one \(i \in \lambda\).

We define
\[ \Sigma := \left\{ \{J^*_i\}_{i \in \nu} : \left[ |i \in \kappa; J^*_i = J_i\right] < \nu \land (J^*_i \neq J_i \Rightarrow J^*_i \in \pi_1(J_i \cap J)) \right\}. \]

\[ \Rightarrow |\Sigma| \leq \mu^\nu \leq \mu^\lambda \leq \mu^{<\kappa} \]

For all \(\sigma = \{J^*_i\}_{i \in \nu} \in \Sigma\) we define a family \(\{J^\sigma_i\}_{i \in \lambda}\) as follows:
\[ J^\sigma_i := \begin{cases} J^*_i, & \text{if } i \in \nu \\ J_i, & \text{if } i \geq \nu \end{cases} \]

For all these \(\{J^\sigma_i\}_{i \in \lambda}\) the proposition already holds, so we can choose a set \(Q_\sigma \subseteq W\) with \(|Q_\sigma| \leq \mu^{<\kappa}\) and for all families \(\{j_i; j_i \in J^\sigma_i\}_{i \in \lambda}\) the following holds:
\[ \left( W \cap \bigcap_{i \in \lambda} \text{pr}^{-1}_i(f_i(x_i)) \neq \emptyset \right) \Rightarrow \left( Q_\sigma \cap \bigcap_{i \in \lambda} \text{pr}^{-1}_i(f_i(x_i)) \neq \emptyset \right). \]

Let \(\sigma = \{j_i; j_i \in J_i\}_{i \in \nu}\) be a family with \(W \cap \bigcap_{i \in \lambda} \text{pr}^{-1}_i(f_i(x_i)) \neq \emptyset\).

Then \(\sigma \in \Sigma\) and \(Q_\sigma \cap \bigcap_{i \in \lambda} \text{pr}^{-1}_i(f_i(x_i)) \neq \emptyset\).

We define
\[ Q := \{q\} \cup \bigcup_{\sigma \in \Sigma} Q_\sigma \]
and because \(|Q| \leq \mu^{<\kappa}\) this is the set we were looking for. \(\square\)

**Theorem 2.3.** Let \(\kappa\) and \(\mu\) be two infinite cardinals with \(\mu \geq \kappa\) and let \(\square^\kappa_{i \in I} X_i\) be a \(\kappa\)-box-product with \(|I| \leq 2^\mu\) and \(w(X_i) \leq \mu\) for all \(i \in I\).

Then \(d(W) \leq \mu^{<\kappa}\) holds for every subset \(W \subseteq \prod_{i \in I} X_i\) which is a union of \(\mu\)-cubes.

**Proof.** Let \(|I| = 2^\nu\), so we may assume that \(|I| = 2^\mu\).

Let \(\mathcal{B}^*\) be a base of the \(\kappa\)-box-product \(\square^\kappa_{i \in \mu} D\) of the discrete space \(D = \{0; 1\}\) with \(|\mathcal{B}^*| = \mu^{<\kappa}\).

For all \(i \in 2^\mu\) let \(\mathcal{B}_i\) be a base of the topology on \(X_i\) with \(|\mathcal{B}_i| = \mu\), \(X\) be a set with \(|X| = \mu\), \(\{f_i; f_i : X \rightarrow \mathcal{B}_i\}_{i \in 2^\mu}\) be a family of surjective functions and \(\psi : 2^\mu \rightarrow \prod_{i \in \mu} D\) be a bijection. We define
\[ \Sigma := \left\{ \langle\{x_i\}_{i \in \lambda}; \{J_i\}_{i \in \lambda}\rangle : \lambda < \kappa \land \forall i, j \in \lambda: \right. \]
\[ x_i \in X \land \emptyset \neq J_i \subseteq 2^\mu \land \psi(J_i) \in \mathcal{B}^* \land (i \neq j \Rightarrow J_i \cap J_j = \emptyset) \} \]

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and choose for every $\sigma \in \Sigma$ a set $Q_\sigma \subseteq W$ with all the properties as stated in proposition 2.2. We define $Q := \bigcup_{\sigma \in \Sigma} Q_\sigma$. Because of $|\mathcal{B}| = \mu$ we have $|\Sigma| \leq \mu^{<\kappa}$ and therefore $|Q| \leq \mu^{<\kappa}$. We will now show that $Q$ is dense in $W$.

Let $O$ be a nonempty open set in $W$ and $U$ an element of the canonical base $\mathcal{B}$ of $\bigotimes_{i \in \mathbb{N}} X_i$ with $\emptyset \neq U \cap W \subseteq O$. Then there exists a set 
\[ \{ j_i; i \in \lambda \} \in \mathcal{B}_C(2^\mu) \] and a family $\{ U_i ; U_i \in \mathcal{B}_i \}_{i \in \lambda}$ with $U = \bigcap_{i \in \lambda} \psi_{j_i}(U_i)$.

We can choose for all $i \in \lambda$ pairwise disjoint open sets $B_i^* \in \mathcal{B}^*$ with $\psi(j_i) \in B_i^*$ and $x_i \in X$ with $j_i(x_i) = U_i$.

Obviously $\sigma := \langle \{ x_i \}_{i \in \lambda}; \{ j_i \}_{i \in \lambda} \rangle$ is an element of $\Sigma$ and we have the condition $\emptyset \neq W \cap \bigcap_{i \in \lambda} \psi_{j_i}^{-1}(j_i(x_i))$, thus $Q_\sigma \cap U \neq \emptyset$.

Therefore $Q$ is dense in $W$ and we have $d(W) \leq |\kappa| \leq \mu^{<\kappa}$. \hfill $\square$

The following is a slight generalisation of the Hewitt-Marczewski-Pondiczery theorem:

**Theorem 2.4.** Let $\kappa$ and $\lambda$ be two infinite cardinals with $\mu \geq \kappa$ and let $\square_{\kappa}^\kappa \bigotimes_{i \in I} X_i$ a $\kappa$-box-product with $|I| \leq 2^{\mu}$ and $d(X_i) \leq \mu$ for all $i \in I$.

Then $d(\square_{\kappa}^\kappa \bigotimes_{i \in I} X_i) \leq \mu^{<\kappa}$.

**Proof.** Obviously there is a set $D$ which is dense in $\square_{\kappa}^\kappa \bigotimes_{i \in I} X_i$ and $|\psi_i(D)| \leq \mu$ for all $i \in I$.

Let $\square_{\kappa}^\kappa W_i$ be the $\kappa$-box-product of discrete spaces $W_i$ with $|W_i| = \mu$ and let $\psi : \prod_{i \in I} W_i \to D$ be a continuous and surjective function. Because $\prod_{i \in I} W_i$ itself is an union of $\mu$-cubes and due to theorem 2.3 there is a dense subset $Q$ of $W$ with $|Q| \leq \mu^{<\kappa}$.

Let $O$ be a nonempty open set in $\bigotimes_{i \in I} X_i$. Then $D \cap O \neq \emptyset$ and $f^{-1}(D \cap O)$ is open in $\square_{\kappa}^\kappa W_i$.

So $Q \cap f^{-1}(D \cap O) \neq \emptyset$ and $\emptyset \neq f(Q \cap f^{-1}(D \cap O)) \subseteq f(Q) \cap O$. Therefore $f(Q)$ is dense in $\square_{\kappa}^\kappa X_i$ and $d(\square_{\kappa}^\kappa X_i) \leq \mu^{<\kappa}$.

Following Wanjun Hu we define:

**Definition 2.5.** Let $S$ be an infinite set, $\kappa$, $\lambda$ and $\theta$ be three cardinals with $\kappa \geq \aleph_0$ and $\lambda \geq 2$. A family $J = \{ J_\alpha \}_{\alpha \in \Sigma}$, of partitions $J_\alpha = \{ J_\alpha^\beta ; \beta \in \lambda \}$ of $S$ is called a $(\kappa, \theta, \lambda)$-generalized independent family, if following holds:

\[ \forall J \in P_{\kappa}(\tau) \forall f : J \to \lambda : \left| \left\{ \bigcap J_\alpha^{f(\alpha)} ; \alpha \in J \right\} \right| \geq \theta \]

We can now apply 2.4 on this theorem and we receive the following:

**Corollary 2.6.** Let $\kappa$ and $\lambda$ be two infinite cardinals with $\mu \geq \kappa$.

On every set with at least $\mu^{<\kappa}$ elements exists a $(\kappa, 1, \mu)$-generalized independent family of cardinality $2^\mu$.

**Proof.** Let $S$ be a set of cardinality $\mu^{<\kappa}$.

For every family $\{ X_i \}_{i \in \mu}$ of topological spaces with $d(X_i) \leq \lambda$ the following
holds with theorem 2.4:
\[ d \left( \bigcap_{i \in \mu} X_i \right) \leq |S| \]
Wanjun Hu proved in theorem 3.2 in [4] that this is equivalent to the existence of a \((\kappa, 1, \mu)\)-generalized independent family of cardinality \(2^\mu\) on \(S\). \[\Box\]

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References


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